Section 3.4: Derivatives of Logarithmic Functions

<u>Theorem</u>: For any positive number $b \neq 1$,

$$[\log_b(x)]' = \frac{1}{x\ln(b)}$$

In particular,

$$[\ln(x)]' = \frac{1}{x}.$$

<u>Proof</u>: If $y = \log_b(x)$ then $x = b^y$ so

$$\frac{d}{dx}[x] = \frac{d}{dx}[b^y]$$

$$1 = b^y \ln(b) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln(b)}$$

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x\ln(b)}.$$

eg $f(x) = \log_4(x)\ln(x)$

$$f'(x) = [\log_4(x)]'\ln(x) + \log_4(x)[\ln(x)]'$$

$$= \frac{1}{x\ln(4)} \cdot \ln(x) + \log_4(x) \cdot \frac{1}{x}$$

$$\boxed{= \frac{\ln(x)}{x\ln(4)} + \frac{\log_4(x)}{x}}$$

Recall the basic properties of logarithms:

① $\ln(xy) = \ln(x) + \ln(y)$

② $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

③ $\ln(x^k) = k\ln(x)$

eg $f(x) = \ln(x^3)$

We could use the Chain Rule:

$$f'(x) = \frac{1}{x^3} \cdot [x^3]'$$

$$= \frac{1}{x^3} \cdot 3x^2 \boxed{= \frac{3}{x}}$$

Alternatively, we could first rewrite $f(x)$:

$$f(x) = 3\ln(x)$$

$$f'(x) = 3\cdot\frac{1}{x} \boxed{= \frac{3}{x}}$$

eg $y = \ln\left(\frac{\sqrt{2x^3-1}}{x(x^2+5)^4}\right)$

Rewrite: $y = \ln(\sqrt{2x^3-1}) - \ln(x(x^2+5)^4)$

$$= \ln((2x^3-1)^{1/2}) - [\ln(x) + \ln((x^2+5)^4)]$$

$$= \frac{1}{2}\ln(2x^3-1) - \ln(x) - 4\ln(x^2+5)$$

Differentiate: $y' = \frac{1}{2}\cdot\frac{1}{2x^3-1}\cdot[2x^3-1]' - \frac{1}{x} - 4\cdot\frac{1}{x^2+5}\cdot[x^2+5]'$

$$= \frac{1}{2}\cdot\frac{1}{2x^3-1}\cdot 6x^2 - \frac{1}{x} - 4\cdot\frac{1}{x^2+5}\cdot 2x$$

$$\boxed{= \frac{3x^2}{2x^3-1} - \frac{1}{x} - \frac{8x}{x^2+5}}$$

When we want to differentiate a logarithmic function it is usually beneficial to first apply all appropriate properties of logarithms to simplify the given function before differentiating.

Now consider a function $y=f(x)$ which consists of many products, quotients and/or composite functions, but is not a logarithmic function. Then we could rewrite the given function by introducing logarithms:

$$\ln(y) = \ln(f(x))$$

and then rewrite the righthand side using the properties of logarithms. Then we could differentiate implicitly and find $\frac{dy}{dx}$. This is logarithmic differentiation.

eg $\quad y = \dfrac{(x^3+1)^4}{\sqrt{2x+7}}$

We will use logarithmic differentiation:

$$\ln(y) = \ln\left(\frac{(x^3+1)^4}{\sqrt{2x+7}}\right)$$

$$= \ln\left((x^3+1)^4\right) - \ln\left(\sqrt{2x+7}\right)$$

$$= 4\ln(x^3+1) - \tfrac{1}{2}\ln(2x+7)$$

Now we differentiate implicitly:

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[4\ln(x^3+1) - \tfrac{1}{2}\ln(2x+7)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 4\cdot\frac{1}{x^3+1}\cdot\frac{d}{dx}\left[x^3+1\right] - \frac{1}{2}\cdot\frac{1}{2x+7}\cdot\frac{d}{dx}\left[2x+7\right]$$

$$= 4\cdot\frac{1}{x^3+1}\cdot 3x^2 - \frac{1}{2}\cdot\frac{1}{2x+7}\cdot 2$$

$$= \frac{12x^2}{x^3+1} - \frac{1}{2x+7}$$

$$\frac{dy}{dx} = y\left(\frac{12x^2}{x^3+1} - \frac{1}{2x+7}\right)$$

$$\boxed{= \frac{(x^3+1)^4}{\sqrt{2x+7}}\left(\frac{12x^2}{x^3+1} - \frac{1}{2x+7}\right)}$$

While logarithmic differentiation is optional in most cases, it is effectively necessary in order to differentiate functions of the form $[f(x)]^{g(x)}$.

eg $y = x^{\sin(x)}$

We cannot apply the Power Rule because it requires the power to be a constant.

Likewise, the derivative of b^x needs a constant b.

Instead, we apply logarithmic differentiation:

$$\ln(y) = \ln\left(x^{\sin(x)}\right)$$
$$= \sin(x)\ln(x)$$

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[\sin(x)\ln(x)]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}[\sin(x)]\cdot\ln(x) + \sin(x)\cdot\frac{d}{dx}[\ln(x)]$$

$$= \cos(x)\ln(x) + \sin(x)\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = y\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right]$$

$$\boxed{= x^{\sin(x)}\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right]}$$