Section 1.3: The Properties of Limits

Theorem: Basic Properties of Limits

Suppose $f(x)$ and $g(x)$ are functions for which $\lim_{x \to p} f(x)$ and $\lim_{x \to p} g(x)$ both exist. Then

1. $$\lim_{x \to p} [f(x) + g(x)] = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$$

2. $$\lim_{x \to p} [f(x) - g(x)] = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)$$

3. $$\lim_{x \to p} [k \cdot f(x)] = k \cdot \lim_{x \to p} f(x)$$

   for any constant k

④ $\lim_{x\to p} [f(x)\cdot g(x)] = [\lim_{x\to p} f(x)]\cdot[\lim_{x\to p} g(x)]$

⑤ $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)}$ if $\lim_{x\to p} g(x) \neq 0$

eg Suppose $\lim_{x\to 4} A(x) = 3$ and $\lim_{x\to 4} B(x) = -7$.

Calculate $\lim_{x\to 4} [5A(x) - B(x)]$.

We write $\lim_{x\to 4} [5A(x) - B(x)]$

$= \lim_{x\to 4} [5A(x)] - \lim_{x\to 4} B(x)$ by Property ②

$= 5\cdot \lim_{x\to 4} A(x) - \lim_{x\to 4} B(x)$ by Property ③

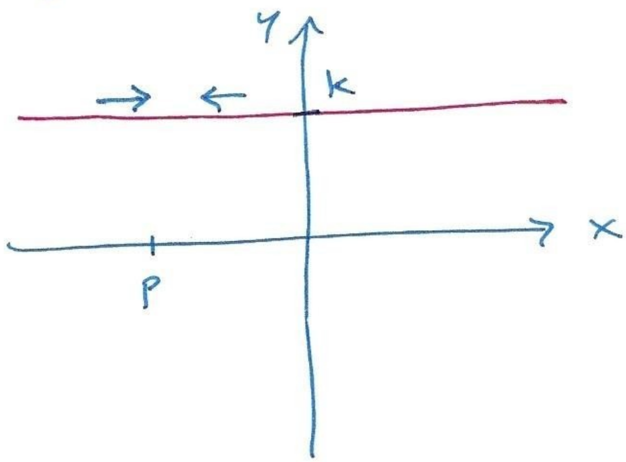
$= 5\cdot 3 - (-7)$

$\boxed{= 22}$

Theorem: Given a constant k,

$$\lim_{x \to p} k = k.$$

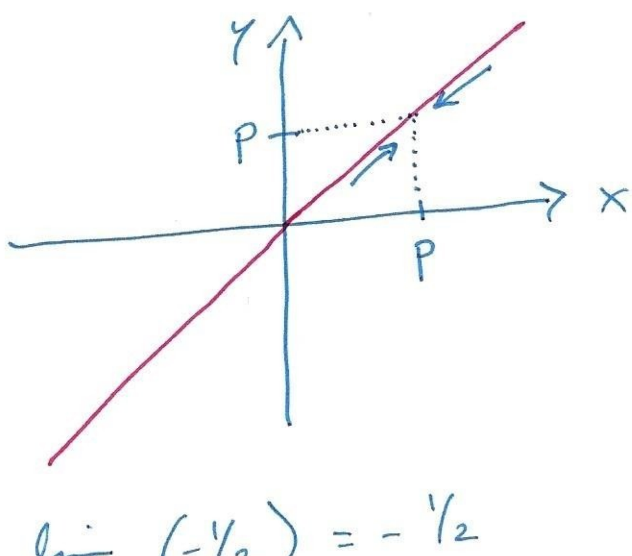
Proof: The graph of $f(x) = k$ is a horizontal line:



$$\lim_{x \to p} k = k$$

Theorem: $\lim_{x \to p} x = p$

Proof: The graph of $f(x) = x$ is a diagonal line:



$$\lim_{x \to p} x = p$$

eg

$$\lim_{x \to 6} \left(-\tfrac{1}{2}\right) = -\tfrac{1}{2}$$

$$\lim_{x \to 0} \pi = \pi$$

$$\lim_{x \to -\sqrt{2}} x = -\sqrt{2}$$

Theorem: For any natural number n,

$$\lim_{x \to p} x^n = p^n.$$

Proof: We can write

$$x^n = \underbrace{x \cdot x \cdot x \cdot x \cdots x}_{n \text{ times}}$$

(Using Basic Limit Prop. #4) $$\lim_{x \to p} x^n = \underbrace{\left[\lim_{x \to p} x\right] \cdot \left[\lim_{x \to p} x\right] \cdots \left[\lim_{x \to p} x\right]}_{n \text{ times}}$$

$$= \underbrace{p \cdot p \cdots p}_{n \text{ times}} = p^n$$

eg $$\lim_{x \to -4} x^3 = (-4)^3 \boxed{= -64}$$

Theorem: For any natural number n,

$$\lim_{x \to p} x^{-n} = \lim_{x \to p} \frac{1}{x^n} = \frac{1}{p^n} \quad \text{if } p \neq 0$$

Proof: Using Basic Limit Prop. #5,

$$\lim_{x \to p} x^{-n} = \frac{\lim_{x \to p} 1}{\lim_{x \to p} x^n} = \frac{1}{p^n} \quad \text{if } p \neq 0$$

eg $$\lim_{x \to 5} x^{-2} = 5^{-2} = \frac{1}{5^2} \boxed{= \frac{1}{25}}$$

Theorem: For any natural number n,

$$\lim_{x \to p} x^{1/n} = \lim_{x \to p} \sqrt[n]{x} = \sqrt[n]{p}$$

For all p if n is odd, and for $p > 0$ if n is even.

eg $\lim_{x \to 64} \sqrt{x} = \sqrt{64} \boxed{= 8}$

$\lim_{x \to -64} \sqrt{x}$ does not exist because $p \leq 0$ and $n = 2$ is even

$\lim_{x \to 64} \sqrt[3]{x} = \sqrt[3]{64} \boxed{= 4}$

$\lim_{x \to -64} \sqrt[3]{x} = \sqrt[3]{-64} \boxed{= -4}$

Def'n: A polynomial function is any function which can be written in the form

$$f(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0$$

where $k_0, k_1, \ldots, k_n$ are constants called the coefficients. The natural number n is called the degree of the polynomial.

eg $f(x) = x^3 - 2x + 4$ is a polynomial of degree 3.

eg $\lim_{x\to -3}(x^3-2x+4)$

$= \lim_{x\to -3} x^3 - \lim_{x\to -3} 2x + \lim_{x\to -3} 4$ by Basic Limit Prop. #1 and #2

$= \lim_{x\to -3} x^3 - 2\lim_{x\to -3} x + \lim_{x\to -3} 4$ by Basic Limit Prop. #3

$= (-3)^3 - 2\cdot(-3) + 4$

$= -27 + 6 + 4 \boxed{= -17}$

Def'n: A rational function is any function of the form $f(x)=\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions.

eg $f(x)=\dfrac{8x+5}{3x^2-x+4}$ is a rational function

eg $\lim_{x\to 2}\dfrac{8x+5}{3x^2-x+4}$

$= \dfrac{\lim_{x\to 2}(8x+5)}{\lim_{x\to 2}(3x^2-x+4)}$ by Basic Limit Prop. #5

$= \dfrac{\lim_{x\to 2} 8x + \lim_{x\to 2} 5}{\lim_{x\to 2} 3x^2 - \lim_{x\to 2} x + \lim_{x\to 2} 4}$ by Basic Limit Prop. #1 and #2

$= \dfrac{8\lim_{x\to 2} x + \lim_{x\to 2} 5}{3\lim_{x\to 2} x^2 - \lim_{x\to 2} x + \lim_{x\to 2} 4}$ by Basic Limit Prop. #3

$$= \frac{8 \cdot 2 + 5}{3 \cdot (2^2) - 2 + 4}$$

$$= \frac{21}{14}$$

$$\boxed{= \frac{3}{2}}$$

Theorem: If $f(x)$ is a polynomial function, or a rational function for which $f(p)$ is defined, then $\lim_{x \to p} f(x) = f(p)$.

Proof: Suppose $f(x)$ is a polynomial function of the form

$$f(x) = k_n x^n + k_{n-1} x^{n-1} + \dots + k_1 x + k_0$$

Then

$$\begin{aligned}
\lim_{x \to p} f(x) &= \lim_{x \to p} \left[k_n x^n + k_{n-1} x^{n-1} + \dots + k_1 x + k_0 \right] \\
&= \lim_{x \to p} k_n x^n + \lim_{x \to p} k_{n-1} x^{n-1} + \dots + \lim_{x \to p} k_1 x + \lim_{x \to p} k_0 && \text{(by Basic Limit Prop. \#1)} \\
&= k_n \lim_{x \to p} x^n + k_{n-1} \lim_{x \to p} x^{n-1} + \dots + k_1 \lim_{x \to p} x + \lim_{x \to p} k_0 && \text{(by Basic Limit Prop. \#3)} \\
&= k_n \cdot p^n + k_{n-1} \cdot p^{n-1} + \dots + k_1 \cdot p + k_0 \\
&= f(p)
\end{aligned}$$

Now suppose that $f(x)$ is a rational function of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, and $Q(p) \neq 0$.

Then

$$\begin{aligned}
\lim_{x \to p} f(x) &= \lim_{x \to p} \frac{P(x)}{Q(x)} \\
&= \frac{\lim_{x \to p} P(x)}{\lim_{x \to p} Q(x)} && \text{by Basic Limit Prop. \#5} \\
&= \frac{P(p)}{Q(p)} = f(p)
\end{aligned}$$

eg $\lim_{x \to 1} \left(-3x^7 + x^4 + 5x^3 - \frac{x}{2}\right)$

$= -3 \cdot 1^7 + 1^4 + 5 \cdot 1^3 - \frac{1}{2}$

$= -3 + 1 + 5 - \frac{1}{2} \; \boxed{= \frac{5}{2}}$

eg $\lim_{t \to 4} \frac{t^2 - 2}{4t - 1} = \frac{4^2 - 2}{4 \cdot 4 - 1} \boxed{= \frac{14}{15}}$

Direct substitution can also be applied to most limits of <u>transcendental</u> functions.

<u>Theorem</u> : ① $\lim_{x \to p} \sin(x) = \sin(p)$

② $\lim_{x \to p} \cos(x) = \cos(p)$

③ $\lim_{x \to p} \tan(x) = \tan(p)$

④ $\lim_{x \to p} \cot(x) = \cot(p)$

⑤ $\lim_{x \to p} \sec(x) = \sec(p)$

⑥ $\lim_{x \to p} \csc(x) = \csc(p)$

⑦ $\lim_{x \to p} b^x = b^p$ for $b > 0$

⑧ $\lim_{x \to p} \log_b(x) = \log_b(p)$ for $b > 0$

Here we assume that $x = p$ lies in the domain of the transcendental function.

eg $\lim_{x \to \pi} \cos(x) = \cos(\pi) \boxed{= -1}$

eg $\lim_{x \to 0} 3^x = 3^0 \boxed{= 1}$

Theorem: Given a natural number n,

① $\lim_{x \to p} [f(x)]^n = \left[\lim_{x \to p} f(x)\right]^n$

② $\lim_{x \to p} [f(x)]^{-n} = \lim_{x \to p} \frac{1}{[f(x)]^n} = \frac{1}{\left[\lim_{x \to p} f(x)\right]^n}$

③ $\lim_{x \to p} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to p} f(x)}$

for all p if n is odd, and

for p such that $f(p) > 0$ if

n is even

eg $\lim_{x \to \frac{\pi}{3}} \sin^4(x) = \lim_{x \to \frac{\pi}{3}} [\sin(x)]^4$

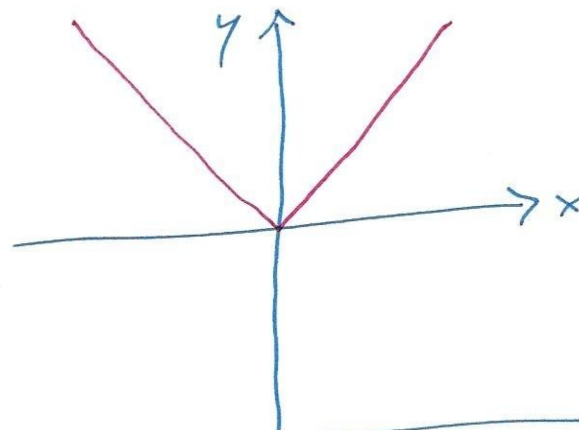
$$= \left[\lim_{x \to \frac{\pi}{3}} \sin(x)\right]^4$$

$$= \left[\sin\left(\frac{\pi}{3}\right)\right]^4$$

$$= \left(\frac{\sqrt{3}}{2}\right)^4 \boxed{= \frac{9}{16}}$$

We can also apply these results to calculate the one-sided limits when investigating piecewise functions.

eg $\lim_{x \to 0} |x|$



$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x)$$
$$= 0$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x$$
$$= 0$$

$\boxed{\text{Hence } \lim_{x \to 0} |x| = 0.}$