1. a) $\lim_{x\to 5} \dfrac{25-x^2}{x^2-2x-15}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 5} \dfrac{-(x^2-25)}{x^2-2x-15}$

$= \lim_{x\to 5} \dfrac{-(x-5)(x+5)}{(x-5)(x+3)}$

$= \lim_{x\to 5} \dfrac{-(x+5)}{x+3} = \dfrac{-10}{8} \boxed{= -\dfrac{5}{4}}$

b) $\lim_{x\to -1} \dfrac{3x}{(x+1)^2}$ does not exist because direct substitution yields a $\frac{k}{0}$ form

From the left: $\lim_{x\to -1^-} \dfrac{3x}{(x+1)^2} = -\infty$

From the right: $\lim_{x\to -1^+} \dfrac{3x}{(x+1)^2} = -\infty$

Thus we can assign $\lim_{x\to -1} \dfrac{3x}{(x+1)^2} \boxed{= -\infty}$.

c) $\lim_{x\to 0} \dfrac{x}{1-\sqrt{1+x}} \cdot \dfrac{1+\sqrt{1+x}}{1+\sqrt{1+x}}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \dfrac{x(1+\sqrt{1+x})}{1-(1+x)}$

$= \lim_{x\to 0} \dfrac{x(1+\sqrt{1+x})}{-x}$

$= \lim_{x\to 0} \dfrac{1+\sqrt{1+x}}{-1} = \dfrac{2}{-1} \boxed{= -2}$

2. a) $f(0) = \dfrac{0^2 - 4}{0^2 - 0 - 2} = 2$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{4}{x+2} = 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{x^2 - 4}{x^2 - x - 2} = 2$

Thus $\lim_{x \to 0} f(x) = 2 = f(0)$ and so $f(x)$ is continuous at $x = 0$.

b) First we set $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2 \quad x = -1 \rightarrow$ OMIT

Now $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^2 - 4}{x^2 - x - 2}$ ($\frac{0}{0}$ form)

$= \lim_{x \to 2} \dfrac{(x-2)(x+2)}{(x-2)(x+1)}$

$= \lim_{x \to 2} \dfrac{x+2}{x+1} = \dfrac{4}{3}$

Hence $x = 2$ is a removable discontinuity.

Next we set $x + 2 = 0$ so $x = -2$.

Now $\lim_{x \to -2} f(x) = \lim_{x \to -2} \dfrac{4}{x+2}$ does not exist because direct substitution yields a $\frac{k}{0}$ form. Thus $x = -2$ is a non-removable discontinuity.

3. a) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1} + \sqrt{2x+1}}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$

$$= \lim_{h \to 0} \frac{(2x+2h+1) - (2x+1)}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$

$$= \frac{2}{2\sqrt{2x+1}}$$

$$\boxed{= \frac{1}{\sqrt{2x+1}}}$$

b) $m = f'(4) = \dfrac{1}{\sqrt{8+1}} = \dfrac{1}{3}$

$y = f(4) = \sqrt{8+1} = 3$

The equation of the tangent line is

$$y - 3 = \frac{1}{3}(x-4)$$

$$y = \frac{1}{3}x - \frac{4}{3} + 3$$

$$\boxed{y = \frac{1}{3}x + \frac{5}{3}}$$

4. a) $y = x^4 e^{x^3}$

$$y' = [x^4]' e^{x^3} + x^4 [e^{x^3}]'$$

$$= 4x^3 e^{x^3} + x^4 \cdot e^{x^3} \cdot [x^3]'$$

$$= 4x^3 e^{x^3} + x^4 \cdot e^{x^3} \cdot 3x^2$$

$$\boxed{= 4x^3 e^{x^3} + 3x^6 e^{x^3}}$$

b) $y = \cos(\sqrt{x}) + \sqrt{\cosh(x)}$

$$y' = [\cos(\sqrt{x})]' + [\sqrt{\cosh(x)}]'$$

$$= -\sin(\sqrt{x}) \cdot [\sqrt{x}]' + \frac{1}{2}(\cosh(x))^{-1/2} \cdot [\cosh(x)]'$$

$$= -\sin(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} + \frac{1}{2}(\cosh(x))^{-1/2} \cdot \sinh(x)$$

$$\boxed{\begin{aligned} &= -\frac{1}{2}x^{-1/2}\sin(\sqrt{x}) + \frac{1}{2}(\cosh(x))^{-1/2}\sinh(x) \\ &= \frac{-\sin(\sqrt{x})}{2\sqrt{x}} + \frac{\sinh(x)}{2\sqrt{\cosh(x)}} \end{aligned}}$$

c) $y - \frac{x}{y} = \tan(y)$

$$y - xy^{-1} = \tan(y)$$

$$\frac{d}{dx}[y - xy^{-1}] = \frac{d}{dx}[\tan(y)]$$

$$\frac{dy}{dx} - \left(\frac{d}{dx}[x]y^{-1} + x \cdot \frac{d}{dx}[y^{-1}]\right) = \sec^2(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} - \left(1 \cdot y^{-1} + x \cdot (-1) y^{-2} \frac{dy}{dx}\right) = \sec^2(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} - y^{-1} + x y^{-2} \frac{dy}{dx} = \sec^2(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} + x y^{-2} \frac{dy}{dx} - \sec^2(y) \frac{dy}{dx} = y^{-1}$$

$$\frac{dy}{dx}\left[1 + x y^{-2} - \sec^2(y)\right] = y^{-1}$$

$$\frac{dy}{dx} = \frac{y^{-1}}{1 + x y^{-2} - \sec^2(y)} \boxed{= \frac{y}{y^2 + x - y^2 \sec^2(y)}}$$

d) $y = \dfrac{(x^2+1)^x}{x}$

$$\ln(y) = \ln\left(\frac{(x^2+1)^x}{x}\right)$$

$$= \ln\left((x^2+1)^x\right) - \ln(x)$$

$$= x \ln(x^2+1) - \ln(x)$$

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}\left[x \ln(x^2+1) - \ln(x)\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}[x] \ln(x^2+1) + x \cdot \frac{d}{dx}\left[\ln(x^2+1)\right] - \frac{1}{x}$$

$$= 1 \cdot \ln(x^2+1) + x \cdot \frac{1}{x^2+1} \cdot \frac{d}{dx}[x^2+1] - \frac{1}{x}$$

$$= \ln(x^2+1) + x \cdot \frac{1}{x^2+1} \cdot 2x - \frac{1}{x}$$

$$\frac{dy}{dx} = y\left[\ln(x^2+1) + \frac{2x^2}{x^2+1} - \frac{1}{x}\right] \boxed{= \frac{(x^2+1)^x}{x}\left[\ln(x^2+1) + \frac{2x^2}{x^2+1} - \frac{1}{x}\right]}$$

5. a) $y = \dfrac{\arctan(x)}{x^2+1}$

$$y' = \frac{[\arctan(x)]'\,(x^2+1) - \arctan(x)\cdot[x^2+1]'}{(x^2+1)^2}$$

$$= \frac{\frac{1}{x^2+1}\cdot(x^2+1) - \arctan(x)\cdot 2x}{(x^2+1)^2}$$

$$= \frac{1 - 2x\arctan(x)}{(x^2+1)^2}$$

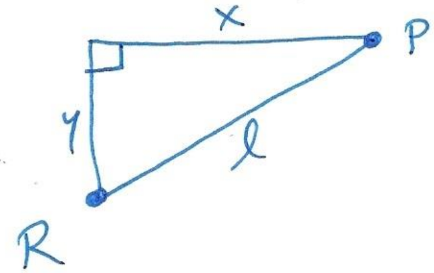
$$y'(1) = \frac{1 - 2\cdot 1\cdot \arctan(1)}{(1^2+1)^2} = \frac{1 - 2\cdot \pi/4}{4}$$

$$\boxed{= \frac{1}{4} - \frac{\pi}{8} = \frac{2-\pi}{8}}$$

b) Observe that the highest power of x in $f(x)$ is x^{49}. If we differentiate $f(x)$ 50 times then every term must differentiate to 0.

Hence $f^{(50)}(x) \boxed{= 0.}$

6.



Let x be the horizontal distance from the radar to the plane.

Let y be the altitude of the plane.

Let l be the distance from the radar to the plane.

We are given that $y = 6$ (constant).

We want to find $\frac{dx}{dt}$ when $l = 10$ and $\frac{dl}{dt} = 200$.

By the Pythagorean theorem,

$$x^2 + y^2 = l^2$$

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[l^2\right]$$

$$2x \cdot \frac{dx}{dt} + 0 = 2l \cdot \frac{dl}{dt}$$

$$x \cdot \frac{dx}{dt} = l \cdot \frac{dl}{dt} \qquad (*)$$

Now $x^2 + y^2 = l^2$ so $x^2 + 6^2 = 10^2$

$$x^2 = 64 \rightarrow x = 8$$

We substitute into Equation $(*)$:

$$8 \cdot \frac{dx}{dt} = 10 \cdot 200$$

$$\frac{dx}{dt} = \frac{2000}{8} = 250$$

The speed of the plane is 250 km/hour.

7. a) We set $(x+1)^2 = 0$

$$x+1 = 0 \rightarrow x=-1$$

Now $\displaystyle\lim_{x\to -1} \frac{4x(x+3)}{(x+1)^2}$ does not exist since we obtain a $\frac{K}{0}$ form by direct substitution. Hence $\boxed{x=-1}$ is a vertical asymptote.

b) $$\lim_{x\to\infty} \frac{4x(x+3)}{(x+1)^2} = \lim_{x\to\infty} \frac{4x^2+12x}{x^2+2x+1} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{4+12/x}{1+2/x+1/x^2}$$

$$= \frac{4+0}{1+0+0} = 4$$

Because this is a rational function, $\displaystyle\lim_{x\to -\infty} f(x) = 4$.

Thus the only horizontal asymptote is $\boxed{y=4}$.

c) For any x-intercepts we set $f(x)=0$ so

$$4x(x+3)=0 \rightarrow x=0,\ x=-3$$

Thus $\boxed{(0,0) \text{ and } (-3,0)}$ are both x-intercepts.

Hence $f(0)=0$ and so $\boxed{(0,0)}$ is also the y-intercept.

d) To find any critical points, we first note that $f'(x)$ is undefined only at $x=-1$ (the vertical asymptote) so we set $f'(x)=0$:

$$4(3-x)=0 \rightarrow x=3$$

	$x<-1$	-1	$-1<x<3$	3	$x>3$
$f'(x)$	$-$	VA	$+$		$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$

$f(x)$ is increasing for $-1<x<3$
decreasing for $x<-1$ and $x>3$

$x=3$ is a relative maximum; this is the point $(3, 9/2)$

e) To find any hypercritical points, we first note that $f''(x)$ is undefined only at $x=-1$ (the vertical asymptote) so we set $f''(x)=0$:

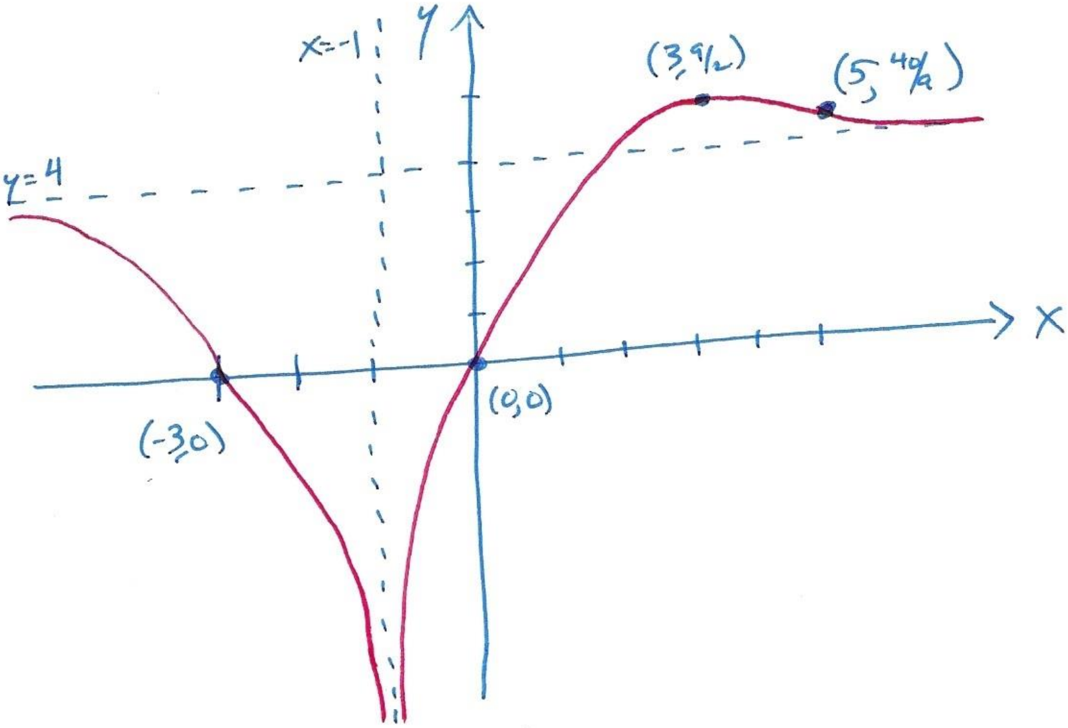
$$8(x-5)=0 \rightarrow x=5$$

	$x<-1$	-1	$-1<x<5$	5	$x>5$
$f''(x)$	$-$	VA	$-$		$+$
$f(x)$	$\frown$		$\frown$		$\smile$

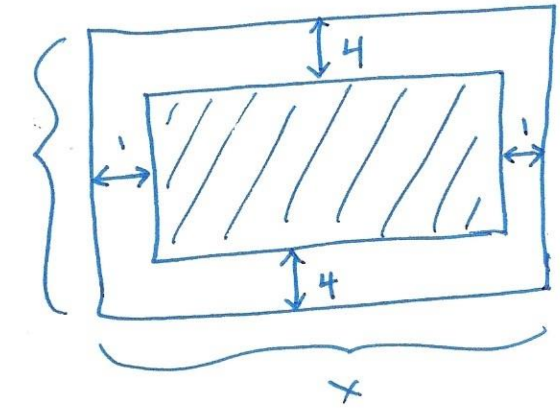
$f(x)$ is concave upward for $x>5$
concave downward for $x<-1$ and $-1<x<5$

$x=5$ is a point of inflection; this is the point $(5, 40/9)$

f)



8.



Let x be the length of the billboard, and y be the width of the billboard. Let a be the area of the billboard, and A be the area of the display.

We are given that $a = 81$, and we want to maximize A.

The primary equation is $A = (x-2)(y-8)$

$$= xy - 8x - 2y + 16$$

The secondary equation is $a = xy = 81$ so $y = \frac{81}{x}$.

The reduced primary equation is

$$A(x) = x\left(\frac{81}{x}\right) - 8x - 2\left(\frac{81}{x}\right) + 16$$

$$= 97 - 8x - \frac{162}{x}, \quad x > 0$$

$$A'(x) = -8 + \frac{162}{x^2}$$

We set $A'(x) = 0$ so $\frac{162}{x^2} = 8 \rightarrow x^2 = \frac{162}{8} = \frac{81}{4}$

$x = \frac{9}{2}$ (CRITICAL POINT)

$A''(x) = -\frac{324}{x^3} \rightarrow A''\left(\frac{9}{2}\right) < 0$ so $x = \frac{9}{2}$ is the absolute maximum

When $x = \frac{9}{2}$, $y = \frac{81}{9/2} = 18$ so the dimensions of the billboard which maximize the display area are $\underline{\frac{9}{2}\text{m} \times 18\text{m}}$.

9. a) $\lim_{x\to 0} \dfrac{e^x - x - 1}{x^2 - x + \sin(x)}$ $\left(\frac{0}{0} \text{ form}\right)$

$\overset{H}{=} \lim_{x\to 0} \dfrac{[e^x - x - 1]'}{[x^2 - x + \sin(x)]'}$

$= \lim_{x\to 0} \dfrac{e^x - 1}{2x - 1 + \cos(x)}$ $\left(\frac{0}{0} \text{ form}\right)$

$\overset{H}{=} \lim_{x\to 0} \dfrac{[e^x - 1]'}{[2x - 1 + \cos(x)]'}$

$= \lim_{x\to 0} \dfrac{e^x}{2 - \sin(x)}$ $\boxed{= \frac{1}{2}}$

b) $\lim_{x\to\infty} \left(\dfrac{x}{x+1}\right)^x$ (1^∞ form)

Let $y = \left(\dfrac{x}{x+1}\right)^x$

$\ln(y) = \ln\left(\left(\dfrac{x}{x+1}\right)^x\right) = x \ln\left(\dfrac{x}{x+1}\right) = \dfrac{\ln\left(\frac{x}{x+1}\right)}{1/x}$

$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \dfrac{\ln\left(\frac{x}{x+1}\right)}{1/x}$ $\left(\frac{0}{0} \text{ form}\right)$

$\overset{H}{=} \lim_{x\to\infty} \dfrac{\left[\ln\left(\frac{x}{x+1}\right)\right]'}{[1/x]'}$

$= \lim_{x\to\infty} \dfrac{\frac{x+1}{x} \cdot \frac{1\cdot(x+1) - x\cdot 1}{(x+1)^2}}{-1/x^2}$

$= \lim_{x\to\infty} \dfrac{\frac{x+1}{x} \cdot \frac{1}{(x+1)^2}}{-1/x^2}$

$$= \lim_{x\to\infty} \frac{1}{x(x+1)} \cdot (-x^2)$$

$$= \lim_{x\to\infty} \frac{-x}{x+1}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{[-x]'}{[x+1]'}$$

$$= \lim_{x\to\infty} \frac{-1}{1} = -1$$

Thus $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x \boxed{= e^{-1}}$.

10. Let $A(x) = kf(x)$ so

$$[kf(x)]' = A'(x)$$

$$= \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

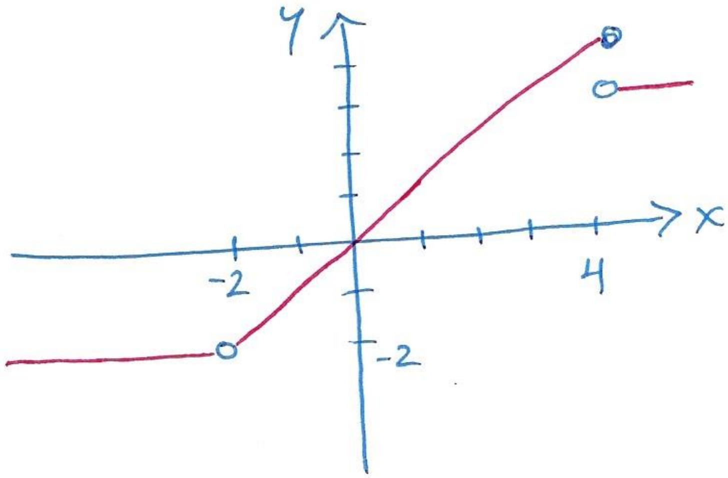
$$= \lim_{h \to 0} \frac{kf(x+h) - kf(x)}{h}$$

$$= k \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= k \cdot f'(x)$$

11.

$$f(x) = \begin{cases} -2, & \text{for } x < -2 \\ x, & \text{for } -2 < x < 4 \\ 3, & \text{for } x > 4 \end{cases}$$



$\lim_{x \to -2^-} f(x) = -2$

$\lim_{x \to -2^+} f(x) = -2$

so $\lim_{x \to -2} f(x) = -2$

$\lim_{x \to 4^-} f(x) = 4$

$\lim_{x \to 4^+} f(x) = 3$

so $\lim_{x \to 4} f(x)$ does not exist