Atlantic Electronic
Journal of Mathematics
Volume **2**, Number **1**, Summer **2007**

http://aejm.ca
http://aejm.ca/rema
pp. **1–13**

A NUMERICAL AND THEORETICAL STUDY OF BLOW-UP FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS USING THE SUNDMAN TRANSFORMATION

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Abstract. The computation of blow-up solutions to a differential equation is often a difficult task. Here, we examine a system of ODEs that is derived from the Maxwell-Debye equations. Blow-up times for solutions to the ODE system are estimated using two approaches – MATLAB event location and a Sundman transformation. The Sundman transformation, whereby a new temporal variable is introduced, results in a system for which solutions exist globally. In addition, it provides a means for simultaneously proving blow-up and finding analytical estimates of blow-up times.

1. **Introduction.** Problems for which solutions develop finite-time singularities occur often in the sciences. Sometimes these situations are of purely theoretical interest, but these singularities can have interesting physical interpretations. As an example, the blow-up of solutions to the nonlinear wave equation describing boson stars is symptomatic of gravitational collapse into a black hole [6]. Here, the equation

$$\left\{ \begin{array}{rcl} y' &=& yz \\ \tau^* z' + z &=& \epsilon y^p \end{array} \right. \tag{1}$$

subject to the initial conditions $(y(0), z(0)) = (y_0, z_0)$ will be considered. A proof of blow-up for solutions to (1) in the case of $\epsilon, p, y_0 > 0$ is presented in the appendix. For a solution to "blow up," some norm of the solution needs to become unbounded in finite time.

The system (1) has its roots in the Maxwell-Debye System (MDS) [7]

$$\left\{ \begin{array}{rcl} \left(\frac{\partial}{\partial z} + \frac{n_0}{c}\frac{\partial}{\partial t}\right) A - \frac{i}{2k}\Delta A + i\frac{\omega_0}{c}\nu A &=& 0 \\ \tau^* \frac{\partial \nu}{\partial t} + \nu &=& n_2 |A|^2. \end{array} \right. \tag{2}$$

This pair of equations, arising in nonlinear optics, models the interaction between an electromagnetic wave and a non-resonant medium. Particularly, A represents the wave envelope of the light wave, and ν is the change in the refractive index of the medium, resulting from the wave. The remaining quantities are physical constants: c is, as usual, the speed of light in the vacuum, n_0 is the initial index of refraction

2000 *Mathematics Subject Classification.* Primary: 65L02; Secondary: 34A02.
Key words and phrases. Sundman transformation, blow-up, event location, ordinary differential equations.
The first author was supported in part by NSERC Canada; the second author was supported in part by NSERC Canada USRA.

of the medium for a light wave with frequency ω_0, k is the wave vector of the incident wave, and n_2 represents the sign and magnitude of the nonlinear coupling of the wave with the medium [2]. An interesting parameter from a mathematical perspective is τ^*. This represents the response time of the medium – an inherent delay. For light pulses longer in duration than this delay, we expect that solutions to (2) will blow up in some finite time [7].

As in [2, 4] we examine the case in which any dependence of A or ν on the longitudinal variable z is dropped. After rearranging and using the rescaling (courtesy of [4])

$$u(x,t) = \left(\frac{\omega_0|n_2|}{n_0}\right)^{1/2} A\left(\left(\frac{c}{kn_0}\right)^{1/2} x, t\right)$$

and

$$v(x,t) = \frac{\omega_0}{n_0}\nu\left(\left(\frac{c}{kn_0}\right)^{1/2} x, t\right),$$

we arrive at the dimensionless *Schrödinger-Debye* (SD) system

$$\left\{ \begin{array}{rcl} i\frac{\partial}{\partial t}u + \frac{1}{2}\Delta u &=& vu, \\ \tau^*\frac{\partial}{\partial t}v + v &=& \epsilon|u|^2. \end{array} \right. \tag{3}$$

System (1) is a simplification of (3) wherein the Laplacian term has been dropped and any coefficients except for τ^* have been set to one. It should be mentioned that due to these changes there will be differences in qualitative behaviour between solutions to (3) and (1). Note that a more obvious choice for an ODE simplification of SD might be the complex-valued system

$$\left\{ \begin{array}{rcl} iy' &=& yz \\ \tau^* z' + z &=& \epsilon|y|^p. \end{array} \right. \tag{4}$$

However, it can be shown that solutions to (4) are of constant modulus and thus do not blow up, so we will not consider (4) further. System (1) is similar to SD in the sense that for $\tau^* = 0$, both reduce to a single equation for which solutions are known to blow up. In the case of (1), this equation is

$$y' = \epsilon y^{p+1} \tag{5}$$

which has exact solution

$$y = \frac{1}{\sqrt[p]{p\epsilon}\sqrt[p]{\frac{1}{p\epsilon y_0^p} - t}}. \tag{6}$$

Blow-up occurs at time t_b when the denominator of (6) is exactly zero, where

$$t_b = \frac{1}{p\epsilon y_0^p},$$

and beyond this time solutions cease to exist. That is, the positive segment of the maximal existence interval of solutions to (5) can be written as $[0, t_b)$.

In [2], it is shown that for initial data (A_0, ν_0) in $H^s \times H^s$ with $s > 1$, as $\tau^* \to 0$, solutions to (3) converge to solutions of the cubic nonlinear Schrödinger equation (NLS),

$$i\frac{\partial}{\partial t}u + \frac{1}{2}\Delta u = \epsilon|u|^2 u. \tag{7}$$

This is a useful relationship, because (7), under certain conditions, is known to have solutions that blow up in finite time (cf.[5]). Consequently, we intuitively expect that for small values of τ^*, solutions to SD (and also to the MDS) will exhibit similar behaviour – that is, they will blow up. Indeed, Besse and Bidégaray [1] generate some nice qualitative numerical results indicating that this is in fact the case. Extrapolating further, we may speculate that as τ^* gets large, solutions to SD and MDS become less like solutions to (7) in the sense that blow-up might be postponed or even disappear. To see why this is reasonable, we rearrange SD to show that

$$\frac{\partial}{\partial t}\nu = -\frac{\nu}{\tau^*} + \frac{\epsilon |u|^2}{\tau^*}.$$

In light of this, it is clear that $\frac{\partial}{\partial t}\nu \to 0$ as $\tau^* \to \infty$, and so in this limit, $\nu(t,x) = \nu_0(x)$. Substituting this result into the first equation of SD, we have

$$i\frac{\partial}{\partial t}u + \frac{1}{2}\Delta u = \nu_0(x)u,$$

which is a linear equation and thus has a global solution for specified initial data.

On the other hand, some results ensuring well-posedness have been established. Besse and Bidégaray [1, 2] use a fixed-point procedure on a Duhamel formulation of SD to show that for initial conditions (u_0, ν_0) belonging to certain Sobolev spaces, solutions exist on $L^\infty(0,T)$ for small enough T. Subsequently, Corcho and Linares [4] proved existence of solutions in particular spaces on any interval $[0,T]$.

In the remainder of the paper, we will consider (1) in the case of $p = 1$, which is sufficient in the sense that this choice of exponent preserves the blow-up properties in which we are interested.

2. **Numerical Detection of Blow-up.** Accurately detecting blow-up solutions is a difficult task [9]. As the blow-up time is approached, the derivative drastically increases and thus smaller and smaller time-steps Δt are required in order to capture the solution with sufficient resolution. If a step is too large, the numerical solution could skip over a blow-up singularity entirely. Supposing that we are able to track the true behaviour of the solution with a reasonable level of accuracy, the question then becomes, how do we find the blow-up time t_b? Since we cannot find the time at which our solution becomes unbounded, one approach is to estimate t_b by noting the time at which the solution reaches some critically high value. We will employ MATLAB's event location feature to realize this idea. A second technique, which will be compared against the event location method, is to employ a *Sundman* transformation, whereby a new temporal variable τ is introduced. Using these methods to solve (1) for various values of the delay parameter, we hope to see the effect of τ^* on the blow-up time.

2.1. **Event Location.** The MATLAB ODE suite is a collection of ODE solvers written for the MATLAB scientific computing environment. These programs are user-friendly and can be used to solve a variety of first-order initial value problems. The user has a choice of solvers, depending on the nature of the problem. Here we will make use of `ode15s.m`, which is an implicit method of variable orders one through five, designed to solve stiff problems.

As a component of its ODE suite, MATLAB contains an event locator option. Of this feature, Shampine and Thompson [8] say, "events are located about as accurately as possible in the precision available." In MATLAB terminology, an event occurs when a user-defined event function attains a value of zero. Since we are trying to estimate a blow-up time, we will choose our event function to be of the form

$$f(y, z, t) = \max(y(t), z(t)) - 10^n, \tag{8}$$

where n is some positive integer.

To test the solver with the event locator option, we refer to the blow-up equation (5) with $p = \epsilon = 1$. The natural choice of $y_0 = 1$ is made. As an event function, we choose (8) with $n = 12$. A level of 10^{12} was selected as a numerical representation of unboundedness somewhat arbitrarily; however, it is noted that increasing n to 13 or 14 does not affect the estimated blow-up times until after the tenth decimal place. At $n = 15$, the ODE solver is unable to satisfy error tolerances. Specifying the relative error to be 10^{-8} using MATLAB's `RelTol` option, (5) is solved with `ode15s.m` and an estimated blow-up time $\widehat{t_b} = 0.99998713167013$ is returned. Recall from (6) that the exact solution to (5) is

$$y(t) = \frac{1}{1-t},$$

with exact blow-up time $t_b = 1$. Estimated blow-up times for various values of τ^* in (1) are presented below. In each case, $p = \epsilon = 1$ with `RelTol` set to 10^{-8}. Note that $\tau^* = 0$ corresponds to the test case (5).

τ^*	$\widehat{t_b}$
0	0.99998713167013
1×10^{-5}	1.00010756938443
1×10^{-3}	1.00725276568088
1×10^{-1}	1.27531513042609
1×10^{1}	3.45546489495704

TABLE 1. Approximate blow-up times $\widehat{t_b}$ for (1) with various τ^*, found through event location.

2.2. **The Sundman Transformation.**

The famous Three-Body Problem regarding the mutual gravitational attraction of three orbiting bodies was first solved by Finnish mathematician Karl Sundman in his 1912 *Memoir on the Three Body Problem*. Therein, Sundman discovered an infinite series solution describing the future motion of three bodies starting from arbitrary positions. However, his infinite series converges much too slowly to be of practical value in predicting the motions of celestial bodies, as originally intended [10]. While his solution to the Three-Body Problem may be of little applicability, we make use of a transformation Sundman employed in his aforementioned work. Such transformations have recently been used, for example, by Budd et al. in [3].

The *Sundman transformation* of a differential equation

$$\frac{du}{dt} = f(u) \tag{9}$$

introduces a fictive temporal variable τ such that the original time variable t is itself considered to be a function of τ. The two are related by

$$\frac{dt}{d\tau} = g(u). \tag{10}$$

According to the chain rule, (9) transforms to

$$\frac{du}{d\tau} = g(u)f(u).$$

By appropriately selecting $g(u)$, we can transform a problem with singular solutions into one with solutions existing globally in the new time variable τ.

3. **A Sundman-Transformed Blow-Up Equation.**

We now implement a Sundman transformation on the simple blow-up equation (5), with $p = 1$. This example will serve as an illustration of the transformation; in addition it will help in determining if this strategy shows promise in detecting blow-up times for (1) and, eventually, the MDS. First, recall that the exact solution to (5) is given by (6), with blow-up at $t_b = 1$. In performing the transformation, $g(y)$ can be chosen as we like, so we select it in a way that makes $y^2 g(y)$ simpler than $f(y) = y^2$. For $f(y) = y^{p+1}$, a suitable transformation function is $g(y) = y^{-p}$ so that $f(y)g(y) = y$. Letting $g(y) = \frac{dt}{d\tau} = \frac{1}{y}$ with the initial condition $t(0) = 0$ we have

$$\frac{dy}{d\tau} = y,\ y(0) = y_0,$$

which is a pleasant result. This equation is solved to yield $y(\tau) = y_0 e^{\tau}$. Substituting into $g(y)$ we have

$$\frac{dt}{d\tau} = \frac{1}{y_0 e^{\tau}}.$$

Thus,

$$t = \frac{1}{y_0}(1 - e^{-\tau}),$$

and clearly $\lim_{\tau \to \infty} t(\tau) = 1$ (since $y_0 = 1$), which is the exact time of blow-up. This example is an ideal case, in that each differential equation arising from the transformation can be solved exactly. This will not always be the situation. In particular, we may not be able to find $g(y)$ for (1) such that $\frac{dy}{d\tau} = f(y)g(y)$ and $\frac{dt}{d\tau} = g(y)$ can be solved analytically. Instead, we will have to rely on approximate solutions. Suppose we have performed the Sundman transformation as above and solved for $dy/d\tau$ using a numerical method such as MATLAB's `ode15s.m` solver. Then a cubic spline interpolant to $y(\tau)$ can be found. Substituting this interpolant into (10) we can perform another `ode15s.m` solve to retrieve $t(\tau)$.

3.1. **Performance for the Test Equation.**

We apply the Sundman transformation to the test equation (5) with $g(y) = 1/y$ as before and solve using a MATLAB script, which operates as follows:

1. Solve $\frac{dy}{d\tau} = f(y)g(y)$ using `ode15s.m`
2. Find cubic spline interpolant to y
3. Solve $\frac{dt}{d\tau} = g(y)$ using `ode15s.m`.

We take the time to which the sequence $\{t_i\}$ appears to converge as our estimate of the blow-up time, $\widehat{t_b}$. For the test problem, with `RelTol` set to 10^{-8} for both `ode15s.m` solves, we find $\widehat{t_b} = 0.99998834051298$. Note that this is more accurate than the solution obtained via event location.

4. **Choosing $g(y, z)$ for the System.** Since the results for the example look promising, we apply a Sundman transformation to (1). Our first task is to select a suitable $g(y, z)$. In [3] it is suggested to choose $g(y, z)$ such that $f_1(y, z)g(y, z)$ and $f_2(y, z)g(y, z)$ are linear. We cannot do this here; instead, we strive to make $\frac{dy}{d\tau} = f_1(y, z)g(y, z)$ and $\frac{dz}{d\tau} = f_2(y, z)g(y, z)$ such that the IVPs are well-posed. This allows us to sidestep the problems that we have thus far encountered in computing blow-up solutions. Our first attempt is to let

$$\frac{dt}{d\tau} = \frac{1}{z}.$$

This results in the transformed system

$$\begin{cases} \frac{dy}{d\tau} & = & y \\ \frac{dz}{d\tau} & = & -\frac{1}{\tau^*} + \frac{\epsilon y}{\tau^* z}. \end{cases} \tag{11}$$

It is claimed that solutions to (11) do not blow up. To see this, observe that (10) is solved to give $y = y_0 e^\tau$. Since $y(\tau)$ cannot experience blow-up in finite τ, this result is substituted into the equation for $\frac{dz}{d\tau}$:

$$\frac{dz}{d\tau} = -\frac{1}{\tau^*} + \frac{\epsilon y_0 e^\tau}{\tau^* z}, \tag{12}$$

with $z_0 = \epsilon y_0$. On the right-hand side of (12), τ is able to take on any real value, while a singularity is encountered at $z(\tau) = 0$. Combining these results with the positivity of z_0, we see that the problem is well-defined on the subset $D = \mathbb{R} \times (0, \infty)$ of the τ-z plane (although negative τ values are ignored). Now suppose that the maximal solution $z(\tau)$ with $z(0) = z_0$ is defined on $I = [0, b)$ where $b < \infty$. Then either $z \to \infty$ as $\tau \to b^-$, or $z \to 0^+$ as $\tau \to b^-$. Suppose the former is true. Then, as $z \to \infty$, $\frac{dz}{d\tau} \to -\frac{1}{\tau^*} < 0$, contradicting the assumption that z is blowing up. On the other hand, suppose that $z \to 0^+$ as $\tau \to b^-$. Since $z_0 > 0$, this must mean that $\frac{dz}{d\tau} < 0$ on some interval (c, b) with $0 \le c < b$. However, (12) indicates that $\frac{dz}{d\tau} \to \infty$ as $\tau \to b^-$. Again we have contradicted our assumption, and hence we conclude that the solution to (12) is defined for $\tau \in I = [0, \infty)$.

With this good behaviour established, we expect that an accurate solution to (11) may be obtained using conventional numerical methods. After having found approximate solution values corresponding to a sequence of τ values, a cubic spline interpolant to $z(\tau)$ is found. We then use `ode15s.m` to solve $dt/d\tau = 1/z$ with the interpolant used in place of z.

τ^*	$\widehat{t_b}$
0	0.99998834051298
1×10^{-5}	1.00010817890966
1×10^{-3}	1.00725294109757
1×10^{-1}	1.27531079880821
1×10^{1}	3.45545384174272

TABLE 2. $\widehat{t_b}$ for various τ^*, found through Sundman transformation.

A plot of $\widehat{t_b}$ versus τ^* over a range of τ^* values is generated and presented below. For each τ^*, $\widehat{t_b}$ is calculated as the final entry in the array of t-values.

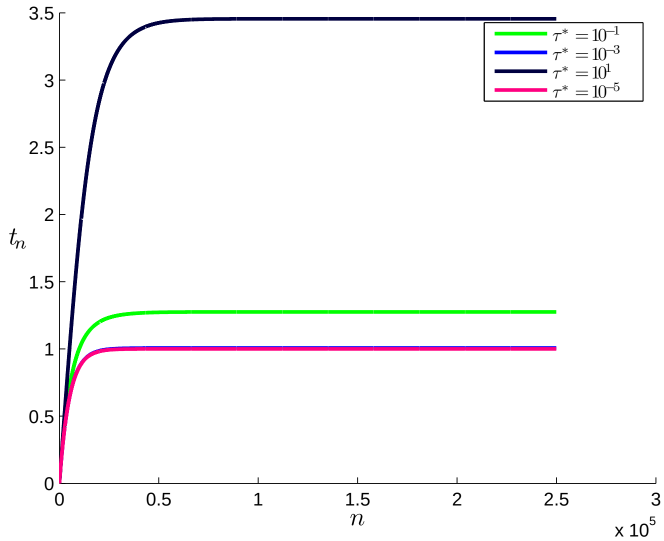

FIGURE 1. t_n versus index n for various τ^*, found through Sundman transformation.

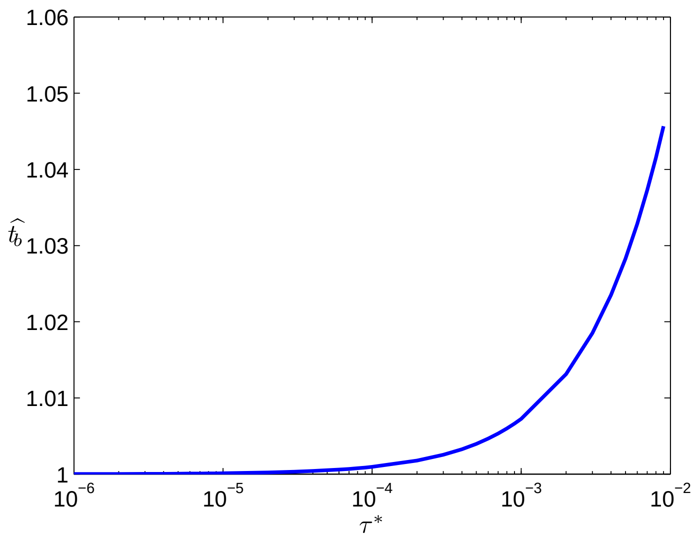

FIGURE 2. $\widehat{t_b}$ vs. τ^*, found through Sundman transformation.

For a direct comparison of the event locator and Sundman transformation methods, a plot of the difference between $\widehat{t_b}$ found using each of the techniques is presented, where the subscripts EL and ST denote results obtained through event location and the Sundman transformation, respectively.

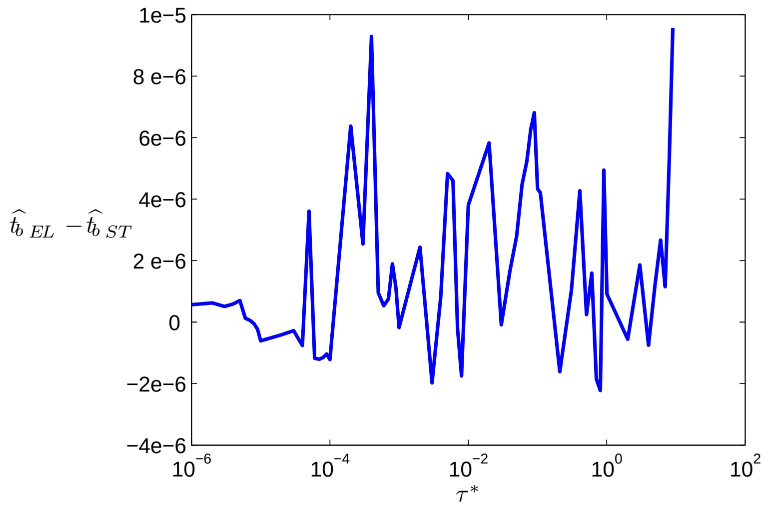

FIGURE 3. Difference in blow-up times, $\widehat{t_b}_{EL} - \widehat{t_b}_{ST}$, versus τ^*.

Figure 3 suggests that the agreement in estimated blow-up times between the Sundman transformation method and the event locator method is very strong. Table 3 below shows just how small the differences are for select values of τ^*.

τ^*	$\widehat{t_b}$ (ST)	$\widehat{t_b}$ (EL)
0	0.99998834051298	0.99998713167013
1×10^{-5}	1.00010817890966	1.00010756938443
1×10^{-3}	1.00725294109757	1.00725276568088
1×10^{-1}	1.27531079880821	1.27531513042609
1×10^{1}	3.45545384174272	3.45546489495704

TABLE 3. A comparison of $\widehat{t_b}$ for ST and EL methods.

The consistency of the results obtained using these two methods is encouraging, as it simultaneously supports the validity of each approach. We place more confidence in the estimates of blow-up time found via the Sundman transformation since this method performed better in the test case. Average run-times for both the EL and ST programs are given below.

τ^*	run-time (sec) (ST)	run-time (sec) (EL)
1×10^{-5}	1.328	1.287
1×10^{-3}	1.315	1.232
1×10^{-1}	1.309	1.182
1×10^{1}	1.259	1.156

TABLE 4. A comparison of average run-times for ST and EL methods.

Table 4 indicates that the Sundman transformation is only slightly more computationally costly than the event location method.

5. **Conclusions.** We have performed Sundman transformations both on an equation for which solutions are known to blow up and on a system for which solutions were previously thought (and are now known) to blow up. In each case, the introduction of a temporal variable τ allows us to transform the equations into ones for which solutions (in terms of τ) exist globally. By numerically solving first for τ then for the original time variable t, we obtain estimates of the blow-up time t_b that, at least in the test case, are more accurate than estimates obtained through event location. This improved accuracy comes with only a small increase in computational cost. Furthermore, the transformation allows us to analytically prove (as is demonstrated in the appendix) that blow-up occurs in the case of system (1), and that the blow-up time t_b depends on the delay parameter τ^*.

The Sundman transformation, however, is not a general fix to the problems associated with the computation of blow-up solutions. For it to be successful, one must be able to find a function $dt/d\tau = g(u)$ such that solutions to $du/d\tau = f(u)g(u)$ exist globally. In the case of the Maxwell-Debye System (2) this may or may not be possible.

Appendix. In this appendix we present and then prove a proposition (stated below) which guarantees blow-up of solutions to (1) and provides bounds for the blow-up time that are subject to the value of the delay parameter τ^*. To accomplish this, the Sundman transformation of (1) is considered. In particular, bounds for $dt/d\tau$ are found. These bounds are then integrated, approximately, to yield bounds for the blow-up time t_b.

Proposition. *For solutions to the system*

$$\begin{cases} y' &= yz \\ \tau^* z' + z &= \epsilon y \end{cases}$$

with $y_0, z_0, \tau^* > 0$ *and* $\epsilon \in (0, 1]$ *blow-up occurs. Furthermore, the time of blow-up* t_b *is subject to the bounds*

$$\frac{1}{z_0} \ln(1 + \tau^* z_0) < t_b < \frac{1}{z_0}\tau^* + \frac{1}{\sqrt{B}} \left(e\pi - 2e \tan^{-1}(\sqrt{e^{\tau^*+2} - 1}) \right)$$

where $B = \frac{2\epsilon y_0}{2/z_0 + \tau^*}$.

Recall that the Sundman transformation of (1) leads to the following equation for $\frac{dz}{d\tau}$:

$$\frac{dz}{d\tau} = -\frac{1}{\tau^*} + \frac{\epsilon e^\tau y_0}{\tau^* z}.$$

For concision, define $a = \frac{1}{\tau^*}$, $b = \frac{\epsilon y_0}{\tau^*}$, and $u(\tau) = \frac{1}{z(\tau)}$, with $z(\tau) > 0$. Then,

$$\frac{du}{d\tau} = u' = au^2 - be^\tau u^3, \tag{13}$$

with $u(0) = u_0 = \frac{1}{z_0}$. Since $\tau^* > 0$, $\epsilon > 0$ and $y_0 > 0$ (by assumption), the inequality $e^\tau \geq 1$ leads to

$$u' = au^2 - be^\tau u^3 \leq au^2 - bu^3.$$

Let $U(\tau)$ be the solution to

$$U' = aU^2 - bU^3,\ U(0) = u_0. \tag{14}$$

Since $U(0) = u_0$, we have that $u(\tau) \leq U(\tau)$ (for proof of this result, see Chapter 2, paragraph 9 of [11]). Recall that for the 'consistent' initial conditions in which we are interested, $z_0 = \epsilon y_0 = b/a$. That is, $u_0 = a/b$. By factoring (14) into the form

$$U' = U^2(a - bU)$$

we see that the initial condition $U(0) = a/b$ forces $U' = 0$. Thus, $U = a/b$, so

$$u(\tau) \leq U(\tau) = u_0.$$

That is,

$$u(\tau) \leq \frac{a}{b}.$$

Now we may write the following inequality:

$$au^2 - be^\tau u^3 \geq au^2 - be^\tau u^2(\frac{a}{b}) = au^2(1 - e^\tau).$$

Let U now be the solution of

$$U' = U^2(a - bu_0e^\tau),\ U(0) = u_0. \tag{15}$$

Then,

$$U(\tau) \leq u(\tau).$$

The IVP (15) is solved by separating variables and integrating:

$$u(\tau) \geq U(\tau) = \frac{1}{a(e^\tau - 1) - a\tau + \frac{b}{a}}.$$

Satisfied with this result for now, we turn back to (13) and embark on a similar argument:

$$u' = au^2 - be^\tau u^3 \leq au_0u - be^\tau u^3$$

since $u_0 \geq u(\tau)$. This time around, let $U(\tau)$ be the solution to

$$U' = au_0U - be^\tau U^3,\ U(0) = u_0.$$

This is recognised as a Bernoulli equation, with solution

$$U(\tau) = \frac{1}{\left(Ae^{-2au_0\tau} + Be^\tau\right)^{1/2}}$$

where

$$A = \left(z_0^2 - \frac{2b}{2au_0 + 1}\right)$$

and

$$B = \frac{2b}{2au_0 + 1}.$$

Finally, we have the bounds

$$\frac{1}{bu_0(e^\tau - 1) - a\tau + z(0)} \leq u(\tau) \leq \frac{1}{\left(Ae^{-2au_0\tau} + Be^\tau\right)^{1/2}}. \tag{16}$$

Theoretical blow-up time. The theoretical blow-up time t_b for a Sundman-transformed system can be found by integrating the relation

$$\frac{dt}{d\tau} = \frac{1}{z(\tau)}$$

to yield

$$t_b = \int_0^\infty u(\tau)d\tau. \tag{17}$$

As a demonstration of this concept, consider the test equation (4) with $p = \epsilon = 1$. Recall that solutions have exact blow-up time $t_b = \frac{1}{y_0}$. Using the Sundman transformation for the above as in section 3, (17) gives

$$t_b = \int_0^\infty \frac{1}{y} d\tau = \frac{1}{y_0},$$

as expected.

Thus, by integrating the bounds for $u(\tau)$ in (16), we can find an interval containing the blow-up time for (1). In particular, expressions for upper and lower bounds of the blow-up time in terms of the delay parameter τ^* can be found. We begin with the lower bound,

$$\int_0^\infty \frac{d\tau}{a(e^\tau - 1) - a\tau + z_0} \le t_b.$$

This integral cannot be computed explicitly, so an estimate is obtained by dropping negative terms from the denominator:

$$\int_0^\infty \frac{d\tau}{ae^\tau + z_0} < \int_0^\infty \frac{d\tau}{a(e^\tau - 1) - a\tau + z_0}.$$

Integrating,

$$\int_0^\infty \frac{d\tau}{ae^\tau + z_0} = \frac{1}{z_0} \ln\left(\frac{a + z_0}{a}\right).$$

Rewriting in terms of τ^*, we have

$$t_b > \frac{1}{z_0} \ln(1 + \tau^* z_0). \tag{18}$$

This result is important as it provides proof that $t_b \to \infty$ as $\tau^* \to \infty$. Note that as τ^* tends to zero, this lower bound for the blow-up time approaches zero. While this at least guarantees a positive blow-up time, the crudeness of the estimate is evident, as we know from experience that $t_b \to 1$ as $\tau^* \to 0$. Now we turn to the upper estimate for blow-up time:

$$\int_0^\infty \frac{1}{\left(Ae^{-2au_0\tau} + Be^\tau\right)^{1/2}} d\tau \ge t_b. \tag{19}$$

Again, the integral cannot be computed exactly, so some approximations are made. For brevity, the integrand of (19) shall be denoted $f(\tau)$. It is claimed that

$$\int_0^\infty f(\tau)d\tau < \int_0^{\tau^*} f(\tau)d\tau + \frac{1}{\sqrt{B}} \int_{\tau^*}^\infty \frac{d\tau}{(e^\tau - e^{-2})^{1/2}}. \tag{20}$$

To see this, observe that

$$\int_0^\infty f(\tau)d\tau = \int_0^{\tau^*} f(\tau)d\tau + \int_{\tau^*}^\infty f(\tau)d\tau$$

and consider the integral from τ^* to infinity of $f(\tau)$. Note that $A = z_0^2 - B$. Thus $f(\tau)$ can be written as

$$\frac{1}{\left(z_0^2 e^{-2au_0\tau} + B(e^\tau - e^{-2au_0\tau})\right)^{1/2}}.$$

The positive term $z_0^2 e^{-2au_0\tau}$ can be dropped while preserving the inequality

$$\frac{1}{\sqrt{B}\left(e^\tau - e^{-2au_0\tau}\right)^{1/2}} > u(\tau).$$

Furthermore, since $\epsilon \in (0,1]$, the exponent $-2au_0\tau = -2\frac{1}{\epsilon y_0}\frac{\tau}{\tau^*} \leq -2$ for $\tau \geq \tau^*$ and $0 < y_0 \leq 1/\epsilon$. In this case, $e^{-2au_0\tau} \leq e^{-2}$, and so the claim (20) holds. Now consider each integral on the right hand side of (19) separately. It can be shown that $f(\tau)$ is a decreasing function for $\tau \geq 0$ by noting that its derivative is negative for nonnegative τ. Thus, $f(\tau)$ attains its maximum value u_0 at $\tau = 0$. For small τ^*, then,

$$\int_0^{\tau^*} f(\tau)d\tau < u_0\tau^*.$$

Now, consider

$$\frac{1}{\sqrt{B}}\int_{\tau^*}^{\infty}\frac{d\tau}{(e^\tau - e^{-2})^{1/2}}.$$

The above integral has an antiderivative, and can be evaluated:

$$\frac{1}{\sqrt{B}}\int_{\tau^*}^{\infty}\frac{d\tau}{(e^\tau - e^{-2})^{1/2}} = \frac{1}{\sqrt{B}}\left(e\pi - 2e\tan^{-1}(\sqrt{e^{\tau^*+2}-1})\right). \tag{21}$$

Combining the above results,

$$\int_0^{\infty} f(\tau)d\tau < u_0\tau^* + \frac{1}{\sqrt{B}}\left(e\pi - 2e\tan^{-1}(\sqrt{e^{\tau^*+2}-1})\right). \tag{22}$$

For several values of τ^* with $\epsilon = y_0 = z_0 = 1$, lower and upper bounds for the blow-up time of solutions to (1) are calculated using (18) and (22). Results are presented in tabular form below, along with estimated blow-up times $\widehat{t_b}$ found in section 4. Note that the limits as $\tau^* \to 0$ of the lower and upper bounds can be computed, respectively, as follows:

$$\lim_{\tau^*\to 0}\frac{1}{z_0}\ln(1+\tau^* z_0) = 0,$$

and

$$\lim_{\tau^*\to 0}\left(u_0\tau^* + \frac{1}{\sqrt{B}}\left(e\pi - 2e\tan^{-1}(\sqrt{e^{\tau^*+2}-1})\right)\right)$$
$$= \sqrt{\frac{2/z_0}{2\epsilon y_0}}\left(e\pi - 2e\tan^{-1}(\sqrt{e^2-1})\right) \approx 2.048103078.$$

τ^*	lower bound	upper bound	$\widehat{t_b}$
1×10^{-5}	0.000010000	2.048107444	1.000108179
1×10^{-3}	0.000999500	2.048539670	1.007252941
1×10^{-1}	0.095310180	2.091594064	1.275310799
1×10^{1}	2.397895273	10.03300910	3.455453842

TABLE 5. Bounds for t_b found with (18) and (22).

The primary value of (18) and (22) is in showing that blow-up for (1) does indeed occur, and that this blow-up can be delayed arbitrarily long by increasing τ^*. For completeness, the original bounds in (16) are numerically integrated using the same parameter values as above.

τ^*	lower bound	upper bound	$\widehat{t_b}$
1×10^{-5}	0.006980435	2.000004996	1.000108179
1×10^{-3}	0.067334942	2.000499938	1.007252941
1×10^{-1}	0.555778249	2.048220641	1.275310799
1×10^{1}	2.783867989	3.533080330	3.455453842

TABLE 6. Bounds for t_b found by numerically integrating (16).

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Received 7 June 2006

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