Schwarz methods for mixed problems

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1 Introduction

Domain decomposition (DD) has been well developed to solve many classes of elliptic PDEs. DD has also been utilized for parabolic PDEs in two ways: 1) semi-discretizing in time and solving the resulting sequence of elliptic problems using DD, for e.g. [6], and 2) decomposing the problem into space-time subdomains and iterating within a Schwarz Waveform Relaxation (SWR) framework, for e.g. [14].

Relatively, there has been much less attention paid to DD for hyperbolic problems. To give a few examples, an early paper by Kopriva [19] provides a spectral DD approach for quasilinear hyperbolic problems, systems of hyperbolic conservation laws are considered in Lucier and Overbeek [21] and Quarteroni [26], and theoretically, SWR is considered for second order hyperbolic problems in Gander et al. [13]. A noniterative domain decomposition method for damped hyperbolic problems is considered in [10].

The study of heterogeneous domain decomposition is closely related to DD for mixed problems, our subject of interest. Heterogeneous DD is used to couple different models (ie. different physics) using (usually) a non-overlapping spatial decomposition. For example, coupled hyperbolic and elliptic problems are solved using a two subdomain method and asymptotics at the interface to determine appropriate transmission conditions in [15, 9]. See also the nice article of Mathew and Chan [7] for an introduction.

In this paper we consider the simple mixed problem studied first by Tricomi in 1923 [29]. Tricomi's equation is given by

$$y u_{xx} + u_{yy} = 0. \tag{1}$$

In any domain $D \subset \mathbb{R}^2$ which contains both positive and negative y–values (1) will be of mixed type: the problem is elliptic if $y > 0$ and is hyperbolic if $y < 0$.

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Generalizations of Tricomi's equation are connected to the theory of plane transonic flow, see the early paper by Frankl [12]. There the required solution represents the stream function for transonic jet flows in the hodograph plane, see also [8]. The Tricomi equation also arises in the study of isometric embedding problems in differential geometry [18].

As will be discussed further below, the alternating Schwarz method (ASM) has been used as a theoretical tool to prove existence of solutions for various mixed problems, see for e.g. [20, 27]. The two subdomain splitting proposed in these papers decomposes the mixed domain into an elliptic domain and a mixed domain with a specially chosen boundary (which then acts as an artificial interface in the ASM). Motivated by these theoretically inspired splittings, here we propose and analyze further spatial splittings for the Tricomi problem.

Not surprisingly, there has been some work on numerical approximations for mixed problems. Some of the first papers considering finite difference approximations include [24, 2] and finite element approximations are studied in [11, 3, 28]. Our hope is that our study of possible Schwarz approaches will provide the nexus for future development of numerical approaches for mixed problems.

2 A few preliminaries

The characteristics of (1) are real for $y \leq 0$ and consist of two families of curves satisfying

$$\frac{dy}{dx} = (-y)^{-1/2} \qquad \text{and} \qquad \frac{dy}{dx} = -(-y)^{-1/2}.$$

This yields the characteristics

$$x = C \pm \frac{2}{3}(-y)^{3/2},$$

for some arbitrary constant C.

For positive values of y the normal contours, introduced by Tricomi, play an important role. For this specific PDE these are curves of the form

$$(x-a)^2 + \frac{4}{9}y^3 = b^2$$

for constants a and b.

The characteristics and normal contours are used to define the domain of interest and allows us to specify a well-posed boundary value problem. We consider (1) on a domain, D, that is a bounded, connected, open subset of $\mathbb{R}^2$ with a piecewise smooth boundary. Our intent is to consider choices of D so that equation (1) is of mixed type. The left plot of Figure 1 shows such a domain bounded by the characteristic curves of (1), arcs BC and CA, given by $\xi = x - \frac{2}{3}(-y)^{3/2} = 0$ and $\eta = x + \frac{2}{3}(-y)^{3/2} = 1$, and a yet to be specified upper boundary, σ, in the elliptic portion of the domain. On

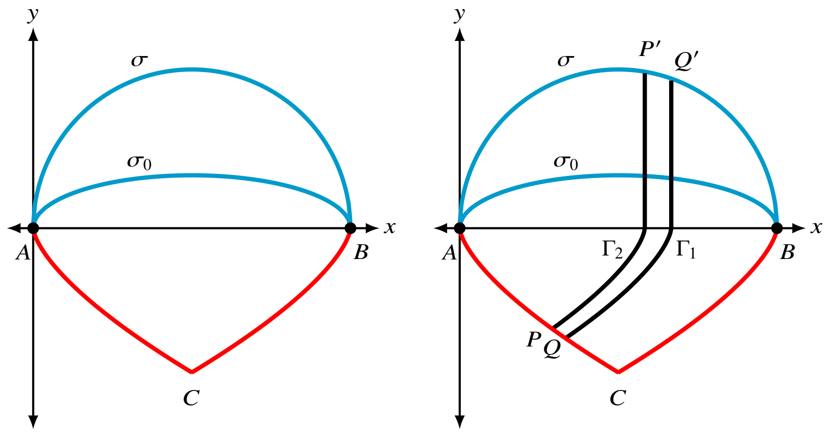

Fig. 1 Partitioning of a mixed domain, with characteristics AC and CB, and normal contour σ_0 interior to elliptic boundary σ.

D equation (1) transitions from elliptic to hyperbolic along the line segment of the x–axis $\overline{AB}$. For any set $X \subset \mathbb{R}^2$, we denote $X \bigcap \{y > 0\}$ as X^+ and $X \bigcap \{y < 0\}$ as X^-. This allows us to define the positive and negative parts of the domain D as D^+ and D^-.

The problem of interest, is called *Problem T* in the literature.

Problem T For a mixed domain D we wish to find $u \in C(\overline{D}) \bigcap C^1(D) \bigcap C^2(D^+ \bigcup D^-)$ which satisfies equation (1) subject to $u = \phi$ on $\bar{\sigma}$ and $u = \psi$ on the charateristic curve AC, where ϕ and ψ are given smooth functions. We impose the capability condition $\phi(0,0) = \psi(0,0)$. Such a solution is called a *regular* solution.

Problem T is an example of an open boundary value problem since u is not specified on the entire boundary of D. In general the closed boundary value problem for a mixed equation will not be well-posed [25].

Tricomi [29] proves the existence and uniqueness of solutions for *Problem T* on the domain D_0 bounded by the normal contour σ_0 and arcs AC and AB using a matching approach at the parabolic line of degeneracy $y = 0$. His analysis primarily makes use of representations of solutions in terms of singular integral equations and Green's functions. The final result then follows using the Fredholm alternative. For reference we state this result as Lemma 1.

Lemma 1 Problem T *has a unique regular solution on the domain* D_0.

In this short paper we will not concentrate on questions of regularity, leaving this for other venues. At this point we simply note that the notion of a regular solution allows for the more interesting behaviour at the points A and B.

Choosing the domain D_0 bounded above in the elliptic region by a portion of the normal contour σ_0 greatly simplifies the integral representations of the solution expressions and appears to be instrumental in Tricomi's analysis. Since then many authors have worked to lift this restriction on the elliptic arc. As we will see in the next section, Tricomi's technique was later generalized to allow the boundary in the elliptic domain, σ, to be a general curve of Lyanponuv class which ends in arbitrarily small arcs of a *normal contour*. Indeed, a novel splitting using a normal contour as an artificial interface followed by an alternating Schwarz analysis utilizing various maximum principles for mixed problems has been instrumental in extending Tricomi's result to general mixed domains.

We begin with a known result for elliptic equations.

Lemma 2 *([17, 5]) The elliptic problem, (1) on D^+ (with specified smooth boundary data on ∂D^+), has a unique regular solution.*

The following results are direct consequences of the well-known maximum principle for elliptic equations.

Lemma 3 *If u is a regular solution to (1) on any subdomain, Ω^+, of D^+, then unless u is a constant, then the maximum of u on $\overline{\Omega^+}$ occurs on $\partial\Omega^+$.*

Lemma 4 *If u is a regular solution to (1) satisfying $u = 0$ on σ, then for $(x, y) \in D^+ \bigcup \sigma$*

$$|u(x, y)| \leq \theta \max_{x \in \overline{AB}} |u(x, 0)|, \qquad 0 < \theta < 1.$$

Unlike the elliptic case, hyperbolic problems generally do not satisfy a maximum principle. But for *Problem T* in the hyperbolic domain we have the following result.

Lemma 5 *([1]) Suppose u is a solution to (1) in the so-called characteristic triangle, D^-, with $u = 0$ on AC, then the maximum of u over D^- occurs on $\overline{AB}$.*

Maximum principles for some mixed problems, including *Problem T* above, were first obtained by Germain and Bader [16], and then generalized by many authors, see for e.g. [1, 23, 27]. We will need the following result.

Lemma 6 *([1]) Suppose u is a regular solution of (1) on D. If $u = 0$ (ie. $\psi = 0$) on the characteristic arc AC, then $u \leq M$ on ∂D imples $u < M$ in D.*

A nice review of these results (and others!) may be found in [22]. As is typical, these maximum principles also directly lead to uniqueness results.

Before proceeding to our main result in Section 3, we give a quick overview, following [5] and [27], of the use of the ASM to obtain existence results for *Problem T* for mixed domains bounded by quite general elliptic arcs σ.

Consider *Problem T* on a domain D as shown in the left of Figure 1 now bounded above by a general curve σ in the elliptic portion of the domain and chacteristic curves AC and BC. We introduce a normal contour σ_0 as an artificial interface, interior to D^+ and connecting A to B. We assume that σ_0 and σ do not share common tangents near A and B. We define D_0 to be the region bounded by σ_0 and the portions of the characteristics AC and BC.

Suppose $u_1(x, y)$ is a solution to (1) in D_0 subject to $u_1 = 0$ on σ_0 and $u_1 = \psi$ on arc AC. This solution exists and is unique since it is a solution to Tricomi's problem on a mixed domain with a normal contour as a boundary in the elliptic domain (Lemma 1). Assume v_1 satisfies (1) on D^+ subject to $v_1 = \phi$ on σ and $v_1 = u_1$ on $\overline{AB}$. The existence of solutions of elliptic problems bounded by a parabolic line of degeneracy is guaranteed by Lemma 2. We recursively define $u_n, n = 2, 3, \ldots$, as regular solutions of (1) satisfying $u_n = v_{n-1}$ on σ_0 and $u_n = \psi$ on arc $\underline{AC}$, while $v_n, n = 2, 3, \ldots$, satisfy (1) and $v_n = \phi$ on σ and $v_n = u_n$ on line segment $\overline{AB}$.

We show the sequences $\{u_n\}$ and $\{v_n\}$ converge uniformly on $\overline{D^+}$ and $\overline{D_0}$. By linearity the differences $u_{n+1} - u_n$ and $v_{n+1} - v_n$ solve (1) and have values of zero on σ and on the arc AC respectively, and hence Lemma 4 and 5 may be applied. These results ensure

$$|u_{n+1}(x, y) - u_n(x, y)| \leq \max_{x \in \overline{AB}} |v_{n+1}(x, 0) - v_n(x, 0)| \leq \max_{\overline{D_0}} |v_{n+1} - v_n|$$

and

$$|v_{n+1} - v_n| \leq \max_{\overline{\sigma_0}} |u_n - u_{n-1}| \leq \theta \max_{x \in \overline{AB}} |u_n(x, 0) - u_{n-1}(x, 0)| \leq \theta \max_{\overline{D_0}} |v_n - v_{n-1}|.$$

Combining these results and iterating we have

$$|v_{n+1} - v_n| \leq \theta^n \max_{\overline{D_0}} |v_1 - v_0|.$$

This gives the uniform convergence of $v_n(x, y) \to v(x, y)$ in $\overline{D_0}$, and $u_n(x, y) \to u(x, y)$ in $\overline{D^+}$, where u and v are regular solutions of (1). Moreover, the limiting functions agree in the overlap region.

As mentioned above the argument above can be further modified to allow σ to be more general, coinciding with σ_0 as σ approaches $y = 0$. A further extension to allow σ to approach $y = 0$ at an arbitrary angle is possible using the construction demonstrated in the proof of Theorem 1 in [4].

Please refer to [5, 27] for details of the regularity of iterates u_n and v_n and the limiting functions.

3 A further Schwarz iteration

We now proceed to consider further splittings of the mixed problem and obtain our main convergence result.

In the right plot of Figure 1 we further decompose D into overlapping subdomains Ω_1 and Ω_2 where Ω_1 is bounded by the arcs AQ, $\underline{QQ'}$, and $\underline{Q'A}$, and Ω_2 is bounded by the arcs PC, CP', and $P'P$. Hence we have $\overline{D} = \overline{\Omega_1} \bigcup \overline{\Omega_2}$ and $\Omega_1 \bigcap \Omega_2 = \emptyset$. We will also need the domains Ω_1^+, Ω_2^+, $\Omega_{10} = \Omega_1 \bigcup D_0$, and $\Omega_{20} = \Omega_2 \bigcup D_0$. The interfaces $\Gamma_2 \equiv PP'$ and $\Gamma_1 \equiv QQ'$ are chosen to be (strictly) interior to Ω_1 and Ω_2 respectively.

In addition to Lemmas 3 and 4, we also need the following classical result in the elliptic portions of the subdomains.

Lemma 7 *If u is a solution to (1) on Ω_1^+ satisfying $u = 0$ on σ , then for $(x, y) \in \Gamma_2 \bigcup \Omega_1^+$*

$$|u(x, y)| \leq \gamma \max_{\Gamma_1} |u(x, y)|,$$

and if u is a solution of (1) on Ω_2^+, then for $(x, y) \in \Gamma_1 \bigcup \Omega_2^+$

$$|u(x, y)| \leq \gamma \max_{\Gamma_2} |u(x, y)|$$

for some constant $\gamma < 1$.

With the partitioning shown in Figure 1 many iterations are possible. Here we outline just one possibility which builds upon the theoretical pieces described above. We sequentially compute solutions u_1^n, u_2^n, z_2^n and z_1^n in the domains $\Omega_1^+, \Omega_2^+, \Omega_{20}$ and Ω_{10} respectively.

Given initial guesses for the solutions on Ω_{20} and Ω_1^+ we compute approximate solutions using the alternating Schwarz iteration: for $n = 1, 2, \ldots$

$$\begin{aligned} yu^n_{1xx} + u^n_{1yy} &= 0, \quad \text{on} \quad \Omega_1^+, \\ u_1^n &= u_2^{n-1} \quad \text{on} \quad \Gamma_1 \cup \partial\Omega_1^+, \\ u_1^n &= z_1^{n-1} \quad \text{on} \quad \overline{AB} \cup \partial\Omega_1^+, \end{aligned} \qquad \begin{aligned} yu^n_{2xx} + u^n_{2yy} &= 0, \quad \text{on} \quad \Omega_2^+, \\ u_2^n &= u_1^{n-1} \quad \text{on} \quad \Gamma_2 \cup \partial\Omega_2^+, \\ u_2^n &= z_2^{n-1} \quad \text{on} \quad \overline{AB} \cup \partial\Omega_2^+, \end{aligned}$$

$$\begin{aligned} yz^n_{2xx} + z^n_{2yy} &= 0, \quad \text{on} \quad \Omega_{20}, \\ z_2^n &= z_1^{n-1} \quad \text{on} \quad \Gamma_2 \cup \partial\Omega_{20}, \\ z_2^n &= u_1^{n-1} \quad \text{on} \quad \sigma_0 \cup \partial\Omega_{20}, \end{aligned} \qquad \begin{aligned} yz^n_{1xx} + z^n_{1yy} &= 0, \quad \text{on} \quad \Omega_{10} \\ z_1^n &= z_2^{n-1} \quad \text{on} \quad \Gamma_1 \cup \partial\Omega_{10}, \\ z_1^n &= u_1^{n-1} \quad \text{on} \quad \sigma_0 \cup \partial\Omega_{10}. \end{aligned}$$

Theorem 1 *The Schwarz iteration defined above is well-posed, generating regular approximations of solutions for* Problem T *on each subdomain.*

Proof. The well-posedness of the subdomain problems for u_1^n and u_2^n follows from Lemma 2, while Lemma 1 gives the well-posedness of the subdomain problems for z_1^n and z_2^n. □

The maximum principles (Lemma 3 and 4) give the following inequalities.

Theorem 2 *For any initial guesses on the interior interfaces, Γ_1 and Γ_2, the subdomain solutions defined above satisfy the inequalities*

$$
\begin{aligned}
|u_1^{n+1} - u_1^n| &\le \max\left(\max_{\overline{D_0}\cap\overline{\Omega_1}} |z_1^n - z_1^{n-1}|, \gamma \cdot \max_{\Gamma_2} |u_2^n - u_2^{n-1}|\right), \quad on\ \Omega_1^+,\\
|u_2^{n+1} - u_2^n| &\le \max\left(\max_{\overline{D_0}\cap\overline{\Omega_2}} |z_2^n - z_2^{n-1}|, \gamma \cdot \max_{\Gamma_1} |u_1^{n+1} - u_1^n|\right), \quad on\ \Omega_2^+,\\
|z_2^{n+1} - z_2^n| &\le \theta \max_{\overline{AB}\cap\overline{\Omega_2}} |u_2^{n+1} - u_2^n| \quad on\ \Omega_{20},\\
|z_1^{n+1} - z_1^n| &\le \theta \max_{\overline{AB}\cap\overline{\Omega_1}} |u_1^{n+1} - u_1^n| \quad on\ \Omega_{10}.
\end{aligned}
$$

This leads to our main result.

Theorem 3 *The alternating Schwarz iteration for* Problem T *on the domain D is convergent.*

Proof. The uniform contraction on the interfaces σ_0 and $\overline{AB}$ follows by combining the inequalities in Theorem 2. Convergence in the interior of the domains follows by applying the appropriate maximum principle. The argument mimics that given at the end of Section 3. □

4 Concluding Remarks

In this short note we give an initial result indicating convergence of standard (overlapping) ASMs for mixed problems. At a distance the usual tools are used, however, the well-posedness and convergence of the iteration is complicated due to the unfamliar nature of the required results for mixed problems. The details of our analysis are obtained for the simplest of the mixed problems and extensions to more practical problems of interest, details of regularity, and numerical experiments will follow.

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