Fully Discrete Schwarz Waveform Relaxation on Two Bounded Overlapping Subdomains

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1 Introduction

Overlapping Schwarz waveform relaxation (SWR) provides space–time parallelism by iteratively solving parabolic partial differential equations (PDEs) over a time window on overlapping spatial subdomains. SWR has been studied for many problems at the continuous and discrete levels. Gander and Stuart [4] and Giladi and Keller [5] have analyzed SWR for the for the heat equation on a finite spatial domain in the continuous and semi-discrete (in time) cases. Semi-discrete (in space) analysis for reaction diffusion equations on an infinite spatial domain can be found in [8]. Closely related work on applications of WR methods to RC type circuits can be found in [3, 2, 1] (continuous in time analysis), [9] (infinite circuit, discrete in time), [6, 7] (fractional order, infinite circuit, discrete and continuous resp. in time), and [10] (Volterra integro-PDEs, infinite spatial domain). We provide an analysis of a full space–time discretization of SWR for the heat equation on two overlapping, bounded subdomains, which does not appear to be in the literature.

Consider the one dimensional heat equation $u_t = u_{xx} + f(x,t)$ for $-L < x < L$ and $0 < t \leq T$ subject to initial and boundary conditions $u(x,0) = u_0(x), u(-L,t) = h_1(t)$, and $u(L,t) = h_2(t)$. Discretizing in space with central finite differences on $\Omega^h = \{x_m : x_{m+1} = x_m + \Delta x, m = -N, ..., N\}$, where $\Delta x = \frac{L}{2N}$ and $x_{-N} = -L$, leads to the IVP

$$\frac{d\mathbf{u}(t)}{dt} = A\mathbf{u}(t) + \mathbf{f}(t), \, 0 < t \leq T, \, \mathbf{u}(0) = \mathbf{u}_0, \tag{1}$$

where $\mathbf{u}(t)$ is the solution vector on the interior of Ω_h with components $u_m(t), m = -(N-1), \ldots, (N-1)$, which are the semi-discrete approximations of $u(x,t)$ at $x = x_m$. Here $A = \frac{1}{\Delta x^2}\text{tridiag}\{-1, 2, 1\} \in \mathbb{R}^{(2N-1)\times(2N-1)}$,

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$$\mathbf{f}(t) = (f(x_{-(N-1)},t) + \frac{1}{\Delta x^2} h_1(t), f(x_{-(N-2)},t), \ldots, f(x_{(N-2)},t), f(x_{(N-1)},t) + \frac{1}{\Delta x^2} h_2(t))^T$$

and

$$\mathbf{u}_0 = (u_0(x_{-(N-1)}), \ldots, u_0(x_{(N-1)}))^T.$$

2 Semi–discretized SWR

To obtain the classical SWR solution of (1), we decompose Ω^h into two overlapping subdomains: $\Omega_1^h = \{x_{-N}, x_{-(N-1)}, \ldots, x_M\}$ and $\Omega_2^h = \{x_{-M}, x_{-(M-1)}, \ldots, x_N\}$ where the quantity $M \geq 1$ is an integer that determines the overlap size. If $\mathbf{w}$ is any vector in $\mathbb{R}^{2N-1}$, then let $\bar{\mathbf{w}}_1 \in \mathbb{R}^{N+M-1}$ be the first $N+M-1$ components of $\mathbf{w}$ and let $\bar{\mathbf{w}}_2 \in \mathbb{R}^{N+M-1}$ be the last $N+M-1$ components of $\mathbf{w}$.

The classical semi-discrete SWR algorithm on the two subdomains, Ω_1^h and Ω_2^h, can be written as : for $k = 1, 2, \ldots$

$$\begin{aligned} \frac{d\mathbf{u}_1^k(t)}{dt} &= A_1 \mathbf{u}_1^k(t) + \mathbf{f}_1^k(t), \qquad 0 < t \leq T, \\ \frac{d\mathbf{u}_2^k(t)}{dt} &= A_2 \mathbf{u}_2^k(t) + \mathbf{f}_2^k(t), \qquad 0 < t \leq T, \end{aligned} \tag{2a}$$

where

$$\mathbf{u}_1^k(t) = (u_{1,-(N-1)}^k(t), u_{1,-(N-2)}^k(t), \ldots, u_{1,(M-1)}^k(t))^T, \tag{2b}$$

and

$$\mathbf{u}_2^k(t) = (u_{2,(-M+1)}^k(t), u_{2,(-M+2)}^k(t), \ldots, u_{2,(N-1)}^k(t))^T, \tag{2c}$$

are the subdomain iterates on the interior nodes of Ω_1^h and Ω_2^h. Here, for $j = 1, 2$, $A_j = \frac{1}{\Delta x^2} \text{tridiag}\{-1, 2, -1\} \in \mathbb{R}^{N+M-1, N+M-1}$. The vectors $\mathbf{f}_j^k \in \mathbb{R}^{N+M-1}$, for $j = 1, 2$, are defined by

$$\mathbf{f}_1^k(t) = \bar{\mathbf{f}}_1(t) + \frac{1}{\Delta x^2} u_{1,M}^k(t) \boldsymbol{\delta_1} \quad \text{and} \quad \mathbf{f}_2^k(t) = \bar{\mathbf{f}}_2(t) + \frac{1}{\Delta x^2} u_{2,-M}^k(t) \boldsymbol{\delta_2}, \tag{2d}$$

where $\delta_j \in \mathbb{R}^{N+M-1}$ for $j = 1, 2$, are the unit column vectors

$$\boldsymbol{\delta_1} = (0, \ldots, 0, 1)^T \quad \text{and} \quad \boldsymbol{\delta_2} = (1, 0, \ldots, 0)^T. \tag{2e}$$

The system (2a) is supplemented with an initial condition

$$\mathbf{u}_j^k(0) = \bar{\mathbf{u}}_j(0), \quad j = 1, 2, \tag{2f}$$

and boundary and transmission conditions

$$\begin{aligned} u_{1,-N}^k(t) &= h_1(t), \qquad u_{1,M}^k(t) = u_{2,M}^{k-1}(t), \qquad 0 < t \leq T, \\ u_{2,-M}^k(t) &= u_{1,-M}^{k-1}(t), \quad u_{2,N}^k(t) = h_2(t), \qquad 0 < t \leq T. \end{aligned} \tag{2g}$$

Here $u^k_{j,m}(t)$ represents the numerical approximation of $u(x,t)$ at $x = x_m$ over Ω_j using the SWR algorithm at the k^{th} iteration. To get the iteration started we must pick initial guesses for $u^0_{2,M}(t)$ and $u^0_{1,-M}(t)$.

3 Convergence Analysis

To analyze the fully discrete SWR we begin with a lemma which desribes the single domain discrete solution of (1) using a backward Euler integrator.

Lemma 1. *The single domain solution at $t = t_n$, $\mathbf{u}(n)$, restricted to the interior of Ω^h_j, $\bar{\mathbf{u}}_j(n)$, for $j = 1,2$, using a backward Euler integrator for the semi–discrete heat equation (1), is the unique solution of the subsystems*

$$(I_1 - \Delta t A_1)\bar{\mathbf{u}}_\mathbf{1}(n) - \mu u_M(n)\delta_\mathbf{1} = \bar{\mathbf{u}}_\mathbf{1}(n-1) + \Delta t \bar{\mathbf{f}}_\mathbf{1}(n) \tag{3}$$

$$(I_2 - \Delta t A_2)\bar{\mathbf{u}}_\mathbf{2}(n) - \mu u_{-M}(n)\delta_\mathbf{2} = \bar{\mathbf{u}}_\mathbf{2}(n-1) + \Delta t \bar{\mathbf{f}}_\mathbf{2}(n), \tag{4}$$

for $n = 1,2,\ldots$. Here $\mu = \Delta t/\Delta x^2$, $\delta_\mathbf{j}$, for $j = 1,2$, are defined in (2e), $u_M(n)$ and $u_{-M}(n)$ are the single domain solutions at the interior interface nodes at time t_n, and $I_{1,2}$ are $(N+M-1)\times(N+M-1)$ identity matrices. Here $\bar{\mathbf{f}}_\mathbf{j}(\mathbf{n}) \equiv \bar{\mathbf{f}}_\mathbf{j}(\mathbf{t_n})$ for $j = 1,2$.

Similar expressions for the SWR approximations are given in the next lemma.

Lemma 2. *The solution of (2a)–(2g) using a backward Euler integrator at $t = t_n$, $\mathbf{u}^k_j(n)$, for $j = 1,2$, at the k^{th} iteration, are the unique solutions of the subsystems*

$$(I_1 - \Delta t A_1)\mathbf{u}^\mathbf{k}_\mathbf{1}(n) = \mathbf{u}^\mathbf{k}_\mathbf{1}(n-1) + \Delta t \mathbf{f}^\mathbf{k}_\mathbf{1}(n), \tag{5}$$

$$(I_2 - \Delta t A_2)\mathbf{u}^\mathbf{k}_\mathbf{2}(n) = \mathbf{u}^\mathbf{k}_\mathbf{2}(n-1) + \Delta t \mathbf{f}^\mathbf{k}_\mathbf{2}(n), \tag{6}$$

for $n = 1,2,\ldots$. Here $\mathbf{f}^k_j(n) \equiv \mathbf{f}^k_j(t_n)$, for $j = 1,2$, where $\mathbf{f}^\mathbf{k}_\mathbf{j}(t)$ are defined in (2d).

We denote the error between the single domain and SWR solutions at time step n by $\mathbf{e}^k_j(n) = \mathbf{u}^k_j(n) - \bar{\mathbf{u}}_j(n)$ for $j = 1,2$. Simply subtracting the representations of the single domain and SWR solutions from the previous two lemmas gives the following result.

Lemma 3. *For $j = 1,2$, $k = 1,2,\ldots$ and $n = 1,2,\ldots$ the errors $\mathbf{e}^k_j(n)$ satisfy*

$$(I_1 - \Delta t A_1)\mathbf{e}^\mathbf{k}_\mathbf{1}(n) = \mathbf{e}^\mathbf{k}_\mathbf{1}(n-1) + \mu \mathbf{e}^\mathbf{k}_\mathbf{1,M}(n)\delta_\mathbf{1}, \tag{7}$$

$$(I_2 - \Delta t A_2)\mathbf{e}^\mathbf{k}_\mathbf{2}(n) = \mathbf{e}^\mathbf{k}_\mathbf{2}(n-1) + \mu \mathbf{e}^\mathbf{k}_\mathbf{2,-M}(n)\delta_\mathbf{2}, \tag{8}$$

with initial condition

$$\mathbf{e}^k_j(0) = \bar{\mathbf{0}}_j, \quad \textit{for } j = 1,2, \tag{9}$$

and boundary conditions

$$\begin{aligned} e^k_{1,M}(n) &= e^{k-1}_{2,M}(n), \qquad e^k_{1,-N}(n) = 0, \\ e^k_{2,-M}(n) &= e^{k-1}_{1,-M}(n), \qquad e^k_{2,N}(n) = 0. \end{aligned} \tag{10}$$

Here $\bar{\mathbf{0}}_{\mathbf{j}} \in \mathbb{R}^{N+M-1}$*, for* $j = 1,2$ *is the zero vector.*

Using the boundary values and the definition of $A_{1,2}$ and $\delta_{\mathbf{1,2}}$ we obtain the following lemma.

Lemma 4. *Component-wise, for* $j = 1,2$*,* $k = 1,2,\ldots$ *and* $n = 1,2,\ldots$ *the errors* $\mathbf{e}^k_{j,m}(n)$ *satisfy*

$$\begin{aligned} -\mu e^k_{1,m-1}(n) + (1+2\mu)e^k_{1,m}(n) - \mu e^k_{1,m+1}(n) &= e^k_{1,m}(n-1), \text{ for } m = -(N-1),\ldots,M-1, \\ -\mu e^k_{2,m-1}(n) + (1+2\mu)e^k_{2,m}(n) - \mu e^k_{2,m+1}(n) &= e^k_{2,m}(n-1), \text{ for } m = -(M-1),\ldots,N-1. \end{aligned}$$

To analyze these recursions for the error we need the discrete Laplace transform. The discrete Laplace transform for a general vector $\upsilon = (\upsilon(0), \upsilon(1), \ldots)^T$, defined on a regular grids with time step Δt is

$$\hat{\upsilon}(s) = \frac{\Delta t}{\sqrt{2\pi}} \sum_{n=0}^{\infty} z^{-n} \upsilon(n), \tag{11}$$

where $z = e^{s\Delta t}$, $s = \sigma + i\omega$, $\sigma > 0$ and $-\pi/T \le \omega \le \pi/\Delta t$.

The recursions for the discrete Laplace transforms are recorded in the next lemma.

Lemma 5. *For* $j = 1,2$*,* $k = 1,2,\ldots$ *and* $n = 1,2,\ldots$ *the discrete Laplace transform of errors* $\hat{\mathbf{e}}^k_{j,m}(n)$ *satisfy*

$$\mu \hat{e}^k_{1,m-1}(s) - (2\mu + \eta)\hat{e}^k_{1,m}(s) + \mu \hat{e}^k_{1,m+1}(s) = 0, \quad m = -(N-1),\ldots,(M-1) \tag{12}$$

and

$$\mu \hat{e}^k_{2,m-1}(s) - (2\mu + \eta)\hat{e}^k_{2,m}(s) + \mu \hat{e}^k_{2,m+1}(s) = 0, \quad m = -(M-1),\ldots,(N-1). \tag{13}$$

The Laplace transform of the initial error gives

$$\hat{\mathbf{e}}^k_j(0) = \mathbf{0}_j, \quad \text{for } j = 1,2 \tag{14}$$

and the Laplace transforms of the boundary conditions are

$$\begin{aligned} \hat{e}^k_{1,M}(s) &= \hat{e}^{k-1}_{2,M}(s), \qquad \hat{e}^k_{1,-N}(s) = 0, \\ \hat{e}^k_{2,-M}(s) &= \hat{e}^{k-1}_{1,-M}(s), \qquad \hat{e}^k_{2,N}(s) = 0, \end{aligned} \tag{15}$$

where $\mu = \frac{\Delta t}{\Delta x^2}$*,* $\eta = \frac{z-1}{z}$ *and* $z = e^{s\Delta t}$*.*

The general solution of these recursion relations is given in the next two lemmas.

Lemma 6. *The general solution of the recursions for the Laplace transforms of the error is given by*

$$\hat{e}^k_{j,m}(s) = a^k_j \lambda_+^m + b^k_j \lambda_+^{-m}, \quad \textit{for } j = 1,2, \tag{16}$$

where λ_+ *solves* $\mu - (2\mu+\eta)\lambda + \mu\lambda^2 = 0$ *and is given explicitly by* $\lambda_+ = \frac{(2\mu+\eta)+\sqrt{(2\mu+\eta)^2-4\mu^2}}{2\mu}$, $\mu = \frac{\Delta t}{\Delta x^2}$, $\eta = \frac{z-1}{z}$ *and* $z = e^{s\Delta t}$ *where the coefficients* $(a^k_j, b^k_j)^T =: \mathbf{c}^k_j$ *are shown to satisfy a simple fixed point iteration in the next lemma.*

Note: in the expression above for λ_+ we have chosen the square root with positive real part.

Lemma 7. *The coefficients in the general solution for the Laplace transform of the error,* $\mathbf{c}^k_j = (a^k_j, b^k_j)^T$, *for* $j = 1,2$, *satisfy*

$$\begin{pmatrix} \mathbf{c}^k_1 \\ \mathbf{c}^k_2 \end{pmatrix} = \Gamma \begin{pmatrix} \mathbf{c}^{k-2}_1 \\ \mathbf{c}^{k-2}_2 \end{pmatrix}, \tag{17}$$

where the contraction matrix, Γ, *is the block diagonal matrix*

$$\Gamma = \begin{pmatrix} S_1 & \\ & S_2 \end{pmatrix}, \tag{18}$$

where

$$S_1 = \Lambda_1^{-1}\Theta_1\Lambda_2^{-1}\Theta_2 \quad \textit{and} \quad S_2 = \Lambda_2^{-1}\Theta_2\Lambda_1^{-1}\Theta_1, \tag{19}$$

and

$$\Lambda_1 = \begin{pmatrix} \lambda_+^{-N} & \lambda_+^{N} \\ \lambda_+^{M} & \lambda_+^{-M} \end{pmatrix}, \Lambda_2 = \begin{pmatrix} \lambda_+^{-M} & \lambda_+^{M} \\ \lambda_+^{N} & \lambda_+^{-N} \end{pmatrix}, \Theta_1 = \begin{pmatrix} 0 & 0 \\ \lambda_+^{M} & \lambda_+^{-M} \end{pmatrix}, \Theta_2 = \begin{pmatrix} \lambda_+^{-M} & \lambda_+^{M} \\ 0 & 0 \end{pmatrix}. \tag{20}$$

To ultimately show convergence of the discrete SWR algorithm we show that for $j = 1,2$, $\mathbf{c}^k_j$ tends to zero as k tends to infinity. A straightforward, but slightly tedious calculation, gives the following explicit representation of $\rho(\Gamma)$.

Lemma 8. *The spectral radius of the contraction matrix* Γ *above,* $\rho(\Gamma)$, *is*

$$\rho(\Gamma) = \left| \frac{\lambda_+^{(N-M)} - \lambda_+^{-(N-M)}}{\lambda_+^{(N+M)} - \lambda_+^{-(N+M)}} \right|^2, \tag{21}$$

where $\lambda_+ = \frac{(2\mu+\eta)+\sqrt{(2\mu+\eta)^2-4\mu^2}}{2\mu}$, $\mu = \frac{\Delta t}{\Delta x^2}$, $\eta = \frac{z-1}{z}$ *and* $z = e^{s\Delta t}$.

From the form of the contraction factor in the previous lemma it is not clear that the algorithm converges. To see this we first rewrite the contraction factor.

Lemma 9. *Using the mapping,* $\lambda_+ = e^v$, *the spectral radius of the contraction matrix,* $\rho(\Gamma)$, *can be written as*

$$\rho(\Gamma) = \left| \frac{\sinh((N-M)v)}{\sinh((N+M)v)} \right|^2. \tag{22}$$

Using $v = \zeta + i\varphi$, we may write $\rho(\Gamma)$ as

$$\rho(\zeta,\varphi) = \frac{2p(\zeta,\varphi) - \sin(2N\varphi)\sin(2M\varphi) - \sinh(2N\zeta)\sinh(2M\zeta)}{2p(\zeta,\varphi) + \sin(2N\varphi)\sin(2M\varphi) + \sinh(2N\zeta)\sinh(2M\zeta)}, \tag{23}$$

where

$$\begin{aligned} p(\zeta,\varphi) = {} & \sinh^2(N\zeta)\cosh^2(M\zeta) + \sinh^2(M\zeta)\cosh^2(N\zeta) \\ & + \sin^2(N\varphi)\cos^2(M\varphi) + \sin^2(M\varphi)\cos^2(N\varphi). \end{aligned} \tag{24}$$

Proof. Using the substitution $\lambda_+ = e^v$ and the definition of the hyperbolic sine function we arrive at (22). Now using $v = \zeta + i\varphi$ and hyperbolic trignometric identities, the contraction rate (22) can be written as

$$\rho(\zeta,\varphi) = \left| \frac{\sinh((N-M)\zeta)\cos((N-M)\varphi) + i\cosh((N-M)\zeta)\sin((N-M)\varphi)}{\sinh((N+M)\zeta)\cos((N+M)\varphi) + i\cosh((N+M)\zeta)\sin((N+M)\varphi)} \right|^2. \tag{25}$$

Simplifying the modulus in (25) gives

$$\rho(\zeta,\varphi) = \frac{\sinh^2((N-M)\zeta)\cos^2((N-M)\varphi) + \cosh^2((N-M)\zeta)\sin^2((N-M)\varphi)}{\sinh^2((N+M)\zeta)\cos^2((N+M)\varphi) + \cosh^2((N+M)\zeta)\sin^2((N+M)\varphi)}. \tag{26}$$

Again using hyperbolic trigonometric identities we arrive at (23) where p is as defined in (24).

To show the spectral radius is strictly less one a more detailed analysis of λ_+ from Lemma 8 is necessary.

Lemma 10. *The quantity $\eta = (z-1)/z$ in the expression for λ_+ satisfies $Re(\eta) > 0$ and hence $Re(\lambda_+) > 1$.*

Proof. From Lemma 8, $\eta = (z-1)/z$ where $z = e^{s\Delta t}$, $s = \sigma + i\omega$, $\sigma > 0$ and $\pi/T \leq |\omega| \leq \pi/\Delta t$. The real part of η is given by

$$\mathrm{Re}(\eta) = 1 - e^{-\sigma\Delta t}\cos(\omega\Delta t)$$

which is easily seen to be positive for $\sigma > 0$ and $\pi/T \leq |\omega| \leq \pi/\Delta t$. The real part of λ_+ is given by

$$\mathrm{Re}(\lambda_+) = 1 + \frac{\mathrm{Re}(\eta)}{2\mu} + \frac{\mathrm{Re}(\sqrt{\eta^2 + 4\mu\eta})}{2\mu}.$$

The conclusion $\mathrm{Re}(\lambda_+) > 1$ then follows from the fact that $\mathrm{Re}(\eta) > 0$ and the choice of the square root in λ_+.

The following inequality will finally lead us to the main result.

Lemma 11. *If $\lambda_+ = e^{\zeta+i\varphi}$ then the following inequality*

$$\sinh(K\zeta) > |\sin(K\varphi)|, \tag{27}$$

holds where the quantity $K \geq 1$ is an integer.

Proof. Recall that λ_+ satisfies $\mu - (2\mu+\eta)\lambda_+ + \mu\lambda_+^2 = 0$. Substituting $\lambda_+ = e^{\zeta+i\varphi}$, multiplying by $e^{-(\zeta+i\varphi)}$ and dividing by 2μ, we find

$$\frac{e^{\zeta+i\varphi} + e^{-(\zeta+i\varphi)}}{2} = 1 + \frac{\eta}{2\mu}.$$

Using the definition of the hyperbolic cosine function and splitting the real and the imaginary parts of η we have

$$\cosh(\zeta + i\varphi) = \left(1 + \frac{\mathrm{Re}(\eta)}{2\mu}\right) + i\frac{\mathrm{Im}(\eta)}{2\mu}. \tag{28}$$

Since $\mathrm{Re}(\eta) > 0$ then clearly $|\cosh(\zeta + i\varphi)|^2 > 1$.

Induction is used to prove (27). Using Euler's formula, hyperbolic trignometric identities and simplifying the square of the modulus, $|\cosh(\zeta + i\varphi)|^2 > 1$ becomes

$$\cosh^2(\zeta)\cos^2(\varphi) + \sinh^2(\zeta)\sin^2(\varphi) > 1,$$

which simplifies to $\sinh^2(\zeta) > \sin^2(\varphi)$. Since we know $Re(\lambda_+) > 1$ and $Re(\lambda_+) = e^{\zeta}cos(\varphi) > 1$, then $\zeta > 0$ and hence $\sinh(\zeta) > 0$. Taking the square root of both sides of the inequality $\sinh^2(\zeta) > \sin^2(\varphi)$ then gives the base case in the induction argument.

The induction step then follows using the base inequality, hyperbolic trig identities, properties of the hyperbolic trignometric and trignometric functions and the triangle inequality.

We now arrive at the final and main result.

Theorem 1. *The fully discrete SWR algorithm which results from applying the backward Euler time integrator to (2a)–(2g) converges to the single domain discrete solution on the interior of Ω_j, for $j = 1, 2$.*

Proof. We are now in a position to prove that $\rho(\Gamma) < 1$. The spectral radius of the contraction matrix, $\rho(\Gamma)$, is given in (23) where p is given in (24). Since $p > 0$, then clearly $\rho(\zeta, \varphi) < 1$ if

$$\sin(2N\varphi)\sin(2M\varphi) + \sinh(2N\zeta)\sinh(2M\zeta) > 0.$$

The above inequality follows from Lemma 11 for $K = 2N$ and $K = 2M$. To see this, we consider different cases for the sign of $\sin(2N\varphi)$ and $\sin(2M\varphi)$. Since $\zeta > 0$ we have $\sinh(2N\zeta) > 0$ and $\sinh(2M\zeta) > 0$. There are two cases to consider: if $\sin(2N\varphi)$ and $\sin(2M\varphi)$ have the same or opposite signs. If they have the same sign then the inequality above is obvious. If they have opposite signs then Lemma 11 gives the result.

4 Conclusions

In this paper we have obtained an explicit contraction rate for the discrete Laplace transform of the error for the fully discretized SWR algorithm applied to the heat equation on two overlapping bounded domains. Further analysis, with other families of time integrators and an arbitrary number of subdomains will appear elsewhere.

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