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Linearized domain decomposition approaches for nonlinear boundary value problems

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ABSTRACT

Linearized (Schwarz) domain decomposition approaches for nonlinear boundary value problems of the form $u'' = f(x, u, u')$, subject to Dirichlet boundary conditions, are proposed and analyzed. In the presence of subsolutions and supersolutions, we construct a globally convergent, linear, monotone iteration suitable for implementation in a distributed computing environment. These iterations provide an alternative to the typical, locally convergent, approach of discretizing and solving the resulting non-linear algebraic equations using a Newton iteration. The work also extends previous results obtained in the case where f has no dependence on the derivative of the solution. The Schwarz iteration is first proposed and studied in detail on two subdomains. The result is then generalized to an arbitrary number of subdomains. Both alternating and parallel Schwarz iterations are analyzed. Numerical results are provided to demonstrate the theory and the utility of the proposed iterations.

1. Introduction

In this paper we consider the solution of nonlinear boundary value problems (BVPs) of the form

$$u'' = f(x, u, u'),\ 0 < x < 1, \quad u(0) = u(1) = 0. \tag{1.1}$$

A standard numerical approach is to discretize (1.1) using finite differences or finite elements and solve the resulting system of nonlinear algebraic equations using a Newton style iteration. Difficult BVPs require a good initial guess to ensure the (local) convergence of the Newton iteration. Assuming the existence of subsolutions and supersolutions of (1.1), we propose linearized iterations which are globally convergent to a solution (at the continuous level). The iterations utilize a divide and conquer approach, solving the problem on subdomains, potentially to split the computational load across multiple processors. Moreover, the approximate subdomain solutions are found by solving a linear problem during each iteration.

Monotonic iterative approaches for BVPs of the form

$$u'' = f(x, u), \qquad u(a) = u(b) = 0, \tag{1.2}$$

without u' dependence, were first introduced by Picard [1] under the assumption that $f(x, u)$ is continuous and decreasing in u and $f(x, 0) = 0$. Picard demonstrates that there exists positive function $\alpha_{(0)}$, known as a subsolution of the BVP, which satisfies

$$\alpha''_{(0)} > f(x, \alpha_{(0)}), \qquad \alpha_{(0)}(a) = \alpha_{(0)}(b) = 0. \tag{1.3}$$

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He then considers the iteration

$$\alpha''_{(n)} = f(x, \alpha_{(n-1)}), \qquad \alpha_{(n)}(a) = \alpha_{(n)}(b) = 0. \tag{1.4}$$

The sequence $\alpha_{(n)}$ is shown to converge monotonically to a solution u of (1.2). A supersolution, $\beta_{(0)}$, of (1.2) would satisfy (1.3) with the inequality reversed. Of course these schemes were originally used as theoretical tools, to prove the well-posedness of the BVP. It was much later that the computational utility of the iterations would be seen.

Babkin [2] extended Picard's iterative scheme. For given subsolution $\alpha_{(0)}$ and supersolution $\beta_{(0)}$ satisfying boundary conditions, he considered the approximation schemes

$$\begin{aligned} \alpha''_{(n)} + K\alpha_{(n)} &= f(x, \alpha_{(n-1)}) + K\alpha_{(n-1)}, \qquad & \alpha_{(n)}(a) = 0, \alpha_{(n)}(b) = 0, \\ \beta''_{(n)} + K\beta_{(n)} &= f(x, \beta_{(n-1)}) + K\beta_{(n-1)}, \qquad & \beta_{(n)}(a) = 0, \beta_{(n)}(b) = 0, \end{aligned}$$

where K is a non-positive constant chosen so that $f(x, u) + Ku$ is decreasing in u. Then the sequences $\alpha_{(n)}$ and $\beta_{(n)}$ are monotonically convergent to solutions of the BVP. Each iteration requires the solution of a linear BVP; the nonlinear BVP is replaced by a convergent sequence of linear BVPs.

Courant and Hilbert [3], generalized Babkin's iteration, and introduced a monotonic iterative approach for the solution of partial differential equations of the form

$$\triangle u = f(x, u), \qquad x \in \Omega \subset \mathbb{R}^n, \qquad u = \Phi \text{ on } \partial\Omega. \tag{1.5}$$

Here f is defined and has continuous first derivatives for all $x \in \bar{\Omega}$ and for all u. The boundary and boundary values are assumed to be smooth. The function f is assumed to be bounded on $\bar{\Omega}$ by a constant $N > 0$, that is $| f(x, u) | \leq N$. Assuming an initial approximation $\alpha_0(x)$ solving $\triangle \alpha_0 = -N$, $\alpha_0 = 0$ on $\partial\Omega$, the authors show that the sequence $\alpha_{(n)}$ satisfying

$$\triangle \alpha_{(n)} + K\alpha_{(n)} = f(x, \alpha_{(n-1)}) + K\alpha_{(n-1)}, \qquad \alpha_{(n)} = 0 \text{ on } \partial\Omega,$$

decreases monotonically to a solution $u_{\max}$ for a suitable choice of K. If the iteration is started from $-\alpha_0$ then the sequence is monotonically increasing to a solution $u_{\min}$. If u is any other solution of (1.5) then $u_{\min} \leq u \leq u_{\max}$. Again each iterate is obtained by solving a linear boundary value problem. Subsequently, Parter [4] extended Courant's work by imposing weaker smoothness requirements. He goes on to consider the convergence of such schemes at the discrete level. Two-sided convergence and resulting error bounds are considered in [5].

In the late 1960s and early 1970s, see, for example [6–9], the analysis was extended to provide linear monotone schemes for problems of the form $\mathcal{L}u = -f(x, u), x \in \Omega, \mathcal{B}u = 0, x \in \partial\Omega$, where $\mathcal{B}$ is a linear boundary operator and $\mathcal{L}$ is a uniformly elliptic second order partial differential operator for which the strong maximum principle holds. Stuart [10] provided a monotone scheme for the BVP (1.2) in the case that f is of bounded variation on the interval. The resulting scheme, however, requires nonlinear solves at each iteration. More recently, problems with nonlinear boundary conditions have been considered, see [11].

The problem with u' dependence in the right-hand side function is more difficult. The first results, assuming the existence of subsolutions and supersolutions, were obtained by Dragoni [12] who proved that solutions exist for the BVPs of the form (1.1) when f is continuous and bounded. The proof was not constructive, no iteration was provided for the computation of a solution. Generalizing Babkin's iteration, Gendzojan [13] developed a linear monotone iterative method for the BVP of the form (1.1). For a given subsolution $\alpha_{(0)}$ and supersolution $\beta_{(0)}$ he considered the iterations: for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} -\alpha''_{(n)} + K(x)\alpha'_{(n)} + l(x)\alpha_{(n)} &= -f(x, \alpha_{(n-1)}, \alpha'_{(n-1)}) \\ &\quad + K(x)\alpha'_{(n-1)} + l(x)\alpha_{(n-1)}, \quad \text{on } \Omega, \\ \alpha_{(n)} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

$$\begin{aligned} -\beta''_{(n)} + K(x)\beta'_{(n)} + l(x)\beta_{(n)} &= -f(x, \beta_{(n-1)}, \beta'_{(n-1)}) \\ &\quad + K(x)\beta'_{(n-1)} + l(x)\beta_{(n-1)}, \quad \text{on } \Omega, \\ \beta_{(n)} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.7}$$

Here the functions $K(x)$ and $l(x)$, depend on f, and are chosen in order to get explicit representations of the iterates. The function $f(x, u, u')$ is assumed to be continuous with continuous derivatives $f_u, f_{u'}$ and $0 \leq f_u \leq M$, $|f_{u'}| \leq M$ for some M, for all $(u, u') \in \mathbb{R}^2$.

Other authors have constructed monotone schemes for problem (1.1) in the presence of subsolutions and supersolutions. Chandra and Davis [14] developed an iterative method to solve a problem that depends linearly on the derivative of the solution. Following that, Bernfeld and Chandra [15] generalized this method for a right hand side function f that depends on the derivative of the unknown solution nonlinearly. The scheme produces a sequence of iterates which converges monotonically but computationally we are forced to solve a sequence of nonlinear problems if f is nonlinear in u'. Later Omari [16] developed an iterative scheme for the BVP of the form $u'' = f(x, u, u')$, $x \in \Omega$, $\mathcal{B}u = r$, $x \in \partial\Omega$, where $\mathcal{B}$ is a first order linear continuous boundary operator. For a given subsolution $\alpha_{(0)}$ and supersolution $\beta_{(0)}$, he constructed a monotone, but again nonlinear, scheme.

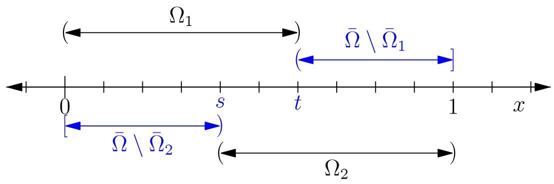


Fig. 1. Partitioning of $\Omega = [0, 1]$ into two overlapping subdomains.

Cherpion et al. [17] proceed as in [13] in the presence of ordered subsolutions and supersolutions. Cherpion proposed a linearized iteration scheme with a slightly simpler choice of the coefficients for problem (1.1). The iteration scheme is given by: for $n = 1, 2, 3, \ldots$

$$\begin{aligned} -\alpha''_{(n)} + \sqrt[3]{l}K(x)\alpha'_{(n)} + l\alpha_{(n)} &= -f(x, \alpha_{(n-1)}, \alpha'_{(n-1)}) \\ &\quad + \sqrt[3]{l}K(x)\alpha'_{(n-1)} + l\alpha_{(n-1)}, \quad \text{on } \Omega, \\ \alpha_{(n)} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.8}$$

$$\begin{aligned} -\beta''_{(n)} + \sqrt[3]{l}K(x)\beta'_{(n)} + l\beta_{(n)} &= -f(x, \beta_{(n-1)}, \beta'_{(n-1)}) \\ &\quad + \sqrt[3]{l}K(x)\beta'_{(n-1)} + l\beta_{(n-1)}, \quad \text{on } \Omega \\ \beta_{(n)} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

where $\alpha_{(0)}$ and $\beta_{(0)}$ are the subsolution and supersolution of BVP (1.1) respectively, $K(x)$ is an antisymmetric function on Ω and $l > 0$ is a constant depending on f. Cherpion's analysis makes less restrictive assumptions on f than required by Gendzojan [13]. The function f is assumed to be Lipschitz in u' and one-sided Lipschitz in u. We will extend Cherpion's scheme to multiple domains and prove convergence of the resulting domain decomposition algorithms in Section 3.

On a single domain other types of boundary conditions have been considered for (1.1). Periodic problems have been studied in, for example, Leela [18], Bellen [19], Omari and Trombetta [20], and Cherpion [21]. Neumann problems are considered in [22,23].

To take advantage of parallel computing, one method to solve BVPs is domain decomposition (DD). DD is based on a divide and conquer philosophy, it divides a big problem into several subproblems on smaller overlapping or non-overlapping subdomains. These subdomains form a partition of the original domain. There are several general classes of DD approaches for nonlinear problems. Detailed discussion of DD methods applied to some nonlinear BVPs can be found in [24–26]. In [26], for example, the authors proposed a nonlinear DD method, where on each subdomain a nonlinear BVP with Dirichlet boundary conditions is solved, for a specific BVP related to mesh generation. The drawback of nonlinear Schwarz methods discussed therein is that in each iteration the solution of many nonlinear systems of equations is required (one for each subdomain). Linearized DD methods avoid these costly nonlinear solves and, as we will see, in the presence of subsolutions and supersolutions, can also provide monotonic convergence to a solution. Here we develop a linearized domain decomposition method which can solve the BVPs of the form $u'' = f(x, u, u')$, where $f(x, u, u')$ depends on u' nonlinearly.

Similar work, developing linearized, monotone domain decomposition methods for nonlinear problems, exists. Lui [27] considers the PDE

$$-\Delta u = f(x, u) \text{ on } \Omega \subset \mathbb{R}^N, \quad u = h \text{ on } \partial\Omega, \tag{1.10}$$

studied earlier by Courant. In one dimension, for example, the domain Ω is decomposed into two overlapping subdomains, $\Omega_1 = (0, t)$ and $\Omega_2 = (s, 1)$ with $s < t$, as shown in Fig. 1. Here $\bar{\Omega} \setminus \bar{\Omega}_1$ denotes the portion of the domain Ω_2 that does not overlap with $\bar{\Omega}_1$, and $\bar{\Omega} \setminus \bar{\Omega}_2$ denotes the portion of Ω_1 that does not overlap with $\bar{\Omega}_2$.

If f is Hölder continuous function, with Hölder exponent α, that is $f \in C^\alpha(\bar{\Omega})$, and there exists $c(x) \in C^\alpha(\bar{\Omega})$ such that $-c(x)(u - v) \le f(x, u) - f(x, v)$ on Ω, then for a given subsolution $\underline{u}$ or supersolution $\bar{u}$ with $\alpha_{(0)} = \underline{u}$ (or analogously with $\beta_{(0)} = \bar{u}$), Lui considers the two subdomain iteration scheme: for $n = 0, 1, 2, \ldots$

$$\begin{aligned} -\Delta\alpha_{(n+\frac{1}{2})} + c\alpha_{(n+\frac{1}{2})} &= f(\alpha_{(n-\frac{1}{2})}) + c\alpha_{(n-\frac{1}{2})} \text{ on } \Omega_1, \\ \alpha_{(n+\frac{1}{2})} &= \alpha_{(n)} \text{ on } \partial\Omega_1, \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} -\Delta\alpha_{(n+1)} + c\alpha_{(n+1)} &= f(\alpha_{(n)}) + c\alpha_{(n)} \text{ on } \Omega_2, \\ \alpha_{(n+1)} &= \alpha_{(n+\frac{1}{2})} \text{ on } \partial\Omega_2. \end{aligned} \tag{1.12}$$

This is an example of a multiplicative Schwarz domain decomposition method. Lui [27] proved that if this iteration starts from a subsolution then it will converge to a minimal solution $u_{\min}$ and if the iteration starts from a supersolution then it will converge to a maximal solution $u_{\max}$. If u is any other solution of the problem then $u_{\min} \le u \le u_{\max}$.

In this paper, we follow Lui and propose and prove convergence of a multiplicative Schwarz version of the iterations (1.8) and (1.9) proposed by Cherpion to solve BVPs of the form (1.1). We will also provide an analysis of a parallel Schwarz variant of this iteration and extend the analysis to an arbitrary number of subdomains. Our one-dimensional results may at first glance seem modest when compared to the more general results of Lui. The development of monotone schemes (even on a single domain) for problems with an explicit gradient dependence, however, is much more difficult. And as we will see the derivative dependence also complicates the multi-domain analysis. We note that in [27], the author refers to the parallel Schwarz style iteration considered here on multiple subdomains as an *additive Schwarz* iteration. Additive Schwarz now commonly refers to a particular choice of a preconditioner used to solve a discretized PDE, see [28]. We also note that the parallel Schwarz iteration considered here and in [27] uses different transmission conditions at the boundary to ensure the monotonicity of the scheme.

The remainder of this paper is organized as follows. In Section 2 we define the nonlinear BVP of interest and clearly state our required assumptions. We then specify the alternating and parallel linearized Schwarz iterations that we analyze. In Section 3 we prove the well-posedness of the Schwarz iterations and prove the main convergence results. Section 4 provides numerical results to demonstrate the theory developed in Section 3. We make some conclusions and comments in the final section.

2. Preliminaries

We consider the Dirichlet problem

$$-u''+f(x,u,u')=0 \text{ on } \Omega=(a,b), \qquad u=0 \text{ on } \partial\Omega. \tag{2.1}$$

We assume the existence of subsolutions and supersolutions of (2.1); these functions are defined in Definition 2.1 and Definition 2.2.

Definition 2.1 (*Subsolution*)**.** A function $\underline{\alpha}\in\mathcal{C}^2([a,b])$ is a subsolution of (2.1) if
(I) for all $x\in[a,b]$, $\underline{\alpha}''(x)\geq f(x,\underline{\alpha}(x),\underline{\alpha}'(x))$;
(II) $\underline{\alpha}(a)\leq 0$, $\underline{\alpha}(b)\leq 0$.

Definition 2.2 (*Supersolution*)**.** A function $\bar{\beta}\in\mathcal{C}^2([a,b])$ is a supersolution of (2.1) if
(I) for all $x\in[a,b]$, $\bar{\beta}''(x)\leq f(x,\bar{\beta}(x),\bar{\beta}'(x))$;
(II) $\bar{\beta}(a)\geq 0$, $\bar{\beta}(b)\geq 0$.

We state our assumptions and conditions on the function f in (2.1) and the problem data in Definition 2.3.

Definition 2.3. [C1] Let $\underline{\alpha}$ and $\bar{\beta}\in\mathcal{C}^2([a,b])$ be ordered subsolutions and supersolutions of (2.1) such that $\underline{\alpha}\leq\bar{\beta}$. We define the set $\mathcal{D}$ as

$$\mathcal{D}=\left\{(x,u,u')\in[a,b]\times\mathbb{R}^2 \mid \underline{\alpha}(x)\leq u\leq\bar{\beta}(x)\right\},$$

and assume $f:\mathcal{D}\to\mathbb{R}$ is a continuous function.
[C2] One Sided Lipschitz Condition in u: Assume there exists a constant $M\geq 0$ such that for all $(x,u_1,v),(x,u_2,v)\in\mathcal{D}$ and for all $u_1\leq u_2$,

$$f(x,u_2,v)-f(x,u_1,v)\leq M(u_2-u_1).$$

[C3] Lipschitz Condition in u': Assume there exists a constant $N\geq 0$ such that for all $(x,u,v_1),(x,u,v_2)\in\mathcal{D}$,

$$|\,f(x,u,v_2)-f(x,u,v_1)\,|\leq N\,|\,v_2-v_1\,|\,.$$

[C4] For a given $u(x)$ we define $F^u(x):=f(x,u(x),u'(x))$. At points in the paper we will assume that F^u is a Hölder continuous function with exponent α.
[C5] Assume $K\in\mathcal{C}[a,b]$, with $K(a)>0$, is anti-symmetric on $[a,b]$, that is K satisfies $K(x)=-K(a+b-x)$.
[C6] Assume $l\in\mathbb{R}$ satisfies $l>\max\left\{M,\frac{N^3}{(K(a))^3}\right\}$.

The constant l will appear in our iterations to solve (2.1) and will require further restrictions to ensure monotone convergence. The stronger Hölder continuity requirement will be used to establish $\mathcal{C}^2$ convergence of our monotone schemes.

Remark 1. Assuming the existence of subsolutions and supersolutions of (2.1) satisfying Definitions 2.1 and 2.2 and assuming f satisfies conditions [C1], [C2] and [C3] then Cherpion et al. prove the existence of solutions of (2.1), see [17,21].

Here we introduce some preliminary results useful in our analysis. The following result is classical and the proof may be found in [21], see also [29].

Lemma 2.4 (*Maximum Principle*). *Suppose $p, q, h \in L^1(a, b)$ (integrable functions), with $q(x) \geq 0$ and $h(x) \leq 0$ on $[a, b]$. And assume $A \geq 0$ and $B \geq 0$. If u is a non-constant solution of*

$$u'' - p(x)u' - q(x)u = h(x)$$
$$u(a) = A, u(b) = B,$$

then u satisfies $u(x) > 0$ on (a, b). Furthermore, if $u(a) = 0$ then $u'(a) > 0$ and if $u(b) = 0$ then $u'(b) < 0$.

The following inequality is used numerous times in our analysis.

Lemma 2.5. *Assume conditions [C2]–[C3] are satisfied, $u, v \in \mathcal{C}^1([a, b])$, $(x, u, u'), (x, v, v') \in \mathcal{D}$ (as defined in [C1]) and $u \leq v$, then*

$$\begin{aligned} f(x, v, v') - f(x, u, u') &\leq M(v - u) + N|v' - u'| \\ &= M(v - u) + N\mathrm{sign}(v' - u') \cdot (v' - u'). \end{aligned}$$

Proof. If $u \leq v$ then conditions [C2] and [C3] imply

$$\begin{aligned} f(x, v, v') - f(x, u, u') &= f(x, v, v') - f(x, u, v') + f(x, u, v') - f(x, u, u') \\ &\leq M(v - u) + N|v' - u'|. \quad \blacksquare \end{aligned}$$

Using Lemmas 2.4 and 2.5 we may prove the following result which characterizes the difference between subsolutions and supersolutions of (2.1).

Lemma 2.6. *Assume $\underline{\alpha}$ and $\bar{\beta}$, with $\underline{\alpha} \leq \bar{\beta}$, are a subsolution and supersolution of (2.1), as defined in Definitions 2.1 and 2.2, which satisfy the boundary conditions $\underline{\alpha}(a) = \bar{\beta}(a) = 0$ and $\underline{\alpha}(b) = \bar{\beta}(b) = 0$. If $\underline{\alpha}$ and $\bar{\beta}$ are not solutions of (2.1), then $\bar{\beta} - \underline{\alpha} > 0$ on (a, b), $\bar{\beta}'(a) - \underline{\alpha}'(a) > 0$ and $\bar{\beta}'(b) - \underline{\alpha}'(b) < 0$. Furthermore, if $K(x)$ satisfies condition [C5] and f satisfies [C2] and [C3] then*

$$f(x, \bar{\beta}, \bar{\beta}') - f(x, \underline{\alpha}, \underline{\alpha}') - \sqrt[3]{l}K(x)(\bar{\beta}' - \underline{\alpha}') - l(\bar{\beta} - \underline{\alpha}) \leq 0 \tag{2.2}$$

is satisfied for l sufficiently large.

Proof. Since $\underline{\alpha} \leq \bar{\beta}$, $\underline{\alpha}'' \geq f(x, \underline{\alpha}, \underline{\alpha}')$ and $\bar{\beta}'' \leq f(x, \bar{\beta}, \bar{\beta}')$ then Lemma 2.5 ensures

$$\bar{\beta}'' - \underline{\alpha}'' \leq M(\bar{\beta} - \underline{\alpha}) + N|\bar{\beta}' - \underline{\alpha}'|.$$

Then $w = \bar{\beta} - \underline{\alpha}$ satisfies $w'' - N|w'| - Mw \leq 0$ on $\Omega = (a, b)$, and $w = 0$ on $\partial\Omega$, since $\underline{\alpha}$ and $\bar{\beta}$ agree at a and b. Since $|w'| = \mathrm{sign}(w')w'$ and $\mathrm{sign}(w') \in L^1(a, b)$ then by Lemma 2.4 we have $w = \bar{\beta} - \underline{\alpha} > 0$ on Ω, $\bar{\beta}'(a) - \underline{\alpha}'(a) > 0$ and $\bar{\beta}'(b) - \underline{\alpha}'(b) < 0$.

Define $G(l, x) := f(x, \bar{\beta}, \bar{\beta}') - f(x, \underline{\alpha}, \underline{\alpha}') - \sqrt[3]{l}K(x)(\bar{\beta}' - \underline{\alpha}') - l(\beta - \alpha)$. Since K is chosen so that $K(a) > 0$ and $K(b) = -K(a) < 0$ then $K(a)(\bar{\beta}'(a) - \underline{\alpha}'(a)) := \gamma_a > 0$ and $K(b)(\bar{\beta}'(b) - \underline{\alpha}'(b)) =: \gamma_b > 0$. For given $\underline{\alpha}, \bar{\beta} \in \mathcal{C}^2$, with f continuous on $\mathcal{D}$, the mapping $x \to f(x, \bar{\beta}(x), \bar{\beta}'(x)) - f(x, \underline{\alpha}(x), \underline{\alpha}'(x))$ is continuous on $[a, b]$. Hence $f(x, \bar{\beta}(x), \bar{\beta}'(x)) - f(x, \underline{\alpha}(x), \underline{\alpha}'(x))$ achieves its upper bound, $\mathcal{F}$ say. Define $\tilde{G}(l, x) := \mathcal{F} - \sqrt[3]{l}K(x)(\bar{\beta}' - \underline{\alpha}') - l(\bar{\beta} - \underline{\alpha})$.

Since $\gamma_a, \gamma_b > 0$ and $\bar{\beta}(a) - \underline{\alpha}(a) = \bar{\beta}(b) - \underline{\alpha}(b) = 0$ then clearly $\tilde{G}(l, a) = \mathcal{F} - \sqrt[3]{l}\gamma_a \leq 0$ and $\tilde{G}(l, b) = \mathcal{F} - \sqrt[3]{l}\gamma_b \leq 0$, for l sufficiently large. By continuity there exists a $\delta_a > 0$ so that $K(x)(\bar{\beta}'(x) - \underline{\alpha}'(x)) \geq \frac{1}{2}\gamma_a > 0$ for $x \in [a, a + \delta_a]$. On $[a, a + \delta_a]$, $\bar{\beta} - \underline{\alpha} \geq 0$, so $\tilde{G}(l, x) \leq \mathcal{F} - \frac{\sqrt[3]{l}}{2}\gamma_a \leq 0$, for l sufficiently large. Likewise, there exists a $\delta_b > 0$ so that $K(x)(\bar{\beta}' - \underline{\alpha}') \geq \frac{1}{2}\gamma_b > 0$ on $[b - \delta_b, b]$. On $[b - \delta_b, b]$, $\bar{\beta} - \underline{\alpha} \geq 0$, so $\tilde{G}(l, x) \leq \mathcal{F} - \frac{\sqrt[3]{l}}{2}\gamma_b \leq 0$ for l sufficiently large. On $[a + \delta_a, b - \delta_b]$, $\bar{\beta} - \underline{\alpha}$ is continuous so it attains its positive minimum, m, that is $\bar{\beta} - \underline{\alpha} \geq m > 0$. Likewise, $K(x)(\bar{\beta}' - \underline{\alpha}')$ attains it minimum, d, on $[a + \delta_a, b - \delta_b]$. So we have $\tilde{G}(l, x) \leq \mathcal{F} - \sqrt[3]{l}d - lm$ on $[a + \delta_a, b - \delta_b]$. If l is sufficiently large, then $\tilde{G} \leq 0$ independent of the sign of d. Since $G(l, x) \leq \tilde{G}(l, x)$, $G(l, x) \leq 0$ for $x \in [a, b]$ and l sufficiently large. $\blacksquare$

The following lemma characterizes the solution of the linear boundary value problems we will encounter in our analysis. The non-negativity of w follows from Lemma 2.4. The well-posedness of the linear elliptic boundary value problem can be found in [30]. The proof of the inequality (2.4) is suggested in [21] and can be found in complete detail in [31].

Lemma 2.7. *Consider the problem*

$$w'' - \sqrt[3]{l}K(x)w' - lw = h(x), \qquad w(a) = A, w(b) = B, \tag{2.3}$$

on the interval $[a, b]$, where l is a positive real scalar, $K \in \mathcal{C}[a, b]$, and $A \geq 0$, $B \geq 0$. If h is a continuous function on $[a, b]$ then there exists a unique $\mathcal{C}^2$ solution, $w(x)$, on $[a, b]$. If in addition h is non-positive, then $w(x) \geq 0$ and

$$(M - l)w + (N\,\mathrm{sign}(w') - \sqrt[3]{l}K(x))w' \leq 0, \quad \text{on} \quad [a, b], \tag{2.4}$$

if conditions [C5] and [C6] are satisfied and l is sufficiently large.

Proof. We sketch the proof of the inequality (2.4) here and refer to Ahmed [31] for the complete details.

The solution of (2.3) is given by

$$w(x) = z_2(x)\Big[\int_a^x \frac{z_1(s)h(s)}{z_2'(s)z_1(s) - z_2(s)z_1'(s)}ds\Big] + z_1(x)\Big[\int_x^b \frac{z_2(s)h(s)}{z_2'(s)z_1(s) - z_2(s)z_1'(s)}ds\Big] + \frac{Az_2(x)}{z_2(a)} + \frac{Bz_1(x)}{z_1(b)},$$

where z_1 and z_2 are the solutions of homogeneous problems

$$z_1''(x) - \sqrt[3]{l}K(x)z_1'(x) - lz_1(x) = 0, \qquad z_1(a) = 0, z_1'(a) = 1,$$
$$z_2''(x) - \sqrt[3]{l}K(x)z_2'(x) - lz_2(x) = 0, \qquad z_2(b) = 0, z_2'(b) = -1.$$

One can show that $z_2(x) = z_1(a + b - x)$, for $x \in (a, b]$, $z_1(x) > 0$, and for $x \in [a, b)$ we have $z_1'(x) > 0$, $z_2 > 0$ and $z_2'(x) < 0$. Using these properties and the symmetry of K it is possible to show that z_1 satisfies the inequality

$$(M - l)z_1(x) + (N\ \mathrm{sign}(w') - \sqrt[3]{l}K(x))z_1'(x) \le 0$$

and z_2 satisfies the inequality

$$(M - l)z_2(x) + (N\ \mathrm{sign}(w') - \sqrt[3]{l}K(x))z_2'(x) \le 0,$$

for l sufficiently large. These inequalities for z_1 and z_2 and the expression for w can then be used to show (2.4). ■

The smoothness assumption on the coefficients and the inhomogeneity in (2.3) can be weakened, see [30] for details. In the next section we propose and analyze various Schwarz methods to solve (2.1).

3. Domain decomposition results

3.1. Alternating linear Schwarz on two subdomains

Following Lui [27], we consider an alternating Schwarz iteration to solve (2.1) on $\Omega = (0, 1)$ with $u = 0$ on $\partial\Omega$. We decompose Ω into two overlapping subdomains $\Omega_1 = (0, t)$ and $\Omega_2 = (s, 1)$ with $s < t$ as shown in Fig. 1.

For a given subsolution $\underline{\alpha}$, we set $\alpha_{(-\frac{1}{2})} = \alpha_{(0)} = \underline{\alpha}$ on $\bar{\Omega}$, with $\underline{\alpha} = 0$ on $\partial\Omega$ and iterate on two subdomains as: for $n = 0, 1, 2, \ldots$

$$\begin{aligned} -\alpha''_{(n+\frac{1}{2})} + \sqrt[3]{l}K(x)\alpha'_{(n+\frac{1}{2})} + l\alpha_{(n+\frac{1}{2})} &= -f(x, \alpha_{(n-\frac{1}{2})}, \alpha'_{(n-\frac{1}{2})}) \\ &\quad + \sqrt[3]{l}K(x)\alpha'_{(n-\frac{1}{2})} + l\alpha_{(n-\frac{1}{2})},\ \text{on } \Omega_1, \\ \alpha_{(n+\frac{1}{2})} = \alpha_{(n)} \text{ on } \partial\Omega_1,\ \alpha_{(n+\frac{1}{2})} &= \alpha_{(n)} \text{ on } \bar{\Omega} \setminus \bar{\Omega}_1, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -\alpha''_{(n+1)} + \sqrt[3]{l}K(x)\alpha'_{(n+1)} + l\alpha_{(n+1)} &= -f(x, \alpha_{(n)}, \alpha'_{(n)}) \\ &\quad + \sqrt[3]{l}K(x)\alpha'_{(n)} + l\alpha_{(n)},\ \text{on } \Omega_2, \\ \alpha_{(n+1)} = \alpha_{(n+\frac{1}{2})} \text{ on } \partial\Omega_2,\ \alpha_{(n+1)} &= \alpha_{(n+\frac{1}{2})} \text{ on } \bar{\Omega} \setminus \bar{\Omega}_2. \end{aligned} \tag{3.2}$$

Given a supersolution $\bar{\beta}$, we set $\beta_{(-\frac{1}{2})} = \beta_{(0)} = \bar{\beta}$ on $\bar{\Omega}$, with $\bar{\beta} = 0$ on $\partial\Omega$ and iterate on two subdomains as: for $n = 0, 1, 2, \ldots$

$$\begin{aligned} -\beta''_{(n+\frac{1}{2})} + \sqrt[3]{l}K(x)\beta'_{(n+\frac{1}{2})} + l\beta_{(n+\frac{1}{2})} &= -f(x, \beta_{(n-\frac{1}{2})}, \beta'_{(n-\frac{1}{2})}) \\ &\quad + \sqrt[3]{l}K(x)\beta'_{(n-\frac{1}{2})} + l\beta_{(n-\frac{1}{2})} \text{ on } \Omega_1, \\ \beta_{(n+\frac{1}{2})} = \beta_{(n)} \text{ on } \partial\Omega_1,\ \beta_{(n+\frac{1}{2})} &= \beta_{(n)} \text{ on } \bar{\Omega} \setminus \bar{\Omega}_1, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} -\beta''_{(n+1)} + \sqrt[3]{l}K(x)\beta'_{(n+1)} + l\beta_{(n+1)} &= -f(x, \beta_{(n)}, \beta'_{(n)}) \\ &\quad + \sqrt[3]{l}K(x)\beta'_{(n)} + l\beta_{(n)} \text{ on } \Omega_2 \\ \beta_{(n+1)} = \beta_{(n+\frac{1}{2})} \text{ on } \partial\Omega_2,\ \beta_{(n+1)} &= \beta_{(n+\frac{1}{2})} \text{ on } \bar{\Omega} \setminus \bar{\Omega}_2. \end{aligned} \tag{3.4}$$

We will ultimately show that it is possible to choose the coefficient function $K(x)$ and constant $l > 0$ to guarantee monotonic convergence of these iterations.

The well-posedness of the iteration, as stated in Lemma 3.1, follows directly from Lemma 2.7.

Lemma 3.1. *Assuming $K \in \mathcal{C}[0,1]$ and $l > 0$, then the iterations defined in (3.1)–(3.2) and (3.3)–(3.4) are well-posed if conditions [C2] and ([C3] or [C4]) are satisfied.*

To prove convergence of the alternating Schwarz iteration (3.1) and (3.2) we show the subdomain solutions are bounded and monotonic. Specifically, we show the iterates from (3.1) and (3.2) satisfy

$$\underline{\alpha} \le \alpha_{(n)} \le \alpha_{(n+\frac{1}{2})} \le \alpha_{(n+1)} \le \alpha_{(n+\frac{3}{2})} \le \bar{\beta}, \quad \text{for } n = 0, 1, 2 \dots.$$

This guarantees the pointwise convergence of the sequence of iterates. We then use elliptic regularity theory and a Nagumo type argument to demonstrate that the pointwise limit is indeed a solution to the boundary value problem (2.1) on [0, 1].

We begin with two results which will help our analysis.

Lemma 3.2. *Suppose $\mathcal{C}^2$ functions $y(x)$ and $z(x)$ satisfy*

$$-y'' + \sqrt[3]{l}K(x)y' + ly = -f(x, z, z') + \sqrt[3]{l}K(x)z' + lz, \text{ on } U, \tag{3.5}$$

then

$$w'' - \sqrt[3]{l}K(x)w' - lw = f(x, z, z') - z'' \text{ on } U,$$

where $w = y - z$.

Proof. Simply add $-z''$ to both sides of (3.5) and rearrange to get

$$(y'' - z'') - \sqrt[3]{l}K(x)(y' - z') - l(y - z) = f(x, z, z') - z''.$$

Introducing $w = y - z$ gives the desired result. ∎

Lemma 3.3. *Suppose $\mathcal{C}^2$ functions y and z, with $y \ge z$, satisfy (3.5), then*

$$f(x, y, y') - y'' \le (M - l)w + (N sign(w') - \sqrt[3]{l}K(x))w',$$

where $w = y - z$.

Proof. By Lemma 2.5, if $y \ge z$ and $w = y - z$ then

$$f(x, y, y') - f(x, z, z') \le Mw + N|w'|. \tag{3.6}$$

Adding $f(x, y, y')$ to both sides of (3.5) and rearranging gives

$$f(x, y, y') - y'' = f(x, y, y') - f(x, z, z') - \sqrt[3]{l}K(x)w' - lw.$$

Inequality (3.6) and the fact that $|w'| = \text{sign}(w')w'$ then imply

$$f(x, y, y') - y'' \le (M - l)w + (N\text{sign}(w') - \sqrt[3]{l}K(x))w'. \quad ∎$$

Our main analysis begins with a lemma that shows if we start the iteration on Ω_1 with a subsolution then all the subsequent subdomain iterates remain subsolutions. Further, we show on each subdomain the iterates form a monotonic sequence. We begin with the following assumption.

Assumption 1. In what follows we will assume that l is sufficiently large so that [C6], (2.2) in Lemma 2.6, and (2.4) in Lemma 2.7 holds.

Lemma 3.4. *Consider iteration (3.1)–(3.2) subject to conditions [C1]–[C3] and [C5], and l is sufficiently large so that Assumption 1 holds. Then for all $n = 0, 1, \dots, \alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$ are subsolutions of (2.1) on Ω_1 and Ω_2 respectively. Furthermore the subdomain solutions satisfy $\alpha_{(n)} \le \alpha_{(n+1)}$ and $\alpha_{(n+\frac{1}{2})} \le \alpha_{(n+\frac{3}{2})}$.*

Proof. We will prove this lemma by induction. On Ω_1, for $n = 0$, using $\alpha_{(-1/2)} = \alpha_{(0)}$, Eq. (3.1) becomes

$$-\alpha''_{(\frac{1}{2})} + \sqrt[3]{l}K(x)\alpha'_{(\frac{1}{2})} + l\alpha_{(\frac{1}{2})} = -f(x, \alpha_{(0)}, \alpha'_{(0)}) + \sqrt[3]{l}K(x)\alpha'_{(0)} + l\alpha_{(0)}. \tag{3.7}$$

Using Lemma 3.2 with $y = \alpha_{(\frac{1}{2})}$ and $z = \alpha_{(0)}$ we obtain

$$w'' - \sqrt[3]{l}K(x)w' - lw = f(x, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)}, \tag{3.8}$$

where $w = \alpha_{(\frac{1}{2})} - \alpha_{(0)}$. As $\alpha_{(0)}$ is a subsolution, we know $f(x, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)} \le 0$. The boundary conditions require $\alpha_{(\frac{1}{2})} = \alpha_{(0)}$ on $\partial\Omega_1$, that is, $w = 0$ on $\partial\Omega_1$. Hence from Lemma 2.4 we conclude $w \ge 0$ or $\alpha_{(\frac{1}{2})} \ge \alpha_{(0)}$ on Ω_1. Lemma 3.3 then implies

$$f(x, \alpha_{(\frac{1}{2})}, \alpha'_{(\frac{1}{2})}) - \alpha''_{(\frac{1}{2})} \le (M - l)w + (N\text{sign}(w') - \sqrt[3]{l}K(x))w'.$$

Since w satisfies (3.8), with a non-positive right hand side, and $w = 0$ on $\partial\Omega_1$ then Lemma 2.7 ensures

$$(M - l)w + (N\text{sign}(w') - \sqrt[3]{l}K(x))w' \le 0.$$

This implies

$$f(x, \alpha_{(\frac{1}{2})}, \alpha'_{(\frac{1}{2})}) - \alpha''_{(\frac{1}{2})} \le 0.$$

Hence we conclude that $\alpha_{(\frac{1}{2})}$ is a subsolution of (2.1).

Repeating this analysis on Ω_2, for $n = 0$, we have by Lemma 3.2 with $y = \alpha_{(1)}$ and $z = \alpha_{(0)}$

$$w'' - \sqrt[3]{l}K(x)w' - lw = f(x, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)}, \tag{3.9}$$

where $w = \alpha_{(1)} - \alpha_{(0)}$. Since $\alpha_{(0)}$ is a subsolution of (2.1), $f(x, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)} \le 0$. We have already seen that on Ω_1, $\alpha_{(\frac{1}{2})} \ge \alpha_{(0)}$. Since the left boundary of Ω_2 lies in Ω_1 where $\alpha_{(1)} = \alpha_{(\frac{1}{2})}$, we have $w = \alpha_{(1)} - \alpha_{(0)} \ge 0$ on $\partial\Omega_2$ and therefore by Lemma 2.4, $w \ge 0$, that is $\alpha_{(1)} \ge \alpha_{(0)}$, on Ω_2. Lemma 3.3 then ensures

$$f(x, \alpha_{(1)}, \alpha'_{(1)}) - \alpha''_{(1)} \le (M - l)w + (N\text{sign}(w') - \sqrt[3]{l}K(x))w'.$$

Since w satisfies (3.9), with a non-positive right hand side, and $w = \alpha_{(1)} - \alpha_{(0)} = \alpha_{(\frac{1}{2})} - \alpha_{(0)} \ge 0$ on $\partial\Omega_1$ (as shown earlier), then using Lemma 2.7 we have $f(x, \alpha_{(1)}, \alpha'_{(1)}) - \alpha''_{(1)} \le 0$, and hence we conclude that $\alpha_{(1)}$ is a subsolution.

Now assume that for some n, $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n)}$ are subsolutions of (2.1), and further assume $\alpha_{(n+\frac{1}{2})} \ge \alpha_{(n-\frac{1}{2})}$ and $\alpha_{(n+1)} \ge \alpha_{(n)}$. We will prove that $\alpha_{(n+\frac{3}{2})} \ge \alpha_{(n+\frac{1}{2})}$, $\alpha_{(n+2)} \ge \alpha_{(n+1)}$ and both $\alpha_{(n+\frac{3}{2})}$ and $\alpha_{(n+1)}$ are subsolutions.

Evaluating (3.1) at iteration $n + 1$ we have

$$\begin{aligned} -\alpha''_{(n+\frac{3}{2})} + \sqrt[3]{l}K(x)\alpha'_{(n+\frac{3}{2})} + l\alpha_{(n+\frac{3}{2})} = &-f(x, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) \\ &+ \sqrt[3]{l}K(x)\alpha'_{(n+\frac{1}{2})} + l\alpha_{(n+\frac{1}{2})}. \end{aligned}$$

Using Lemma 3.2 with $y = \alpha_{(n+\frac{3}{2})}$ and $z = \alpha_{(n+\frac{1}{2})}$ we have

$$w'' - \sqrt[3]{l}K(x)w' - lw = f(x, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) - \alpha''_{(n+\frac{1}{2})}, \tag{3.10}$$

where $w = \alpha_{(n+\frac{3}{2})} - \alpha_{(n+\frac{1}{2})}$. As $\alpha_{(n+\frac{1}{2})}$ is a subsolution we have $f(x, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) - \alpha''_{(n+\frac{1}{2})} \le 0$. By the induction hypothesis $w = \alpha_{(n+1)} - \alpha_{(n)} \ge 0$ on $\partial\Omega_1$. Hence Lemma 2.4 ensures $w \ge 0$ on Ω_1, that is, $\alpha_{(n+\frac{3}{2})} \ge \alpha_{(n+\frac{1}{2})}$ on Ω_1. Lemma 3.3 then ensures

$$f(x, \alpha_{(n+\frac{3}{2})}, \alpha'_{(n+\frac{3}{2})}) - \alpha''_{(n+\frac{3}{2})} \le (M - l)w + (N\text{sign}(w') - \sqrt[3]{l}K(x))w', \tag{3.11}$$

where w satisfies (3.10) which has a non-positive right hand side. Using the imposed boundary conditions and the induction hypothesis, $w \ge 0$ at the boundaries, and Lemma 2.7 ensures the right hand side of (3.11) is non-positive. Hence $f(x, \alpha_{(n+\frac{3}{2})}, \alpha'_{(n+\frac{3}{2})}) - \alpha''_{(n+\frac{3}{2})} \le 0$ and by induction $\alpha_{(n+\frac{3}{2})}$ is a subsolution for all n. A similar argument on Ω_2 shows that if $\alpha_{(n+1)} \ge \alpha_{(n)}$ and $\alpha_{(n)}$ is a subsolution then $\alpha_{(n+2)} \ge \alpha_{(n+1)}$ and $\alpha_{(n+1)}$ is a subsolution, which completes the proof. ■

In a similar fashion we can prove the following lemma.

Lemma 3.5. *Consider iteration (3.3)–(3.4) subject to conditions [C1]–[C3] and [C5], and l is sufficiently large so that Assumption 1 holds. Then for all $n = 0, 1, \ldots$, $\beta_{(n+\frac{1}{2})}$ and $\beta_{(n+1)}$ are supersolutions of (2.1) on Ω_1 and Ω_2 respectively and $\beta_n \le \beta_{n+1}$ and $\beta_{n+\frac{1}{2}} \le \beta_{n+\frac{3}{2}}$.*

The next lemma is a technical result which will help us prove the chain of inequalities in Lemma 3.7.

Lemma 3.6. *Assume $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$ are the sequences defined by (3.1) and (3.2) respectively, subject to conditions [C1]–[C3] and [C5], and l is sufficiently large so that Assumption 1 holds. Define the differences $w_{(n)} = \alpha_{(n)} - \alpha_{(n-\frac{1}{2})}$. If $w_{(n)} \ge 0$, then on $\Omega_{12} = \Omega_1 \cap \Omega_2$, $w_{(n+1)}$ satisfies*

$$w''_{(n+1)} - \sqrt[3]{l}K(x)w'_{(n+1)} - lw_{(n+1)} \le (M - l)w_{(n)} + (N\text{sign}(w'_{(n)}) - \sqrt[3]{l}K(x))w_{(n)}.$$

Proof. On Ω_{12} subtract the defining equations for $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$. Using the definition of $w_{(n)}$ and $w_{(n+1)}$ we find

$$\begin{aligned} &-w''_{(n+1)} + \sqrt[3]{l}K(x)w'_{(n+1)} + lw_{(n+1)} = \\ &-f(x, \alpha_{(n)}, \alpha'_{(n)}) + f(x, \alpha_{(n-\frac{1}{2})}, \alpha'_{(n-\frac{1}{2})}) + \sqrt[3]{l}K(x)w'_{(n)} + lw_{(n)}. \end{aligned}$$

Add and subtract $f(x, \alpha_{(n-\frac{1}{2})}, \alpha'_{(n-\frac{1}{2})})$ to the left side of the above inequality. If $w_{(n)} \geq 0$ then using Lemma 2.5 gives the required result. ■

In the following lemma we will continue the chain of inequalities and demonstrate a relation between $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$.

Lemma 3.7. *Assume $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$ are the sequences defined by (3.1) and (3.2) respectively, subject to conditions [C1]–[C3] and [C5], and l is sufficiently large so that Assumption 1 holds. Then for $n = 0, 1, \ldots$,*

$$\underline{\alpha} \leq \alpha_{(n)} \leq \alpha_{(n+\frac{1}{2})} \leq \alpha_{(n+1)} \leq \alpha_{(n+\frac{3}{2})} \quad \text{on } \Omega. \tag{3.12}$$

Furthermore, we show that on $\Omega_{12} \equiv \Omega_1 \cap \Omega_2$, the differences $w_{(n+\frac{1}{2})} = \alpha_{(n+\frac{1}{2})} - \alpha_{(n)}$ and $w_{(n+1)} = \alpha_{(n+1)} - \alpha_{(n+\frac{1}{2})}$ satisfy a BVP of the form (2.3) with non-positive right hand side and non-negative boundary conditions.

Proof. We proceed by induction. Let $\Omega_{12} = \Omega_1 \cap \Omega_2$. By definition $\alpha_{(0)} = \alpha_{(-\frac{1}{2})}$, hence $w_{(0)} = 0$ on Ω_{12}. Hence Lemma 3.6 guarantees

$$w''_{(1)} - \sqrt[3]{l}K(x)w'_{(1)} - lw_{(1)} \leq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\alpha_{(1)} \geq \alpha_{(\frac{1}{2})} = \alpha_{(0)}$ and on $\partial\Omega_2 \cap \Omega_1$, $\alpha_{(1)} = \alpha_{(\frac{1}{2})}$ by definition. Hence $w_{(1)} \geq 0$ or $\alpha_{(1)} - \alpha_{(\frac{1}{2})} \geq 0$ on $\partial\Omega_{12}$. Lemma 2.4 then guarantees that $\alpha_{(1)} \geq \alpha_{(\frac{1}{2})}$ on Ω_{12}. We also have $\alpha_{(1)} = \alpha_{(\frac{1}{2})}$ by definition on $\Omega \setminus \Omega_2$. Lemma 3.4 ensures $\alpha_{(1)} \geq \alpha_{(0)}$ on $\Omega \setminus \Omega_1$ but $\alpha_{(\frac{1}{2})} = \alpha_{(0)}$ on $\Omega \setminus \Omega_1$, hence $\alpha_{(1)} \geq \alpha_{(\frac{1}{2})}$ on $\Omega \setminus \Omega_1$ and so we have $\alpha_{(1)} \geq \alpha_{(\frac{1}{2})}$ on Ω. Furthermore, note that the difference $w_{(1)}$ satisfies (2.3) with a non-positive right hand side and non-negative boundary conditions.

Since $w_{(\frac{1}{2})} \geq 0$ on Ω_{12} then Lemma 3.6 with $n = \frac{3}{2}$ gives

$$w''_{(\frac{3}{2})} - \sqrt[3]{l}K(x)w'_{(\frac{3}{2})} - lw_{(\frac{3}{2})} \leq (M-l)w_{(\frac{1}{2})} + (N\,\mathrm{sign}\,(w'_{(\frac{1}{2})}) - \sqrt[3]{l}K(x))w'_{(\frac{1}{2})}. \tag{3.13}$$

Recall that $w_{(\frac{1}{2})}$ is the solution of (3.8) with non-negative boundary conditions and hence by Lemma 2.7 the right hand side of the above inequality is non-positive. We conclude from Eq. (3.13) that

$$w''_{(\frac{3}{2})} - \sqrt[3]{l}K(x)w'_{(\frac{3}{2})} - lw_{(\frac{3}{2})} \leq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\alpha_{(\frac{3}{2})} = \alpha_{(1)}$ and on $\partial\Omega_2 \cap \Omega_1$ $\alpha_{(\frac{3}{2})} \geq \alpha_{(\frac{1}{2})} = \alpha_{(1)}$. This implies that $w_1 = \alpha_{(\frac{3}{2})} - \alpha_{(1)} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma 2.4 we know $\alpha_{(\frac{3}{2})} \geq \alpha_{(1)}$ on Ω_{12}. Furthermore, by definition $\alpha_{(\frac{3}{2})} = \alpha_{(1)}$ on $\Omega \setminus \Omega_1$ and by Lemma 3.4 $\alpha_{(\frac{3}{2})} \geq \alpha_{(\frac{1}{2})} = \alpha_{(1)}$ on $\Omega \setminus \Omega_2$. Therefore, $\alpha_{(\frac{3}{2})} \geq \alpha_{(1)}$ on Ω. Note again, the difference $w_{(\frac{3}{2})}$ satisfies a BVP of the form (2.3) with a non-positive right hand side and non-negative boundary conditions.

Having established the base case, now assume that on Ω

$$\alpha_{(k+\frac{1}{2})} \leq \alpha_{(k+1)} \leq \alpha_{(k+\frac{3}{2})}, \tag{3.14}$$

for $k = 0, 1, \ldots, n-1$. In addition, assume that the differences $w_{(k+\frac{3}{2})} = \alpha_{(k+\frac{3}{2})} - \alpha_{(k+1)}$ and $w_{(k+1)} = \alpha_{(k+1)} - \alpha_{(k+\frac{1}{2})}$, for $k = 0, 1, \ldots, n-1$, satisfy BVP (2.3) with a non-positive right hand side and with non-negative boundary conditions. We will prove (3.14) holds for $k = n$ or that

$$\alpha_{(n+\frac{1}{2})} \leq \alpha_{(n+1)} \leq \alpha_{(n+\frac{3}{2})}.$$

From the induction hypothesis, $w_{(n)} \geq 0$ on Ω_{12}. Lemma 3.6 then ensures

$$w''_{(n+1)} - \sqrt[3]{l}K(x)w'_{(n+1)} - lw_{(n+1)} \leq (M-l)w_{(n)} + (N\mathrm{sign}(w'_{(n)}) - \sqrt[3]{l}K(x))w'_{(n)}.$$

The induction hypothesis ensures that $w_{(n)}$ satisfies BVP (2.3) with a non-positive right hand side and non-negative boundary conditions on $\partial\Omega_{12}$. By Lemma 2.7 we conclude that

$$w''_{(n+1)} - \sqrt[3]{l}K(x)w'_{(n+1)} - lw_{(n+1)} \leq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\alpha_{(n+1)} \geq \alpha_{(n)} = \alpha_{(n+\frac{1}{2})}$ (the inequality follows from Lemma 3.4 and equality from the enforced boundary condition), while on $\partial\Omega_2 \cap \Omega_1$ $\alpha_{(n+1)} = \alpha_{(n+\frac{1}{2})}$ by definition. This implies $\alpha_{(n+1)} - \alpha_{(n+\frac{1}{2})} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma 2.4 we conclude that $\alpha_{(n+1)} \geq \alpha_{(n+\frac{1}{2})}$ on Ω_{12}. We have $\alpha_{(n+1)} = \alpha_{(n+\frac{1}{2})}$ by definition on $\Omega \setminus \Omega_2$. Lemma 3.4 ensures $\alpha_{(n+1)} \geq \alpha_{(n)}$ on $\Omega \setminus \Omega_1$ but $\alpha_{(n+\frac{1}{2})} = \alpha_{(n)}$ on $\Omega \setminus \Omega_1$. Hence $\alpha_{(n+1)} \geq \alpha_{(n+\frac{1}{2})}$ on Ω.

The non-negativity of $w_{(n+\frac{3}{2})}$, given the induction hypothesis, follows similarly and the details are omitted. ■

We now complete the required inequalities by showing that the sequence is bounded above by the supersolution.

Lemma 3.8. *Assume $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$ are the sequences defined by (3.1) and (3.2) respectively, subject to conditions [C1]–[C3] and [C5], and l is sufficiently large so that Assumption 1 holds. Then for all $n = 0, 1, \ldots$,*

$$\underline{\alpha} \le \alpha_{(n)} \le \alpha_{(n+\frac{1}{2})} \le \alpha_{(n+1)} \le \alpha_{(n+\frac{3}{2})} \le \bar{\beta} \quad \text{on } \Omega. \tag{3.15}$$

Proof. In Lemma 3.4 we have proved that $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n)}$ are monotonically increasing from subsolution $\underline{\alpha}$ in Ω, so these sequences are bounded below by $\underline{\alpha}$ in Ω.

Define

$$h_{(n)} := f(x, \bar{\beta}, \bar{\beta}') - f(x, \alpha_{(n)}, \alpha'_{(n)}) - \sqrt[3]{l}K(x)(\bar{\beta}' - \alpha'_{(n)}) - l(\bar{\beta} - \alpha_{(n)})$$

and define $h_{(n+\frac{1}{2})}$ as above in the obvious way. Here we use induction to show that for all $n = 0, 1, \ldots, \alpha_{(n)} \le \bar{\beta}, \alpha_{(n+\frac{1}{2})} \le \bar{\beta}$, $h_{(n)} \le 0$ and $h_{(n+\frac{1}{2})} \le 0$.

Since $\alpha_{(0)} = \underline{\alpha}$, then $\alpha_{(0)} \le \bar{\beta}$ follows from Lemma 2.6. From (3.1), for $n = 0$, we have

$$\begin{aligned} -\alpha''_{(\frac{1}{2})} + \sqrt[3]{l}K(x)\alpha'_{(\frac{1}{2})} + l\alpha_{(\frac{1}{2})} = &-f(x, \underline{\alpha}, \underline{\alpha}') \\ &+ \sqrt[3]{l}K(x)\underline{\alpha}' + l\underline{\alpha}, \end{aligned} \tag{3.16}$$

as $\alpha_{(-\frac{1}{2})} = \alpha_{(0)} = \underline{\alpha}$. From the definition of a supersolution we have $\bar{\beta}'' \le f(x, \bar{\beta}, \bar{\beta}')$. Adding $-\sqrt[3]{l}K(x)\bar{\beta}' - l\bar{\beta}$ on both sides of this inequality and rearranging, we have

$$-\bar{\beta}'' + \sqrt[3]{l}K(x)\bar{\beta}' + l\bar{\beta} \ge -f(x, \bar{\beta}, \bar{\beta}') + \sqrt[3]{l}K(x)\bar{\beta}' + l\bar{\beta}. \tag{3.17}$$

Now subtracting (3.16) from (3.17) and rearranging, we find

$$\begin{aligned} &-(\bar{\beta}'' - \alpha''_{(\frac{1}{2})}) + \sqrt[3]{l}K(x)(\bar{\beta}' - \alpha'_{(\frac{1}{2})}) + l(\bar{\beta} - \alpha_{(\frac{1}{2})}) \ge f(x, \underline{\alpha}, \underline{\alpha}') - f(x, \bar{\beta}, \bar{\beta}') + \\ &+ \sqrt[3]{l}K(x)(\bar{\beta}' - \alpha'_{(0)}) + l(\bar{\beta} - \alpha_{(0)}). \end{aligned} \tag{3.18}$$

The requirement $\alpha_{(\frac{1}{2})} \le \bar{\beta}$ on Ω_1 now follows from (2.2) and Lemma 2.4. Since $\alpha_{(1/2)} = \alpha_{(0)} \le \bar{\beta}$ on $\bar{\Omega} \setminus \bar{\Omega}_1$ then $\alpha_{(1/2)} \le \bar{\beta}$ on $\bar{\Omega}$. The relation (2.2) also gives us $h_{(0)} \le 0$, while $h_{(\frac{1}{2})} \le 0$ follows from (3.18), Lemma 2.7, conditions [C2] and [C3].

Relation (3.18) is satisfied with $\alpha_{(1/2)}$ replaced by $\alpha_{(1)}$ on Ω_2. Hence $\alpha_{(1)} \le \bar{\beta}$ from (2.2) and Lemma 2.4 since $\bar{\beta} - \alpha_{(1)} = \bar{\beta} - \alpha_{(1/2)} \ge 0$ on $\partial\Omega_2$.

Assume that for some n, $\alpha_{(n+\frac{1}{2})} \le \bar{\beta}, h_{(n+\frac{1}{2})} \le 0$ on Ω and also $\alpha_{(n+1)} \le \bar{\beta}, h_{(n+1)} \le 0$ on Ω. We will prove that $\alpha_{(n+\frac{3}{2})} \le \bar{\beta}$ and $h_{(n+\frac{3}{2})} \le 0$ and also $\alpha_{(n+2)} \le \bar{\beta}$ and $h_{(n+2)} \le 0$.

Subtracting the equation for $\alpha_{(n+\frac{3}{2})}$ from (3.17) we get

$$\begin{aligned} -(\bar{\beta}'' - \alpha''_{(n+\frac{3}{2})}) + \sqrt[3]{l}K(x)(\bar{\beta}' - \alpha'_{(n+\frac{3}{2})}) + l(\bar{\beta} - \alpha_{(n+\frac{3}{2})}) = &-f(x, \bar{\beta}, \bar{\beta}') + \\ f(x, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) + \sqrt[3]{l}K(x)(\bar{\beta}' - \alpha'_{(n+\frac{1}{2})}) + l(\bar{\beta} - \alpha_{(n+\frac{1}{2})}). \end{aligned}$$

The right hand side is simply $-h_{(n+\frac{1}{2})}$. Hence using the induction hypothesis and Lemma 2.7 we conclude $\alpha_{(n+\frac{1}{2})} \le \bar{\beta}$. The requirement that $h_{(n+3/2)} \le 0$ now follows upon using the conditions [C2] and [C3] and the inequality in Lemma 2.7. The proof that $\alpha_{(n+1)} \le \bar{\beta}$ and $h_{(n+2)} \le 0$ is completely analogous. ■

In a similar way we can prove the following lemma.

Lemma 3.9. *Assume $\beta_{(n+\frac{1}{2})}$ and $\beta_{(n+1)}$ are the sequences defined by (3.3) and (3.4) respectively, subject to conditions [C1]–[C3] and [C5], and l is sufficiently large so that Assumption 1 holds. Then for all $n = 0, 1, \ldots$,*

$$\bar{\beta} \ge \beta_{(n)} \ge \beta_{(n+\frac{1}{2})} \ge \beta_{(n+1)} \ge \beta_{(n+\frac{3}{2})} \ge \underline{\alpha} \quad \text{on } \Omega, \tag{3.19}$$

where $\underline{\alpha}$ is a subsolution and $\bar{\beta}$ is a supersolution of (2.1).

The monotonicity and boundedness of the subdomain iterates guarantee the existence of pointwise limits. We conclude this section by considering the regularity of these pointwise limits and showing that the subdomain iterates converge to solutions of the model problem (2.1). Compared to the single domain case presented in [21], the analysis of the DD iteration is complicated by the lack of explicit control over the derivatives of the subdomain solutions at the interior subdomain boundaries.

We begin with a lemma from Schmitt and Thompson [32] which provides an a priori bound on the derivative of solutions of our subdomain problems.

Lemma 3.10. *Suppose $u \in \mathcal{C}^2([a,b])$ and there exists a positive number P so that $\|u\|_\infty \le P$. Assume $\|u''\|_\infty \le \phi(\|u'\|_\infty)$, where $\phi(s)$ is a positive, nondecreasing, continuous real-valued function which satisfies*

$$\lim_{s\to\infty} \frac{\phi(s)}{s^2} = 0. \tag{3.20}$$

Then there exists a constant Q so that $\frac{\phi(s)}{s^2} < \frac{1}{4P}$ when $s > Q$. Furthermore, $\|u'\|_\infty \le \bar{M}$ where $\bar{M} = \max(Q, 8P)$.

Our right-hand side function f, satisfying conditions [C2] and [C3], satisfies a linear growth condition as shown in the following lemma.

Lemma 3.11. *Assume conditions [C1]–[C3] hold, then $|f(x,u,v)| \le j(|v|)$ on the set $\mathcal{D}$, where $j(s) \equiv Ns + d$ for some constants N and d.*

Proof. The proof follows from condition [C3] which guarantees that

$$-N|u'| - f(x,u,0) \le f(x,u,u') \le N|u'| + f(x,u,0).$$

The continuity of f for $\underline{\alpha} \le u \le \bar{\beta}$ ensures that there exists a constant $d > 0$ such that $|f(x,u,0)| \le d$. Hence we have

$$-N|u'| - d \le f(x,u,u') \le N|u'| + d,$$

or $|f(x,u,u')| \le N|u'| + d$ on $\mathcal{D}$, from which the result follows. ■

Motivated by our iterations (3.1)–(3.2), and (3.3)–(3.4), we obtain the following result.

Lemma 3.12. *Suppose $u \in \mathcal{C}^2([a,b])$ and $\eta \in \mathcal{C}^1([a,b])$, satisfy $u, \eta \in [\underline{\alpha}, \bar{\beta}]$. Furthermore, assume u satisfies*

$$u'' = f(x,\eta,\eta') + l(u-\eta) + \sqrt[3]{l}K(x)(u'-\eta'), \tag{3.21}$$

and suppose

$$\|u'\|_\infty \ge \max(\|\eta'\|_\infty, l).$$

Then there exists a $\bar{M}$ such that $\|u'\|_\infty \le \bar{M}$.

Proof. Define $R = max\{\|\underline{\alpha} - \bar{\beta}\|_\infty, \|\underline{\alpha}\|_\infty, \|\bar{\beta}\|_\infty\}$. If $\|u'\|_\infty \ge \max(\|\eta'\|_\infty, l)$ and u satisfies (3.21) then using the triangle inequality and Lemma 3.11, we have $\|u''\|_\infty \le \phi(\|u'\|_\infty)$ where

$$\phi(s) = j(s) + Rs + 2s^{4/3}\|K\|_\infty.$$

By assumption u is bounded (in L^∞) and since

$$\lim_{s\to\infty} \frac{\phi(s)}{s^2} = 0,$$

then by Lemma 3.10 there exists a constant $\bar{M}$ (depending only on ϕ, K, $\underline{\alpha}$, and $\bar{\beta}$, but not η), such that $\|u'\|_\infty \le \bar{M}$. ■

This allows us to the prove the following theorem.

Theorem 3.13. *Let u_n denote the sequence of subdomain iterates defined in (3.1) and (3.2), or in (3.3) and (3.4), subject to conditions [C1]–[C3] and [C5], and suppose l is sufficiently large so that Assumption 1 holds. Then the derivatives of the subdomain iterates are uniformly bounded (in n). Specifically, we have*

$$\|u_n'\|_\infty \le \max(\|u_0'\|_\infty, \bar{M}, l), \quad \textit{for all } n = 0, 1, \ldots$$

Proof. We proceed by induction. By Lemma 3.12 either $\|u_1'\|_\infty \le \max(\|u_0'\|_\infty, l)$ or $\|u_1'\|_\infty \le \bar{M}$. In either case, $\|u_1'\|_\infty \le \max(\|u_0'\|_\infty, \bar{M}, l)$. Assume $\|u_{n-1}'\|_\infty \le \max(\|u_0'\|_\infty, \bar{M}, l)$. Then by Lemma 3.12 either $\|u_n'\|_\infty \le \max(\|u_{n-1}'\|_\infty, l) \le \max(\|u_0'\|_\infty, \bar{M}, l)$ or $\|u_n'\|_\infty \le \bar{M}$. In either case, $\|u_n'\|_\infty \le \max(\|u_0'\|_\infty, \bar{M}, l)$ as required. ■

We are now in a position to prove the first main result.

Theorem 3.14. *Assume conditions [C1]–[C5] hold, and l is sufficiently large so that Assumption 1 holds. Then the alternating Schwarz sequence $\alpha_{(n+\frac{i}{2})}$, defined in (3.1) and (3.2), converges to a solution, u, of (2.1) in $\mathcal{C}^2(\bar{\Omega}_i)$ for $i = 1,2$.*

Proof. The monotonicity and boundedness of the iterates on each subdomain, as demonstrated above if l is sufficiently large, guarantee the point-wise limits $\hat{\alpha}_1$ and $\hat{\alpha}_2$ exist so that

$$\lim_{n\to\infty} \alpha_{(n+1/2)}(x) = \hat{\alpha}_1(x) \text{ on } \Omega_1 \qquad \text{and} \qquad \lim_{n\to\infty} \alpha_{(n+1)}(x) = \hat{\alpha}_2(x) \text{ on } \Omega_2.$$

We now show that $\hat{\alpha}_1 = \hat{\alpha}_2$ is a solution of (2.1) following the general regularity argument given in [33].

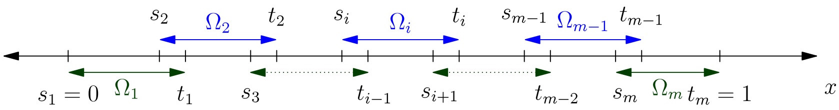


Fig. 2. Domain decomposed into m subdomains.

Conditions [C1]–[C5] and the uniform boundedness of $\alpha_{(n-1/2)}$ and $\alpha'_{(n-1/2)}$ (from Theorem 3.13) ensure the uniform boundedness, in L_∞, of the right hand side of the defining equation for $\alpha_{(n+1/2)}$ in (3.1). This gives the uniform boundedness of the right hand side of the defining equation for $\alpha_{(n+1/2)}$ in $L_p(\Omega_1)$ for $p \geq 1$. Then Lemma 1.1 in Chapter 3 of Pao [33] gives uniform boundedness of $\alpha_{(n+1/2)}$ in $W_p^2(\Omega_1)$. By choosing $p > 1$ so that $\mu = 1 - 1/p > 0$ then the embedding estimate, Lemma 1.2 in Chapter 3 of Pao [33], gives the uniform boundedness of $\alpha_{(n+1/2)}$ in $\mathcal{C}^{1+\mu}(\bar{\Omega}_1)$. The Hölder continuity of f then implies that the right hand side of the defining equation for $\alpha_{(n+1/2)}$ is uniformly bounded in $\mathcal{C}^{\mu}(\bar{\Omega}_1)$. The Schauder estimate, Theorem 1.3 in Chapter 3 of Pao [33], then gives $\alpha_{(n+1/2)}$ in $\mathcal{C}^{2+\mu}(\bar{\Omega}_1)$. The Arzel–Ascoli theorem guarantees a subsequence which converges, in $\mathcal{C}^2$, to a function $\bar{\alpha}_1$. Since $\alpha_{(n+1/2)}$ converges point-wise to $\hat{\alpha}_1$ then $\bar{\alpha}_1 \equiv \hat{\alpha}_1$. The monotonicity of the sequence then ensures the whole sequence $\alpha_{(n+1/2)}$ converges, in $\mathcal{C}^2(\bar{\Omega}_1)$ to $\hat{\alpha}_1$. Likewise $\alpha_{(n+1)}$ converges to $\hat{\alpha}_2$, in $\mathcal{C}^2(\bar{\Omega}_2)$.

Passing to the limit in the inequalities (3.15) we must have $\hat{\alpha}_1 = \hat{\alpha}_2$ on $\bar{\Omega}$. Now, define $u = \hat{\alpha}_1$. Then u solves (2.1) on $\bar{\Omega}$. ■

Similarly we can prove the following theorem.

Theorem 3.15. *Assume conditions [C1]–[C5] hold and l is sufficiently large so that Assumption 1 holds. Then the alternating Schwarz sequence $\{\beta_{(n+\frac{i}{2})}\}$, defined in (3.3) and (3.4), converges to a solution of (2.1) in $\mathcal{C}^2(\bar{\Omega}_i)$ for $i = 1,2$.*

If the BVP has a unique solution then sequence of subsolutions and supersolutions both converge to the unique solution. We note that in [21], $\mathcal{C}^1$ convergence is obtained on a single domain by omitting the stronger Hölder continuity requirement [C4].

3.2. Alternating linear Schwarz on multiple subdomains

Now suppose the domain is decomposed into m overlapping subdomains as shown in Fig. 2. For a given subsolution $\underline{\alpha}$, and initial subdomain solutions $\alpha_{(0,i)} = \underline{\alpha}$, $i = 1, \ldots, m$, the iteration scheme proceeds as: for $n = 0, 1, 2, \ldots$, for $i = 1, \ldots, m$, solve on Ω_i

$$\begin{aligned} \alpha''_{(n+1,i)} - \sqrt[3]{l}K(x)\alpha'_{(n+1,i)} - l\alpha_{(n+1,i)} &= f(x, \alpha_{(n,i)}, \alpha'_{(n,i)}) - \sqrt[3]{l}K(x)\alpha'_{(n,i)} - l\alpha_{(n,i)}, \\ \alpha_{(n+1,i)}(s_i) = \alpha_{(n+1,i-1)}(s_i) \quad &\text{and} \quad \alpha_{(n+1,i)}(t_i) = \alpha_{(n,i+1)}(t_i). \end{aligned} \tag{3.22}$$

For purposes of consistency, we define $\alpha_{(n+1,0)}(s_1) = \alpha_{(n,m+1)}(t_m) = 0$. Outside of Ω_i we define $\alpha_{(n+1,i)} = \alpha_{(n+1,k)}$ for $k < i$ and $\alpha_{(n+1,i)} = \alpha_{(n,k)}$ for $k > i$.

Similarly for a given supersolution $\bar{\beta}$, and initial subdomain solutions $\beta_{(0,i)}$, $i = 1, \ldots, m$, we consider the iteration scheme: for $n = 0, 1, 2, \ldots$, for $i = 1, \ldots, m$, solve on Ω_i

$$\begin{aligned} \beta''_{(n+1,i)} - \sqrt[3]{l}K(x)\beta'_{(n+1,i)} - l\beta_{(n+1,i)} &= f(x, \beta_{(n,i)}, \beta'_{(n,i)}) - \sqrt[3]{l}K(x)\beta'_{(n,i)} - l\alpha_{(n,i)}, \\ \beta_{(n+1,i)}(s_i) = \beta_{(n+1,i-1)}(s_i) \quad &\text{and} \quad \beta_{(n+1,i)}(t_i) = \beta_{(n,i+1)}(t_i). \end{aligned} \tag{3.23}$$

Again, for consistency, we define $\beta_{(n+1,0)} = \beta_{(n,m+1)} = 0$. Outside of Ω_i we define $\beta_{(n+1,i)} = \beta_{(n+1,k)}$ for $k < i$ and $\beta_{(n+1,i)} = \beta_{(n,k)}$ for $k > i$.

Theorem 3.16. *Assume conditions [C1]–[C5] hold and l is sufficiently large so that Assumption 1 holds. Then the alternating Schwarz sequence defined in (3.22), converges from a subsolution to a solution, u, of (2.1) in $\mathcal{C}^2(\bar{\Omega}_i)$ for $i = 1, \ldots, m$.*

Proof. Proceeding as in Section 3.1, we may show by induction that the sequence $\alpha_{(n,i)}$, for $n = 0, 1, \ldots$ and $i = 1, \ldots, m$, defined by (3.22), satisfies the inequalities

$$\begin{aligned} &\underline{\alpha} \leq \alpha_{(n,1)} \leq \alpha_{(n,2)} \leq \ldots \leq \alpha_{(n,i)} \leq \alpha_{(n,i+1)} \\ &\qquad \leq \ldots \leq \alpha_{(n,m)} \leq \alpha_{(n+1,1)} \leq \alpha_{(n+1,2)} \leq \cdots \leq \alpha_{(n+1,m)} \leq \bar{\beta} \quad \text{on } \Omega. \end{aligned} \tag{3.24}$$

As in the proof of Theorem 3.14, the monotonicity and boundedness of the subdomain solutions give the existence of point-wise limits on each subdomain. Repeating the elliptic regularity argument given in the proof of Theorem 3.14 gives convergence of $\alpha_{(n,i)}$ to a solution u of (2.1) on $\bar{\Omega}_i$. ■

Similarly we obtain convergence starting from a supersolution.

Theorem 3.17. *Assume conditions [C1]–[C5] hold and l is sufficiently large so that Assumption 1 holds. Then the alternating Schwarz sequence defined in (3.23), converges from a supersolution to a solution, u, of (2.1) in $C^2(\bar{\Omega}_i)$ for $i = 1, \ldots, m$.*

3.3. Parallel linear Schwarz on multiple subdomains

The following Schwarz variant allows a parallel implementation, as the m subdomain solves per iteration are completely independent.

For a given subsolution $\underline{\alpha}$, and initial subdomain solutions $\alpha_{(0,i)} = \underline{\alpha}$, $i = 1, \ldots, m$, the iteration scheme proceeds as: for $n = 0, 1, 2, \ldots$, for $i = 1, \ldots, m$, solve on Ω_i

$$\begin{aligned} &\alpha''_{(n+1,i)} - \sqrt[3]{l}K(x)\alpha'_{(n+1,i)} - l\alpha_{(n+1,i)} = f(x, \alpha_{(n,i)}, \alpha'_{(n,i)}) - \sqrt[3]{l}K(x)\alpha'_{(n,i)} - l\alpha_{(n,i)}, \\ &\alpha_{(n+1,i)} = \alpha_{(n)} \text{ on } \partial\Omega_i, \;\; \alpha_{(n+1,i)} = \alpha_{(n)} \text{ on } \bar{\Omega} \setminus \bar{\Omega}_i \text{ and } \alpha_{(n)}(x) = \max_{1\le i\le m} \alpha_{(n,i)}(x), \;\; x \in \bar{\Omega}. \end{aligned} \tag{3.25}$$

Similarly for a given supersolution $\bar{\beta}$, and initial subdomain solutions $\beta_{(0,i)}$, $i = 1, \ldots, m$, we consider the iteration scheme: for $n = 0, 1, 2, \ldots$, for $i = 1, \ldots, m$, solve on Ω_i

$$\begin{aligned} &\beta''_{(n+1,i)} - \sqrt[3]{l}K(x)\beta'_{(n+1,i)} - l\beta_{(n+1,i)} = f(x, \beta_{(n,i)}, \beta'_{(n,i)}) - \sqrt[3]{l}K(x)\beta'_{(n,i)} - l\alpha_{(n,i)}, \\ &\beta_{(n+1,i)} = \beta_{(n)} \text{ on } \partial\Omega_i, \;\; \beta_{(n+1,i)} = \beta_{(n)} \text{ on } \bar{\Omega} \setminus \bar{\Omega}_i \text{ and } \beta_{(n)}(x) = \min_{1\le i\le m} \beta_{(n,i)}(x), \;\; x \in \bar{\Omega}. \end{aligned} \tag{3.26}$$

Theorem 3.18. *Assume conditions [C1]–[C5] hold and l is sufficiently large so that Assumption 1 holds. Then the parallel Schwarz sequence defined in (3.25), converges from a subsolution to a solution, u, of (2.1) in $C^2(\bar{\Omega}_i)$ for $i = 1, \ldots, m$.*

Proof. The definition of $\alpha_{(n)}$, and induction arguments similar to those provided in Section 3.1, give the inequalities

$$\underline{\alpha} \le \alpha_{(n,i)} \le \alpha_{(n+1,i)} \le \bar{\beta} \quad \text{on} \quad \bar{\Omega}_i. \tag{3.27}$$

We also have

$$\underline{\alpha} \le \alpha_{(n)} \le \alpha_{(n+1)} \le \bar{\beta} \quad \text{on} \quad \bar{\Omega}.$$

This follows quickly from the inequalities (3.27) as $\alpha_{(n)}(x) = \alpha_{(n,i)}(x)$ for some i, which implies

$$\alpha_{(n)}(x) = \alpha_{(n,i)}(x) \le \alpha_{(n+1,i)}(x) \le \alpha_{(n+1)}(x).$$

And finally we have

$$\alpha_{(n)} \le \alpha_{(n+1,i)} \quad \text{on} \quad \bar{\Omega},$$

for $i = 1, \ldots, m$. This inequality is more delicate. Since $\alpha_{(n)}(x) = \max_{1\le i\le m} \alpha_{(n,i)}(x)$ for $x \in \bar{\Omega}$, then the inequality can be established by showing that for each i, $\alpha_{(n+1,i)} \ge \alpha_{(n,j)}$ for $j = 1, \ldots, m$. The case $j = i$ follows from (3.27) and holds by the definition of $\alpha_{(n+1,i)}$ for j such that $\Omega_j \cap \Omega_i = \emptyset$. Hence, if $i = 1$ then we only have to consider $j = 2$, if $i = m$ then we only consider $j = m - 1$ and for all other i values we have to only consider $j = i - 1$ and $j = i + 1$.

As an example, suppose $1 < i < m$ and let $j = i + 1$, then we must show that on $\Omega_i \cap \Omega_j$ we have $\alpha_{(n+1,i)} \ge \alpha_{(n,i+1)}$ for all $n = 0, 1, \ldots$. The argument proceeds by induction. To show the base case we subtract the defining equations for $\alpha_{(1,i)}$ and $\alpha_{(0)}$ and use the fact that $\alpha_{(0)}$ is a subsolution and the maximum principle to conclude that $\alpha_{(1,i)} \ge \alpha_{(0)}$ in the overlap and furthermore $w = \alpha_{(1,i)} - \alpha_{(0)}$ satisfies $(l - M)w + (\sqrt[3]{l}K - N\text{sign}(w'))w' \ge 0$.

Now assume $\alpha_{(n-1)}(x) \le \alpha_{(n,i)}$ on $\bar{\Omega}$ and $w = \alpha_{(n,i)} - \alpha_{(n-1,i+1)}$ satisfies $(l - M)w + (\sqrt[3]{l}K - N\text{sign}(w'))w' \ge 0$ in the overlap. We complete the proof by showing $\alpha_{(n+1,i)} \ge \alpha_{(n,i+1)}$ in the overlap region.

Subtracting the defining equations for $\alpha_{(n+1,i)}$ and $\alpha_{(n,i+1)}$ and using the induction hypothesis we find that $w = \alpha_{(n+1,i)} - \alpha_{(n,i+1)}$ satisfies $w'' - \sqrt[3]{l}K(x)w' - lw \le 0$ in the overlap region. Also on the left boundary of the overlap we have $\alpha_{(n+1,i)} \ge \alpha_{(n,i)} \ge \alpha_{(n-1)} = \alpha_{(n,i+1)}$. This first inequality follows from (3.27) and the second from the induction hypothesis. At the right boundary of the overlap we have $\alpha_{(n+1,i)} = \alpha_{(n,i+1)}$ by definition. Hence Lemma 2.7 then guarantees $w \ge 0$ in overlap region. The argument for $j = i - 1$ proceeds similarly.

This gives the existence of point-wise limits $\hat{\alpha}_i$ and $\hat{\alpha}_0$ so that

$$\lim_{n\to\infty} \alpha_{(n,i)}(x) = \hat{\alpha}_i(x) \qquad \text{and} \qquad \lim_{n\to\infty} \alpha_{(n)}(x) = \hat{\alpha}_0(x).$$

Repeating Pao's argument we find that $\hat{\alpha}_i$ satisfies the differential equation on $\bar{\Omega}_i$ and the convergence is in the sense of $C^2(\bar{\Omega}_i)$. The above inequalities guarantee $\hat{\alpha}_i \le \hat{\alpha}_0$ on $\bar{\Omega}$ for $i = 1, \ldots, m$. Also for any j we have $\hat{\alpha}_0 \le \hat{\alpha}_j \le \hat{\alpha}_0 \le \hat{\alpha}_i$, hence $\hat{\alpha}_i = \hat{\alpha}_j = \hat{\alpha}_0$ for all $i, j = 1, \ldots, m$. This common function must be a solution of (2.1). ∎

Similarly we obtain convergence starting from a supersolution.

Theorem 3.19. *Assume conditions [C1]–[C5] hold and l is sufficiently large so that Assumption 1 holds. Then the parallel Schwarz sequence defined in (3.26), converges from a supersolution to a solution, u, of (2.1) in $C^2(\bar{\Omega}_i)$ for $i = 1, \ldots, m$.*

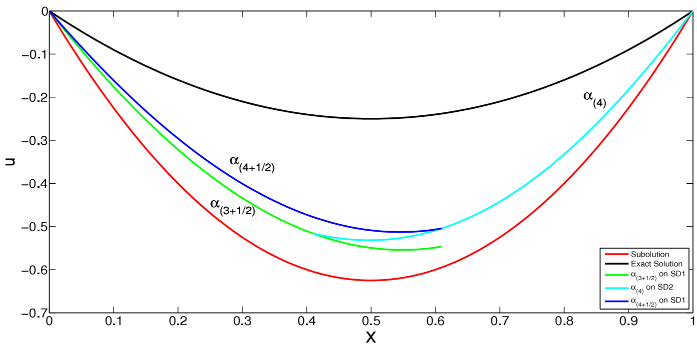


Fig. 3. Plot showing inequality (3.12) for BVP (4.1) for $n = 9$.

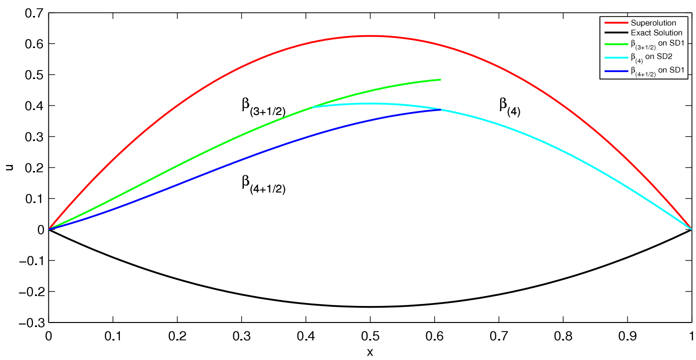


Fig. 4. Plot showing inequality (3.19) for BVP (4.1) for $n = 9$.

4. Numerical results

In this section, we provide some brief numerical results to illustrate the theory developed above.

Example 1. As a simple test example we consider the nonlinear BVP

$$u'' = \sin(u') + 2 - \sin(2x - 1), \qquad u(0) = 0, \quad u(1) = 0, \tag{4.1}$$

whose exact solution is given by $u(x) = x^2 - x$. One is able to quickly check that the BVP satisfies the required regularity conditions. Furthermore, one can explicitly check that the BVP has a subsolution $\underline{\alpha}(x) = \frac{5}{2}x(x-1)$, and a supersolution is $\bar{\beta}(x) = -\frac{5}{2}x(x-1)$, with $\underline{u}(x) \le \bar{u}(x)$.

In Fig. 3 we verify numerically the monotonicity properties indicated in inequality (3.12), for iterates starting from the subsolution.

Similarly Fig. 4 shows that inequality (3.19) is satisfied when the iterates start from the supersolution.

Fig. 5 shows the numerical solution of BVP (4.1) using iterations (3.1) and (3.2) starting from the subsolution. The exact solution of the BVP is plotted in the heavy thick black line.

Fig. 6 shows the numerical solution of BVP (4.1) using iterations (3.3) and (3.4) starting from the supersolution.

Indeed the convergence is monotonic as predicted by the theory. In Figs. 7 and 8 we plot the log of the difference between the exact solution and the subdomain iterates (at $x = 1/4$ for subdomain one on the left and at $x = 3/4$ for subdomain two on the right). In both plots the difference decreases (notice the negative values on the vertical axis in Fig. 8). The difference remains positive if we start the iteration from the subsolution and remains negative if we start the iteration from the supersolution.

Fig. 9 shows the convergence history obtained using the alternating Schwarz iteration (3.22) on 2, 3, 4 and 5 subdomains. As is well-known the convergence rate decreases as the number of subdomains increases.

Fig. 10 shows the convergence history obtained using the parallel Schwarz iterations (3.26) on 2, 3, 4 and 5 subdomains.

Example 2. As an another example we consider the viscous steady state Burgers' equation with a viscosity of 1. The BVP is given by

$$u_{xx} = uu_x, \qquad u(0) = 1, u(1) = -1.$$

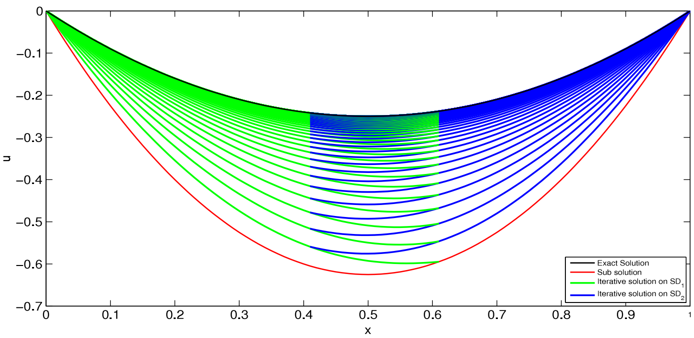


Fig. 5. Linearized DD iterations starting from the subsolution for BVP (4.1).

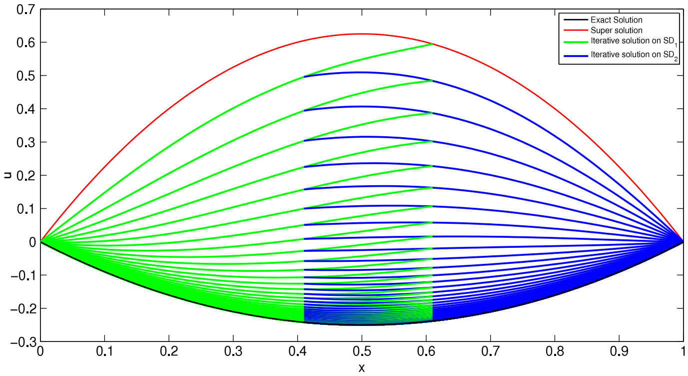
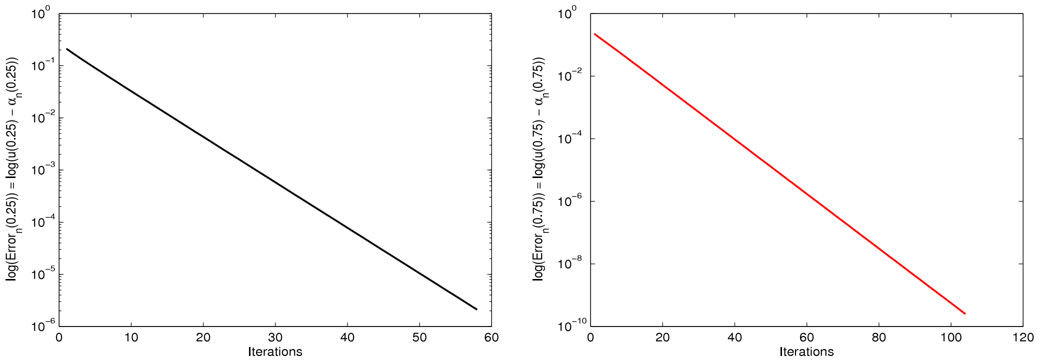


Fig. 6. Linearized DD iterations starting from the supersolution for BVP (4.1).



(a) Monotonicity of the iterates on the first subdomain.

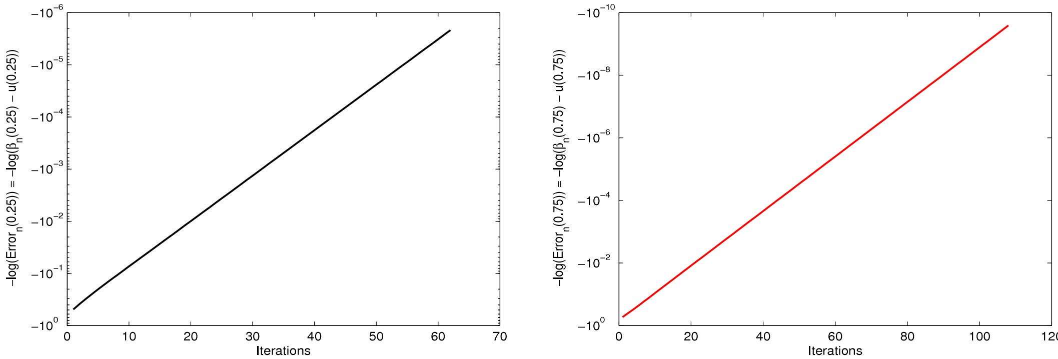
(b) Monotonicity of the iterates on the second subdomain.

Fig. 7. Monotonicity of the iterates starting from subsolution for BVP (4.1).

A change of variables to obtain zero boundary conditions gives the equivalent BVP

$$v_{xx} = (v - 2x + 1)(v_x - 2), \qquad v(0) = v(1) = 0. \tag{4.2}$$

It is easy to verify that $\underline{\alpha} = \gamma x(1-x)$ with $\gamma \geq 2/3$ is a supersolution for this problem, while $\bar{\beta} = \gamma x(1-x)$ is a subsolution if $\gamma \leq -2/3$. It should be noted that generalizing to find subsolutions and supersolutions which satisfy the boundary conditions (as required by the theory) for an arbitrary viscosity for this example is not trivial. This highlights one shortcoming of this approach. Furthermore, this example does not satisfy condition [C2] on the right hand side function f as required by the proof of the theorem. In spite of this, we can indeed generate a monotonically convergent sequence of functions for this example as the numerics illustrate in Figs. 11 and 12. In this case the mild nonlinearity in v_x suggests that



(a) Monotonicity of the iterates on the first subdomain.

(b) Monotonicity of the iterates on second subdomain.

Fig. 8. Monotonicity of the iterates starting from supersolution for BVP (4.1).

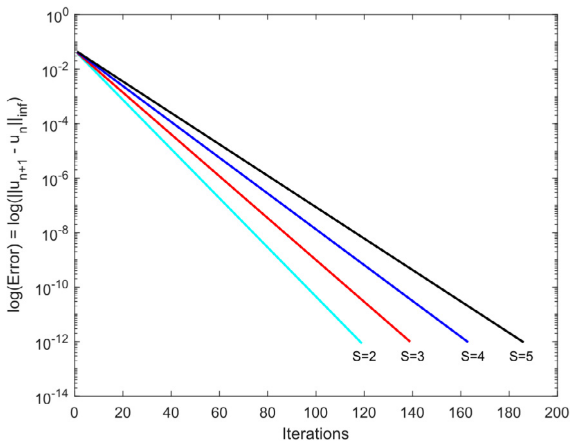


Fig. 9. Convergence history for the linearized alternating Schwarz algorithm for (4.1) on 2, 3, 4 and 5 subdomains.

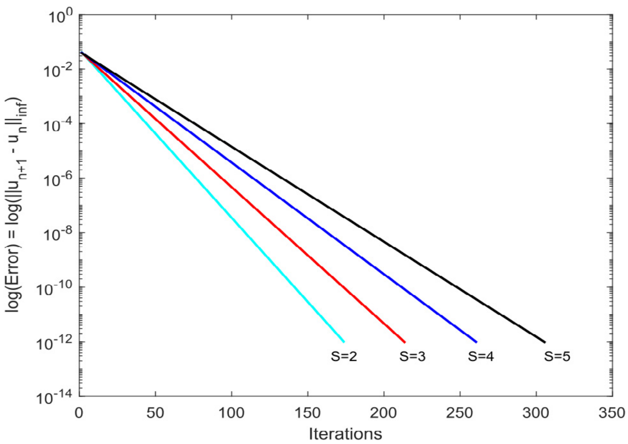


Fig. 10. Convergence history for the linearized parallel Schwarz algorithm for (4.1) on 2, 3, 4 and 5 subdomains.

the problem satisfies a Nagumo condition, allowing us to get a priori C^1 estimates on the solutions of the modified problem, and to recover the monotone convergence of the iteration. Further work in this direction is ongoing.

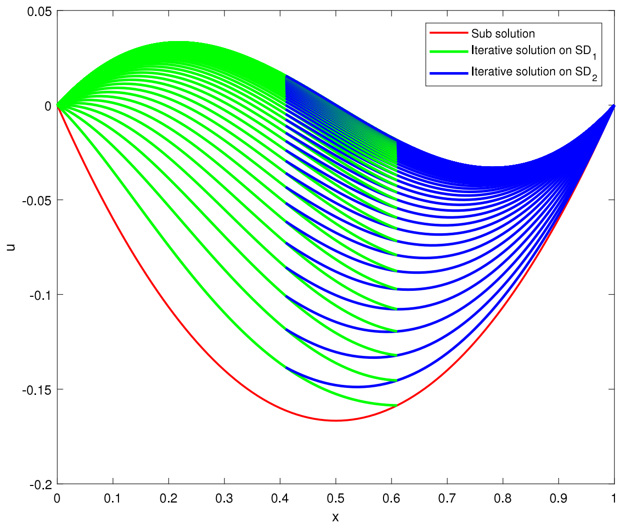


Fig. 11. Linearized DD iterations starting from the subsolution for BVP (4.2).

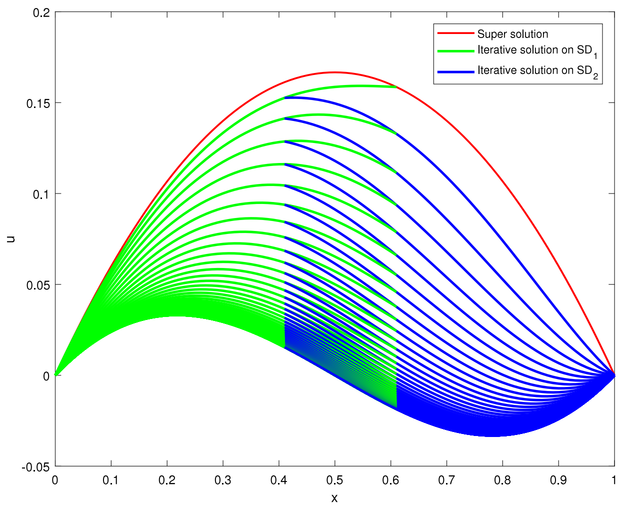


Fig. 12. Linearized DD iterations starting from the supersolution for BVP (4.2).

5. Conclusion and summary

In this paper we have proposed and analyzed multiplicative and parallel, linearized Schwarz iterations for boundary value problems which depend nonlinearly on the derivative of the unknown solution, subject to Dirichlet boundary conditions. Our work extends the single domain monotone schemes proposed by Cherpion [21] for these problems to a solution on multiple subdomains and also extends the linearized Schwarz iterations analyzed by Lui for problems without explicit u' dependence.

The proposed approach is not without its issues — the monotone scheme for this Dirichlet problem relies on the generation of a subsolution or a supersolution which satisfies the boundary conditions. This is not trivial, and may be as difficult as obtaining an initial solution within the basin of attraction for Newton's method. The conditions on f stated here can be relaxed as our last example illustrates. Attempts to relax the restrictions imposed on the growth of the right hand side function with respect to u' and extensions to PDE problems with gradient dependence are underway.

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