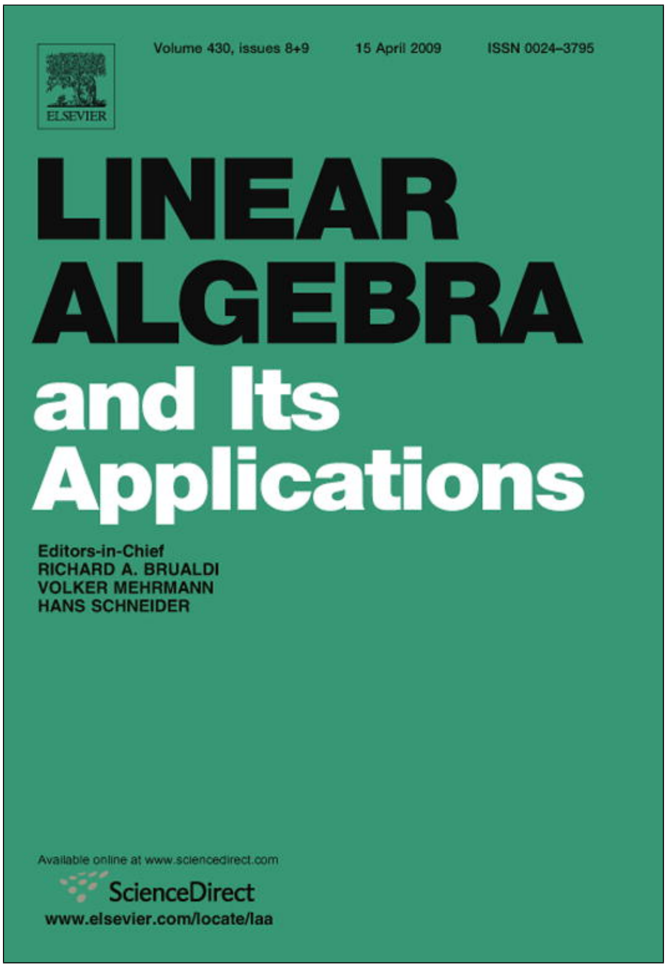
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Inverse positivity of perturbed tridiagonal M-matrices

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A B S T R A C T

A well-known property of an M-matrix M is that the inverse is element-wise non-negative, which we write as $M^{-1} \geqslant 0$. In this paper, we consider element-wise perturbations of non-symmetric tridiagonal M-matrices and obtain bounds on the perturbations so that the non-negative inverse persists. Sufficient bounds are written in terms of decay estimates which characterize the decay of the elements of the inverse of the unperturbed matrix. Results for general symmetric matrices and symmetric Toeplitz matrices are obtained as special cases and compared with known results.

1. Introduction

A rich class of matrices known as M-matrices were introduced by Ostrowski in 1937 [1], with reference to the work of Minkowski [2,3]. Approximately 50 different but equivalent characterizations of M-matrices are given by Bermann and Plemmons [4]. A condition which is easy to check is that a matrix M is an M-matrix if and only if $m_{ij} \leqslant 0$ for $i \neq j$, $m_{ii} > 0$ and M is generalized strictly diagonally dominant. A matrix M is said to be generalized (strictly) diagonally dominant if there exists a diagonal matrix D with positive entries so that MD is (strictly) diagonally dominant. Of particular importance to us is the fact that since M is an M-matrix it is non-singular and $M^{-1} \geqslant 0$, where the inequality is satisfied element-wise.

In this paper, we consider the inverses of perturbed M-matrices. Specifically we consider the effect of changing single elements of M. If these perturbations do not change the M-matrix sign pattern, then a sufficient condition to ensure the inverse is non-negative is obtained by imposing the required diagonal dominance property. We explore perturbations which destroy the sign pattern and ask under what conditions are the inverses of the resulting matrices non-negative. In [5], the authors considered perturbations of the second diagonals (elements $(i, i+2)$ and $(i, i-2)$) of symmetric tridiagonal M-

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matrices and an example of a specific single element-wise perturbation of a non-symmetric tridiagonal M-matrix. In this paper, we extend these results and obtain bounds on the maximum allowable perturbation made to a general element of a non-symmetric tridiagonal M-matrix. We extract the analogous results for symmetric and Toeplitz matrices as special cases.

The remainder of the paper is organized as follows. In Section 2, we review known properties of inverses of tridiagonal M-matrices. Section 3 details the main results for general element-wise perturbations of non-symmetric tridiagonal matrices made to elements outside of the tridiagonal band. In Section 4, we consider the special case of symmetric and Toeplitz M-matrices and show how these results agree with and extend the results found in [5]. We then discuss perturbations made to elements inside the diagonal band in Section 5. We conclude in Section 6 with comments and suggestions for future work.

2. Inverses of tridiagonal M-matrices

Characterizations of the inverses of banded matrices have been considered by many authors, cf. [6–9]. Here we will review the results which are important for the remainder of this paper.

Throughout this paper we use the following notation for a general tridiagonal M-matrix:

$$M = \begin{pmatrix} a_1 & -c_1 & 0 & \cdots & 0 \\ -b_1 & a_2 & -c_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -b_{n-2} & a_{n-1} & -c_{n-1} \\ 0 & \cdots & 0 & -b_{n-1} & a_n \end{pmatrix},$$

where a_i, b_i and $c_i > 0$, and each a_i is large enough that M is strictly diagonally dominant.

The following quantities were originally given by Nabben [10] but were refined for M-matrices by Peluso [11]:

$$\tau_i = \frac{c_i}{a_i - b_{i-1}}, \quad \omega_i = \frac{b_{i-1}}{a_i - c_i},$$
$$\delta_i = \frac{c_i}{a_i}, \quad \gamma_i = \frac{b_{i-1}}{a_i}$$

for $i = 1, \ldots, n$, with $b_0 = c_0 = b_n = c_n = 0$ for consistency. Nabben [10] gives the following result regarding the decay rate of elements of the inverse of M. This result will enable a comparison of any two elements in the same column of the inverse.

Theorem 1. *If m_{ij}^{-1} are elements of M^{-1} then*

$$\delta_i m_{i+1,j}^{-1} \leqslant m_{ij}^{-1} \leqslant \tau_i m_{i+1,j}^{-1}, \quad i = 1, \ldots, j-1$$

and

$$\gamma_i m_{i-1,j}^{-1} \leqslant m_{ij}^{-1} \leqslant \omega_i m_{i-1,j}^{-1}, \quad i = j+1, \ldots, n.$$

For notational convenience we define the following quantities:

$$\delta = \min_{i=2,\ldots,n} \delta_i, \quad \gamma = \min_{i=i,\ldots,n-1} \gamma_i,$$
$$\tau = \max_{i=2,\ldots,n} \tau_i, \quad \text{and} \quad \omega = \max_{i=1\ldots n-1} \omega_i. \tag{1}$$

It is useful to note that δ_i, γ_i, τ_i, and ω_i are all positive values less than 1 due to the assumption of strict diagonal dominance of M. The decay estimates above can be refined iteratively, see [10] for details. These improved estimates can be used in place of $\delta_i, \tau_i, \gamma_i$ and ω_i improving the quality of the bounds obtained in the results below.

The decay estimates also give bounds on the diagonal entries of the inverse which are easy to compute. Since $MM^{-1} = I$, we have

$$-b_{i-1}m^{-1}_{i-1,i} + a_i m^{-1}_{ii} - c_i m^{-1}_{i+1,i} = 1 \quad \text{for } i = 1, \ldots, n,$$

where we set $b_0 = m^{-1}_{0,1} = c_n = m^{-1}_{n+1,n} = 0$ for consistency. Using the decay elements described in Theorem 1, we see

$$(-\tau_{i-1}b_{i-1} + a_i - \omega_{i+1}c_i)m^{-1}_{ii} \leqslant 1$$

or

$$m^{-1}_{ii} \leqslant \frac{1}{a_i - \tau_{i-1}b_{i-1} - \omega_{i+1}c_i}$$

with $\tau_0 = \omega_{n+1} = 0$ for consistency. This result, also found in [10], is stated as Lemma 1.

Lemma 1. *If the diagonal entries of M^{-1} are denoted m^{-1}_{ii} then they are bounded as*

$$m^{-1}_{ii} \leqslant \mu \equiv \max_{i=1,\ldots,n} \frac{1}{a_i - \tau_{i-1}c_{i-1} - \omega_{i+1}b_i}. \tag{2}$$

The decay rates τ_i, ω_i, γ_i and δ_i from Theorem 1 bound ratios of consecutive elements in columns of the inverse but are not dependent on the column index itself (as the single subscript would suggest). The proof of this follows from a result by Capovani [12] which states that for any tridiagonal matrix M, there exist four vectors u, v, x, and y where $u_i v_i = x_i y_i$ for all i, so that m^{-1}_{ij} are given by

$$m^{-1}_{ij} = \begin{cases} u_i v_j, & i \leqslant j, \\ x_i y_j, & i \geqslant j. \end{cases} \tag{3}$$

Therefore

$$\begin{aligned} \frac{m^{-1}_{i,j_1}}{m^{-1}_{i+1,j_1}} &= \frac{u_i v_{j_1}}{u_{i+1}v_{j_1}} = \frac{u_i}{u_{i+1}} = \frac{u_i v_{j_2}}{u_{i+1}v_{j_2}} = \frac{m^{-1}_{i,j_2}}{m^{-1}_{i+1,j_2}} \quad \text{for } i < j_1, j_2 \\ \frac{m^{-1}_{i,j_1}}{m^{-1}_{i-1,j_1}} &= \frac{x_i y_{j_1}}{x_{i-1}y_{j_1}} = \frac{x_i}{x_{i-1}} = \frac{x_i y_{j_2}}{x_{i-1}y_{j_2}} = \frac{m^{-1}_{i,j_2}}{m^{-1}_{i-1,j_2}} \quad \text{for } i > j_1, j_2. \end{aligned} \tag{4}$$

Hence proving the ratios are independent of the column index j.

3. Non-symmetric M-matrices

In this section, we explore the effects of single element perturbations made to tridiagonal M-matrices and find a bound on the size of a general perturbation that ensures the inverse of the perturbed matrix is non-negative.

3.1. Perturbing element $(1, 3)$

Let P be the perturbed matrix, $P = M + E$ where $E = uv^T$ with $u = (h, 0, \ldots, 0)^T$ and $v = (0, 0, 1, 0, \ldots, 0)^T$, so that we perturb the $(1, 3)$ element of M^{-1}. How small must h be in order to ensure that P^{-1} is element-wise non-negative?

In [13], Sherman and Morrison give an explicit formula for the inverse of the perturbed matrix:

$$P^{-1} = (M + uv^T)^{-1} = M^{-1} - \frac{M^{-1}uv^T M^{-1}}{1 + v^T M^{-1} u}.$$

A couple of quick calculations show $v^T M^{-1} u = hm^{-1}_{31}$ and $uv^T M^{-1}$ is a matrix whose first row is h times the third row of M^{-1} with the rest of the rows being zeroes. So $P^{-1} \geqslant 0$ if and only if

$$M^{-1}\cdot\frac{h}{1+hm_{31}^{-1}}\cdot\begin{pmatrix} m_{31}^{-1} & m_{32}^{-1} & \cdots & m_{3n}^{-1}\\ 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}\leqslant M^{-1}.$$

We now use the decay estimates given in Theorem 1 to find an upper bound on h to ensure $P^{-1}\geqslant 0$. Element-wise, the above inequality requires

$$sm_{i,1}^{-1}m_{3,j}^{-1}\leqslant m_{ij}^{-1} \tag{5}$$

for all $1\leqslant i,j\leqslant n$ where $s=h/(1+hm_{31}^{-1})$. We find our bound on h by finding a restriction on s for which (5) will hold for all i and j.

When $(i,j)=(1,1)$ we use the decay estimates in Theorem 1 and find that the following sequence of inequalities hold:

$$sm_{11}^{-1}m_{31}^{-1}=sm_{11}^{-1}\frac{m_{31}^{-1}}{m_{21}^{-1}}\frac{m_{21}^{-1}}{m_{11}^{-1}}m_{11}^{-1}\leqslant s\omega_2\omega_3m_{11}^{-1}m_{11}^{-1}\leqslant s\omega_2\omega_3\mu m_{11}^{-1}.$$

If we then require that $s\omega_2\omega_3\mu m_{11}^{-1}\leqslant m_{11}^{-1}$ then this indicates that $s\leqslant\frac{1}{\mu\omega_2\omega_3}$ is a sufficient requirement to make $sm_{11}^{-1}m_{31}^{-1}\leqslant m_{11}^{-1}$.

For $(i,j)=(1,2)$ we see

$$sm_{11}^{-1}m_{32}^{-1}=sm_{11}^{-1}\frac{m_{32}^{-1}}{m_{22}^{-1}}\frac{m_{22}^{-1}}{m_{12}^{-1}}m_{12}^{-1}\leqslant s\frac{\omega_3}{\delta_1}\mu m_{12}^{-1},$$

so making $s\leqslant\frac{\delta_1}{\mu\omega_3}$ will force $s\frac{\omega_3}{\delta_1}\mu m_{12}^{-1}\leqslant m_{12}^{-1}$, so this restriction on s is sufficient.
When $i=1$ and $j\geqslant 3$ there is an obvious pattern, so we can group these cases together. We find

$$sm_{11}^{-1}m_{3j}^{-1}=sm_{11}^{-1}\frac{m_{3j}^{-1}}{m_{2j}^{-1}}\frac{m_{2j}^{-1}}{m_{1j}^{-1}}m_{1,j}^{-1}\leqslant s\frac{\mu}{\delta_1\delta_2}m_{1j}^{-1}\leqslant m_{1j}^{-1}\quad\text{if } s\leqslant\frac{\delta_1\delta_2}{\mu}.$$

Now consider $j\leqslant i=2$,

$$sm_{21}^{-1}m_{3j}^{-1}=s\frac{m_{2,1}^{-1}}{m_{1,1}^{-1}}m_{11}^{-1}\frac{m_{3,j}^{-1}}{m_{2,j}^{-1}}m_{2,j}^{-1}\leqslant s\mu\omega_2\omega_3m_{ij}^{-1}\leqslant m_{ij}^{-1}\quad\text{if } s\leqslant\frac{1}{\mu\omega_2\omega_3}.$$

Then when $j>i=2$,

$$sm_{21}^{-1}m_{3j}^{-1}=s\frac{m_{2,1}^{-1}}{m_{1,1}^{-1}}m_{11}^{-1}\frac{m_{3,j}^{-1}}{m_{2,j}^{-1}}m_{2,j}^{-1}\leqslant s\mu\omega_2\frac{1}{\delta_2}m_{ij}^{-1}\leqslant m_{ij}^{-1}\quad\text{if } s\leqslant\frac{\delta_2}{\mu\omega_2}.$$

When $j\leqslant 3\leqslant i$ things get a little more complicated. We start off by noting

$$sm_{i1}^{-1}m_{3j}^{-1}=s\prod_{x=2}^{i}\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}m_{11}^{-1}\prod_{y=4}^{i}\frac{m_{y-1,j}^{-1}}{m_{y,j}^{-1}}m_{ij}^{-1}. \tag{6}$$

In Section 2, we demonstrated that the ratio between two consecutive elements in a column is independent of the column index, it is only dependent on the row and which side of the diagonal the elements are on. The result is summarized in (4), and allows for cancellation since $\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}=\frac{m_{y,j}^{-1}}{m_{y-1,j}^{-1}}$ when $x=y$. Therefore, we may simplify (6) as

$$sm_{i1}^{-1}m_{3j}^{-1}=s\frac{m_{2,1}^{-1}}{m_{1,1}^{-1}}\frac{m_{3,1}^{-1}}{m_{2,1}^{-1}}m_{1,1}^{-1}m_{ij}^{-1}\leqslant s\omega_2\omega_3\mu m_{ij}^{-1}.$$

This will be less than m_{ij}^{-1} as long as $s\leqslant\frac{1}{\omega_2\omega_3\mu}$.

When $3 \leqslant j \leqslant i$ we again use (4) and find

$$sm_{i1}^{-1}m_{3j}^{-1} = s\prod_{x=2}^{i}\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}m_{1,1}^{-1}\prod_{y=3}^{j-1}\frac{m_{y,j}^{-1}}{m_{y+1,j}^{-1}}\prod_{z=j+1}^{l}\frac{m_{z-1,j}^{-1}}{m_{z,j}^{-1}}m_{ij}^{-1}$$
$$= s\prod_{x=2}^{j}\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}m_{1,1}^{-1}\prod_{y=3}^{j-1}\frac{m_{y,j}^{-1}}{m_{y+1,j}^{-1}}m_{ij}^{-1} \leqslant s\prod_{x=2}^{j}\omega_x\prod_{y=3}^{j-1}\tau_y\mu m_{ij}^{-1}.$$

Hence we require $s \leqslant \frac{1}{\mu\omega_2\cdots\omega_j\tau_3\cdots\tau_{j-1}}$.

Finally, if $j > i \geqslant 3$ then we have

$$sm_{i1}^{-1}m_{3j}^{-1} = s\prod_{x=2}^{i}\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}\prod_{y=3}^{i-1}\frac{m_{y,j}^{-1}}{m_{y+1,j}^{-1}}m_{ij}^{-1} \leqslant m_{ij}^{-1} \quad \text{if } s \leqslant \frac{1}{\mu\omega_2\cdots\omega_i\tau_3\cdots\tau_{i-1}}.$$

We now wish to find the smallest of these bounds on s, since this will guarantee that (5) holds for all (i,j), and this in turn guarantees the non-negativity of elements of the inverse of the perturbed M-matrix, P. Using the fact that ω_i, τ_i and δ_i are all less than one we are able to show that the smallest restriction is $\frac{\delta_1\delta_2}{\mu}$.

Since $s \leqslant h$, forcing h to be less than $\frac{\delta_1\delta_2}{\mu}$ ensures that s will also be less than this bound. Therefore, $m_{ij}^{-1} \geqslant sm_{i,k}^{-1}m_{l,j}^{-1}$ for all (i,j), which in turn ensures that P^{-1} will be element-wise non-negative. This gives us the following theorem.

Theorem 2. *Assume M is a strictly diagonally dominant tridiagonal M-matrix. Let $u = (h,0,0,\ldots,0,0)^T$ and $v = (0,0,1,0,\ldots,0)^T$ and form the rank-1 matrix uv^T. To ensure the matrix $P = M + uv^T$ has a non-negative inverse (element-wise) it is sufficient that h satisfies*

$$h \leqslant \frac{\delta^2}{\mu},$$

where δ and μ are defined in (1) and (2), respectively.

As an illustration, consider the 40×40 M-matrix where $b_i = c_i = 1$ and $a_i = 4$ for all i. Our bound implies that if $h \leqslant 0.208333$ then the inverse will be non-negative. The actual largest value of h for which the inverse will be element-wise non-negative, is 0.25. The actual value may be computed numerically using an exhaustive search or more efficiently by employing a bisection method to find the h value at which the inverse changes from having all non-negative elements to having at least one negative element.

3.2. Perturbing element $(1,k)$ for $k \geqslant 3$

We will now consider a perturbation in position $(1,k)$. Just as before, we express our perturbed matrix P as $P = M + uv^T$ where M is our unperturbed tridiagonal matrix and u and v are vectors of length n. In this case, to perturb the entry $(1,k)$ we choose $v = (0,\ldots,0,h,0,\ldots,0)^T$, where h is in the kth position, and $u = (1,0,0,\ldots)^T$. Using the Sherman–Morrison formula and simplifying in a similar way, we find that we need

$$m_{ij}^{-1} \geqslant sm_{i,1}^{-1}m_{k,j}^{-1}$$

for all (i,j), where $s = \frac{h}{1+hm_{k,1}^{-1}}$, in order to ensure that the inverse is non-negative. The bound on s is found by considering all combinations of i and j.

When $i \leqslant k$ we have the following possibilities and results:

$$j \leqslant i \quad sm_{i,1}^{-1}m_{k,j}^{-1} \leqslant s\prod_{x=2}^{i}\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}m_{11}^{-1}\prod_{y=i+1}^{k}\frac{m_{y,j}^{-1}}{m_{y-1,j}^{-1}}m_{ij}^{-1}\mu$$

$$\leqslant s\prod_{x=2}^{i}\omega_x\prod_{y=i+1}^{k}\omega_y\mu m_{ij}^{-1}\leqslant m_{ij}^{-1}\quad\text{if } s\leqslant\frac{1}{\omega_2\cdots\omega_k\mu},$$

$$i\leqslant j\leqslant k\quad sm_{i,1}^{-1}m_{k,j}^{-1}\leqslant s\prod_{x=2}^{i}\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}m_{11}^{-1}\prod_{y=j+1}^{k}\frac{m_{y,j}^{-1}}{m_{y-1,j}^{-1}}\prod_{z=i}^{j-1}\frac{m_{z+1,j}^{-1}}{m_{z,j}^{-1}}m_{ij}^{-1}$$

$$\leqslant s\prod_{x=2}^{i}\omega_x\prod_{y=j+1}^{k}\omega_y\prod_{z=i}^{j-1}\frac{1}{\delta_z}\mu m_{ij}^{-1}\leqslant m_{ij}^{-1}$$

$$\text{if } s\leqslant\frac{\delta_i\cdots\delta_{j-1}}{\omega_2\cdots\omega_i\omega_{j+1}\cdots\omega_k\mu},$$

$$k\leqslant j\quad sm_{i,1}^{-1}m_{k,j}^{-1}\leqslant s\prod_{x=2}^{i}\frac{m_{x,1}^{-1}}{m_{x-1,1}^{-1}}m_{11}^{-1}\prod_{y=i}^{k-1}\frac{m_{y+1,j}^{-1}}{m_{y,j}^{-1}}m_{i,j}^{-1}$$

$$\leqslant s\prod_{x=2}^{i}\omega_x\prod_{y=i}^{k-1}\frac{1}{\delta_y}\mu m_{i,j}^{-1}\leqslant m_{i,j}^{-1}\quad\text{if } s\leqslant\frac{\delta_i\cdots\delta_{k-1}}{\omega_2\cdots\omega_i\mu}.$$

Similarly, we look at the possible cases when $k < i$ and the resulting upper bounds on s are shown in the following table:

	$j \leqslant k$	$k \leqslant j < i$	$j \geqslant i$
Bound on s	$\frac{1}{\omega_2\cdots\omega_k\mu}$	$\frac{1}{\omega_2\cdots\omega_j\tau_k\cdots\tau_{j-1}\mu}$	$\frac{1}{\tau_k\cdots\tau_{i-1}\omega_2\cdots\omega_i\mu}$

We now have six categories of restrictions on s. If all of these restrictions are met then $m_{ij}^{-1} \geqslant sm_{i,1}^{-1}m_{k,j}^{-1}$ for all (i,j), and so P^{-1} will be element-wise non-negative. Since τ_i, ω_i and δ_i are always less than 1 we find that the minimum bound on s over the possible (i,j) for a fixed k will be $s \leqslant \frac{\delta_1\cdots\delta_{k-1}}{\mu}$. This gives us the following theorem.

Theorem 3. *Assume M is a strictly diagonally dominant tridiagonal M-matrix. Let v be a vector of zeroes with h in the kth position (where $k \geqslant 3$) and u be a vector of zeroes with 1 in the first position, so that they form the rank-1 matrix uv^T. To ensure the matrix $P = M + uv^T$ has a non-negative inverse (element-wise) it is sufficient that h satisfies*

$$h \leqslant \frac{\delta^{k-1}}{\mu},$$

where δ and μ are defined in (1) and (2), respectively.

This bound indicates that the further down the row we wish to make our perturbation, the smaller the allowable perturbation in order to ensure that the inverse will be non-negative. Note that this result is consistent with the result found in Section 3.1.

3.3. Perturbing a general element (k,l) for $k \geqslant l+2$ or $k \leqslant l-2$

We wish to generalize our result to an arbitrary single element perturbation in position (k,l) where $k \geqslant l+2$ or $k \leqslant l-2$. Similar to the previous cases, we build our perturbed matrix P by defining the vectors u and v such that the matrix uv^T is a matrix of zeroes, except for element (k,l) which has a value of h. Continuing in the same manner as before, we find that we must have $m_{ij}^{-1} \geqslant sm_{i,k}^{-1}m_{l,j}^{-1}$ where

Table 1
Restrictions on s when perturbing above or below the diagonal band.

$l \geqslant k+2$		$k \geqslant l+2$	
	Bound on s		Bound on s
$j \leqslant i \leqslant k < l$	$\frac{1}{\tau_i \cdots \tau_{k-1} \omega_{i+1} \cdots \omega_l \mu}$	$j \leqslant i \leqslant l \leqslant k$	$\frac{1}{\tau_i \cdots \tau_{k-1} \omega_{i+1} \cdots \omega_l \mu}$
$i \leqslant j \leqslant k, l$	$\frac{1}{\tau_j \cdots \tau_{k-1} \omega_{j+1} \cdots \omega_l \mu}$	$i \leqslant j \leqslant l \leqslant k$	$\frac{1}{\tau_j \cdots \tau_{k-1} \omega_{j+1} \cdots \omega_l \mu}$
$i \leqslant k \leqslant j \leqslant l$	$\frac{\delta_k \cdots \delta_{j-1}}{\omega_{j+1} \cdots \omega_l \mu}$	$i \leqslant l \leqslant j \leqslant k$	$\frac{1}{\tau_l \tau_{l+1} \cdots \tau_k \mu}$
$i \leqslant k < l \leqslant j$	$\frac{\delta_k \cdots \delta_{l-1}}{\mu}$	$i \leqslant l \leqslant k \leqslant j$	$\frac{1}{\tau_l \tau_{l+1} \cdots \tau_{k-1} \mu}$
$j \leqslant k \leqslant i \leqslant l$	$\frac{1}{\omega_{k+1} \cdots \omega_l \mu}$	$j \leqslant l \leqslant i \leqslant k$	$\frac{\gamma_{i+1} \cdots \gamma_l}{\tau_i \cdots \tau_{k-1} \mu}$
$k \leqslant j \leqslant i \leqslant l$	$\frac{1}{\omega_{k+1} \cdots \omega_l \mu}$	$l \leqslant j \leqslant i \leqslant k$	$\frac{\gamma_{j+1} \cdots \gamma_i}{\tau_l \cdots \tau_{j-1} \tau_i \cdots \tau_{k-1} \mu}$
$k \leqslant i \leqslant j \leqslant l$	$\frac{\delta_i \cdots \delta_{j-1}}{\omega_{k+1} \cdots \omega_i \omega_{j+1} \cdots \omega_l \mu}$	$l \leqslant i \leqslant j \leqslant k$	$\frac{1}{\tau_l \cdots \tau_{k-1} \mu}$
$k \leqslant i \leqslant l \leqslant j$	$\frac{\delta_i \cdots \delta_{l-1}}{\omega_{k+1} \cdots \omega_i \mu}$	$l \leqslant i \leqslant k \leqslant j$	$\frac{1}{\tau_l \cdots \tau_{k-1} \mu}$
$j \leqslant k < l \leqslant i$	$\frac{1}{\omega_{k+1} \cdots \omega_l \mu}$	$j \leqslant l < k \leqslant i$	$\frac{\gamma_{l+1} \cdots \gamma_k}{\mu}$
$k \leqslant j \leqslant l \leqslant i$	$\frac{1}{\omega_{k+1} \cdots \omega_l \mu}$	$l \leqslant j \leqslant k \leqslant i$	$\frac{\gamma_{j+1} \cdots \gamma_k}{\tau_l \cdots \tau_{j-1} \mu}$
$k < l \leqslant j \leqslant i$	$\frac{1}{\tau_l \cdots \tau_{j-1} \omega_{k+1} \cdots \omega_j \mu}$	$l < k \leqslant j \leqslant i$	$\frac{1}{\tau_l \cdots \tau_{j-1} \omega_{k+1} \cdots \omega_j \mu}$
$k < l \leqslant i \leqslant j$	$\frac{1}{\tau_l \cdots \tau_{i-1} \omega_{k+1} \cdots \omega_i \mu}$	$l < k \leqslant i \leqslant j$	$\frac{1}{\tau_l \cdots \tau_{i-1} \omega_{k+1} \cdots \omega_i \mu}$

$s = \frac{h}{1+hm_{l,k}^{-1}}$ for all (i,j) in order to ensure that the inverse is non-negative. There are many different combinations of i, j, k and l which we need to consider. In particular we must consider separately the cases when $l \geqslant k+2$ and when $k \geqslant l+2$ (in other words when the perturbation is above or below the diagonal). Then given a fixed k and l, we must consider the restriction on s for every possible (i,j) pair, as we did in the previous sections. The resulting restrictions are presented in Table 1. More detail regarding the calculations can be found in [14].

Since for a fixed l and k we require the restrictions to hold for all possible i and j, we compare to find the strictest (smallest) restriction. We know δ, τ and ω are always less than 1, so we find that the smallest restriction on s is

$$s \leqslant \frac{\delta_k \cdots \delta_{l-1}}{\mu} \quad \text{or} \quad s \leqslant \frac{\gamma_{l+1} \cdots \gamma_k}{\mu} \tag{7}$$

if the perturbation is above or below the diagonal, respectively.

We summarize these results in the following theorem.

Theorem 4. *Assume M is a strictly diagonally dominant tridiagonal M-matrix. Let u be a vector of zeroes with h in the kth position and v be a vector of zeroes with a 1 in the position lth, so that they form the rank-1 matrix uv^T. To ensure the matrix $P = M + uv^T$ has a non-negative inverse (element-wise) it is sufficient that h satisfies*

$$h \leqslant \frac{\delta^{l-k}}{\mu} \quad \text{if } l \geqslant k+2$$

or

$$h \leqslant \frac{\gamma^{k-1}}{\mu} \quad \text{if } l \leqslant k-2,$$

where δ, γ and μ are defined in (1) and (2).

We note that the bounds in Theorem 4 are consistent with the results in Sections 3.1 and 3.2. Also, we see that below the diagonal the allowable perturbation gets smaller as we increase the row index or decrease the column index. Similarly when above the diagonal the allowable perturbation gets smaller as we decrease the row index or increase the column index. In other words the further from the diagonal we wish to make our perturbation the smaller it will have to be. Finally, we note that we have found a restriction that does not depend on the placement of the perturbation along a diagonal, although the tightness of the bound will depend on the exact placement. In particular this leads to the following corollary.

Corollary 1. *Assume M is a tridiagonal M-matrix. Let u be a vector of zeroes with h in the kth position and v be a vector of zeroes with a 1 in position $k+2$, so that they form the rank-1 matrix uv^T. To ensure the matrix $P = M + uv^T$ has a non-negative inverse (element-wise) it is sufficient that h satisfies*

$$h \leqslant \frac{\delta^2}{\mu},$$

where δ and μ are defined in (1) and (2).

As indicated above, our bounds (see Theorem 4) suggest that the maximum allowable perturbation to maintain non-negativity of the inverse decreases as the "distance" from the diagonal increases. To demonstrate this we consider a strictly diagonally dominant M-matrix $M = \text{tridiag}\{-1, 4, -2\}$. For each pair of indices (k, l), $|k - l| \geqslant 2$, (8) depicts the actual maximum allowable perturbation (computed numerically) to the entries outside the tridiagonal band. The example supports our theoretical results:

$$\begin{pmatrix} 0.00000 & 0.00000 & 1.00000 & 0.57143 & 0.33333 & 0.19512 & 0.11429 & 0.06695 \\ 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.57143 & 0.33333 & 0.19512 & 0.11429 \\ 0.25000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.57143 & 0.33333 & 0.19512 \\ 0.07143 & 0.25000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.57143 & 0.33333 \\ 0.02083 & 0.07143 & 0.25000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.57143 \\ 0.00610 & 0.02083 & 0.07143 & 0.25000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 \\ 0.00179 & 0.00610 & 0.02083 & 0.07143 & 0.25000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00052 & 0.00179 & 0.00610 & 0.02083 & 0.07143 & 0.25000 & 0.00000 & 0.00000 \end{pmatrix} \tag{8}$$

4. Symmetric and Toeplitz cases

If the matrix M is symmetric ($m_{ij} = m_{ji}$) or Toeplitz ($a_i = a_j$, $b_i = b_j$ and $c_i = c_j$ for all i and j), then the bounds on h may be tightened.

If M is symmetric, then we have

$$\frac{m_{ij}^{-1}}{m_{i\pm1,j}^{-1}} = \frac{m_{ji}^{-1}}{m_{j,i\pm1}^{-1}}.$$

This allows us to compare elements along rows as well as columns and hence we gain much flexibility in our analysis.

As an example we consider the effect of symmetry when perturbing the $(1, 3)$ entry. In our analysis of this case we required

$$sm_{i,1}^{-1} m_{3,j}^{-1} \leqslant m_{ij}^{-1}$$

for all i and j, or equivalently

$$s \leqslant \frac{m_{ij}^{-1}}{m_{i,1}^{-1} m_{3,j}^{-1}} = \begin{cases} \frac{1}{m_{11}^{-1}} \frac{m_{11}^{-1}}{m_{2,1}^{-1}} \cdots \frac{m_{i-1,1}^{-1}}{m_{i,1}^{-1}} \frac{m_{ij}^{-1}}{m_{i+1,j}^{-1}} \cdots \frac{m_{2,j}^{-1}}{m_{3,j}^{-1}} & \text{if } i < 3, \\ \frac{1}{m_{11}^{-1}} \frac{m_{11}^{-1}}{m_{2,1}^{-1}} \cdots \frac{m_{i-1,1}^{-1}}{m_{i,1}^{-1}} \frac{m_{ij}^{-1}}{m_{i-1,j}^{-1}} \cdots \frac{m_{4,j}^{-1}}{m_{3,j}^{-1}} & \text{if } i \geqslant 3. \end{cases}$$

Using what we know about the ratio of two consecutive elements of a column, it is possible to show that the tightest restriction on s arises when $i = 1$ and $j \geqslant 3$. Therefore, we consider only this case.

When $i = 1$ and $j \geqslant 3$ we require

$$sm_{11}^{-1}m_{3j}^{-1} \leqslant m_{1,j}^{-1}.$$

Symmetry allows us to manipulate our inequalities in additional ways. For example, we could compare m_{11}^{-1} to $m_{1,j}^{-1}$ and m_{3j}^{-1} to m_{33}^{-1} along rows. With some cancellation we see

$$sm_{11}^{-1}m_{3j}^{-1} = s\prod_{x=2}^{3}\frac{m_{x-1,1}^{-1}}{m_{x,1}^{-1}}m_{1,j}^{-1}m_{3,3}^{-1} \leqslant s\frac{\mu}{\gamma_2\gamma_3}m_{1,j}^{-1} \leqslant m_{1,j}^{-1} \quad \text{if } s \leqslant \frac{\gamma_2\gamma_3}{\mu}.$$

Another option is to compare m_{11}^{-1} to $m_{1,j}^{-1}$ and m_{3j}^{-1} to m_{22}^{-1} using both column-wise and row-wise comparisons. This gives

$$sm_{11}^{-1}m_{3j}^{-1} = s\frac{m_{1,1}^{-1}}{m_{2,1}^{-1}}\frac{m_{3,j}^{-1}}{m_{2,j}^{-1}}m_{1,j}^{-1}m_{2,2}^{-1} \leqslant s\frac{\mu}{\gamma_2\delta_2}m_{1,j}^{-1} \leqslant m_{1,j}^{-1} \quad \text{if } s \leqslant \frac{\gamma_2\delta_2}{\mu}.$$

Yet another possibility is to compare m_{11}^{-1} to $m_{1,j}^{-1}$ and m_{3j}^{-1} to m_{jj}^{-1},

$$sm_{11}^{-1}m_{3j}^{-1} \leqslant s\prod_{x=2}^{j}\frac{1}{\gamma_x}\prod_{y=3}^{j-1}\tau_y\mu m_{1,j}^{-1} \leqslant m_{1,j}^{-1} \quad \text{if } s \leqslant \frac{\gamma_2\cdots\gamma_j}{\tau_3\cdots\tau_{j-1}\mu}.$$

There are many other possible manipulations, however, in other cases we found the resulting restrictions on s to be very similar to those obtained above.

Previously we found restrictions on s for all possible i and j which had to hold simultaneously, however, here we are dealing with a specific i and j so we simply need the largest restriction to hold. Which of these bounds is the largest depends on the specific matrix. For example, if

$$M = \begin{pmatrix} 10 & -1 & 0 & 0 & 0 \\ -1 & 50 & -8 & 0 & 0 \\ 0 & -8 & 100 & -1 & 0 \\ 0 & 0 & -1 & 20 & -8 \\ 0 & 0 & 0 & -8 & 10 \end{pmatrix},$$

we have

$$\frac{\delta_1\delta_2}{\mu} = \frac{1008}{9500} \approx 0.1061, \qquad \frac{\gamma_2\gamma_3}{\mu} = \frac{1008}{95000} \approx 0.01061,$$
$$\frac{\delta_2\gamma_2}{\mu} = \frac{1008}{47500} \approx 0.02122 \quad \text{and} \quad \frac{\gamma_2\cdots\gamma_5}{\tau_3\tau_4\mu} = \frac{14095872}{152000000} = 0.092736.$$

As we can see, our potential bounds vary greatly due to the path of comparison and the accuracy of the estimates δ_i, γ_i and τ_i. The largest of these is 0.1061, so in this particular case we can perturb the entry $(1,3)$ by as much as 0.1061 and still be sure that the inverse will be element-wise nonnegative. Numerical tests tell us the actual maximum perturbation allowable is approximately 0.16. Symmetry tells us this is also the largest we can perturb entry $(3,1)$. If we had tried to get a bound a perturbation of entry $(3,1)$ without taking symmetry into account, Theorem 4 would have told us that $h \leqslant 0.01061$ which is far from the actual bound of 0.16. In this case symmetry helped us tighten our bound considerably.

As a second example consider the symmetric matrix

$$M = \begin{pmatrix} 50 & -1 & 0 & 0 & 0 \\ -1 & 40 & -1 & 0 & 0 \\ 0 & -1 & 30 & -1 & 0 \\ 0 & 0 & -1 & 20 & -1 \\ 0 & 0 & 0 & -1 & 10 \end{pmatrix}.$$

Here we find

$$\frac{\delta_1\delta_2}{\mu} \approx 0.004973, \quad \frac{\delta_2\gamma_2}{\mu} \approx 0.006217,$$
$$\frac{\gamma_2\gamma_3}{\mu} \approx 0.008289, \quad \text{and} \quad \frac{\gamma_2\gamma_3\gamma_4\gamma_5}{\tau_3\tau_4\mu} \approx 0.02284.$$

Numerically we find that the largest h is 0.025. So in this case choosing to use $\frac{\gamma_2\gamma_3\gamma_4\gamma_5}{\tau_3\tau_4\mu}$ as a bound for h is best.

Now suppose the matrix M is both symmetric *and* Toeplitz. This means that all the diagonal elements have the same value, a, and all the elements on the sub and super diagonals have the same value, b. Our decay parameters will be

$$\tau_i = \omega_i = \frac{b}{a-b}, \quad \delta_i = \gamma_i = \frac{b}{a}$$

for all i, except $i = 1$ where $\tau_1 = \delta_1 = \frac{b}{a}$ and $\omega_1 = \gamma_1 = 0$. This means that three of our potential bounds, $\frac{\delta_1\delta_2}{\mu}$, $\frac{\delta_2\gamma_2}{\mu}$ and $\frac{\gamma_2\gamma_3}{\mu}$ will be equal. The fourth potential bound which we have considered in this section will be smaller than the others since it simply has extra factors of the form $\frac{\gamma_i}{\tau_{i-1}} \leqslant 1$. We can therefore say that when M is a symmetric Toeplitz matrix, then we can perturb the entry $(1, 3)$ by as much as $\frac{\tau^2}{\mu}$ where $\tau = \frac{b}{a}$, and still be sure that the inverse of the perturbed matrix will be element-wise non-negative.

Indeed, symmetry and the Toeplitz structure can be used to obtain following two corollaries.

Corollary 2. *Let M be a strictly diagonally dominant symmetric tridiagonal M-matrix. A general element (k, l), where $k \geqslant l + 2$ or $k \leqslant l - 2$, can be perturbed by as much as*

$$\frac{\gamma^{|k-l|}}{\mu} \quad or \quad \frac{\delta^{|k-l|}}{\mu}$$

and the non-negative inverse will persist. Here γ, δ and μ are defined in (1) and (2).

Corollary 3. *Let M be a strictly diagonally dominant symmetric tridiagonal Toeplitz M-matrix. A general element (k, l), where $k \geqslant l + 2$ or $k \leqslant l - 2$, can be perturbed by as much as*

$$\frac{\tau^{|k-l|}}{\mu}$$

and the non-negative inverse will persist. Here $\tau = \frac{b}{a}$ and μ is defined in (2).

5. Perturbations inside the diagonal band

We have now found bounds on the size of a perturbation made to a general element outside of the tridiagonal band, however, we have not addressed the effects of perturbing an element within the band. Recall that we are only looking at perturbations to elements in the band that are of a magnitude large enough to break the sign pattern.

When perturbing an element within the band of the matrix M, the resulting matrix, $P = \text{tridiag}\{-b'_i, a'_i, -c'_i\}$, will be a tridiagonal matrix, so from [10] we know there exist vectors u', v', x' and y' such that

$$p^{-1}_{ij} = \begin{cases} u'_i v'_j, & i \leqslant j, \\ x'_j y'_i, & i \geqslant j \end{cases}$$

and

$$u'_1 = 1, \quad u'_2 = \frac{a'_1}{c'_1}, \quad u'_i = \frac{a'_{i-1}u'_{i-1} - b'_{i-2}u'_{i-2}}{c'_{i-1}} \qquad \text{for } i = 3, \ldots, n,$$

$$x'_1 = 1, \quad x'_2 = \frac{a'_1}{b'_1}, \quad x'_i = \frac{a'_{i-1}x'_{i-1} - c'_{i-2}x'_{i-2}}{b'_{i-1}} \qquad \text{for } i = 3, \ldots, n,$$

$$y'_n = \frac{1}{a'_n x'_n - c'_{n-1}x'_{n-1}}, \quad y'_{n-1} = \frac{a'_n}{b'_{n-1}}y'_n,$$

$$y'_i = \frac{a'_{i+1}y'_{i+1} - c'_{i+1}y'_{i+2}}{b'_i} \quad \text{for } i = n-2, \ldots, 1,$$

$$v'_n = \frac{1}{a'_n u'_n - b'_{n-1}u'_{n-1}}, \quad v'_{n-1} = \frac{a'_n}{c'_{n-1}}v'_n,$$

$$v'_i = \frac{a'_{i+1}v'_{i+1} - b'_{i+1}v'_{i+2}}{c'_i} \quad \text{for } i = n-2, \ldots, 1.$$

The elements of M^{-1} are also defined in an analogous manner, using vectors u, v, x, and y as in (3).

Now, in order for each p_{ij}^{-1} to be non-negative, we must have $v_i \geqslant 0$ and $y_i \geqslant 0$ for all i since $u_1 = x_1 = 1$ and we want $u_1 v_i \geqslant 0$ and $x_1 y_i \geqslant 0$. This then means that $u_i \geqslant 0$ and $x_i \geqslant 0$ for all i.

If we wish to perturb element $(1, 2)$ and change its sign then we add a positive quantity, h, so that $-c'_1 = -c_1 + h > 0$ and $c'_1 < 0$. However, we still have $a_1 = a'_1 \geqslant 0$ so $u'_2 = \frac{a'_1}{c'_1} < 0$ and hence the inverse cannot be element-wise non-negative. Similarly we cannot perturb element $(2, 1)$ and maintain inverse non-negativity.

If the element being perturbed is $(k, k-1), k \geqslant 2$, then $-b'_{k-1} = -b_{k-1} + h < 0$ and so $b'_{k-1} < 0$. Also $a'_{k-1}x'_{k-1} - c'_{k-2}x'_{k-2} = a_{k-1}x_{k-1} - c_{k-2}x_{k-2} \geqslant 0$ since no other elements were perturbed and $m_{ij}^{-1} \geqslant 0$. This implies $x'_i < 0$ so P^{-1} cannot be element-wise non-negative. Similarly if element $(k, k+1)$, $k \geqslant 2$ is being perturbed the inverse cannot be non-negative.

When perturbing an element on the diagonal, (k, k), in order to change the sign pattern we need to make it negative so we subtract a positive quantity, h. This would imply $a'_k = a_k - h < 0$, so if $2 \leqslant k \leqslant n-1$, then $a'_k x'_k - c'_{k-1}x'_{k-1} < 0$ since x'_i and c'_i are positive for all $i \leqslant k$. This would require $x'_{k+1} < 0$. Similarly we find if $k = 1$, x'_2 is negative, and if $k = n$, y'_n is negative. Therefore, we cannot perturb the diagonal and still have a non-negative inverse.

This leads to the following theorem.

Theorem 5. *If a single element within the band of a tridiagonal M-matrix is perturbed by a large enough amount to change the element's sign, then the inverse of the perturbed matrix cannot be element-wise non-negative.*

6. Conclusions

In this paper, we have perturbed a general element of a tridiagonal M-matrix and have found sufficient upper bounds on the size of this perturbation which ensure that the inverse of the perturbed matrix is non-negative (element-wise). We have discovered when outside the band, the further from the diagonal we wish to make the perturbation the smaller the perturbation has to be to preserve inverse non-negativity. These results were consistent with previous findings, cf. [5]. When perturbing an element inside the diagonal band we found that it is not possible to maintain the non-negative inverse property without also maintaining the M-matrix sign pattern.

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