Domain Decomposition Approaches for PDE Based Mesh Generation

Ronald D. Haynes

1 Introduction

Automatically adaptive and possibly dynamic meshes are often introduced to solve partial differential equations (PDEs) whose solutions evolve on disparate space and time scales. In this paper we will review a class of PDE based mesh generators in 1D and 2D—a PDE is formulated and its solution provides the mesh used to approximate the solution of the physical PDE of interest. The physical PDE and mesh PDE are coupled and are solved in a simultaneous or decoupled manner. The hope is that the cost of computing the mesh, by solving the mesh PDE, should not substantially increase the total computational burden and ideally the mesh solution strategy should fit within the overall solution framework.

Meshes which automatically react to the solution of the physical PDE fall into (at least) two broad categories: *hp*-refinement and *r*-refinement—PDE based mesh generation which evolves a fixed number of mesh points with a fixed topology. The choice of mesh generator is often predicated on the class of problem and experience of the practitioner. The PDE based mesh generators, motivated by *r*-refinement, discussed here, can be designed to capture dynamical physics, Lagrangian behaviour, symmetries, conservation laws or self-similarity features of the physical solution, and achieve global mesh regularity.

In this overview paper, we review parallel solution strategies for the mesh PDE and the coupled system using domain decomposition (DD) and survey various known theoretical results. The analysis of the optimized Schwarz method (OSM) uses several classical tools including Peaceman-Rachford iterations and monotone convergence using the theory of M-functions. We present previews of

R. D. Haynes (✉)
Memorial University, St. John’s, NF, Canada
e-mail: rhaynes@mun.ca

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two extensions of our previous work. We provide an analysis of OSM on two subdomains using the theory of M-functions. We also introduce a coarse correction for the mesh PDE to improve convergence of DD as the number of subdomains increases.

In this paper we provide a brief review of PDE based mesh generation (Sect. 2), an overview of, and theoretical convergence results for, Schwarz methods to solve the mesh PDE (Sect. 3), a new strategy for the analysis of OSM and a new coarse correction algorithm to solve the nonlinear mesh PDE (Sect. 4).

2 PDE Based Mesh Generation

We consider PDEs whose numerical solution can benefit from automatically chosen non-uniform meshes. r-refinement adapts an initial grid by relocating a fixed number of mesh nodes. The mesh is determined by solving a mesh PDE simultaneously, or in an iterative fashion, with the physical PDE. Suppose the PDE defined on the physical domain $x \in \Omega_p = [0, 1]$ is difficult to solve in the physical co-ordinate x. We compute a mesh transformation, $x = x(\xi, t)$, so that solving the problem on a uniform mesh $\xi_i = \frac{i}{N}, \quad i = 0, 1, \ldots, N$, with moderate N, is sufficient. In one dimension, such a mesh transformation can be constructed by the equidistribution principle of de Boor [7]. Given some measure of the error in the physical solution, M (called the mesh density function), we require

$$\int_{x_{i-1}(t)}^{x_i(t)} M(t, \tilde{x}, u)\, d\tilde{x} = \frac{1}{N} \int_0^1 M(t, \tilde{x}, u)\, d\tilde{x},$$

which says that the error in the solution is equally distributed across all intervals.

If we assume some approximation to the physical solution u is given, then in the steady case a continuous form of the mesh transformation can be found by solving the nonlinear boundary value problem (BVP)

$$\frac{\partial}{\partial \xi}\left\{ M(x(\xi)) \frac{\partial}{\partial \xi} x(\xi) \right\} = 0, \quad \text{subject to} \quad x(0) = 0 \text{ and } x(1) = 1. \qquad (1)$$

The boundary conditions ensure mesh points at the boundaries of the physical domain. This is equivalent to minimizing the functional $I[x] = \frac{1}{2} \int_0^1 \left(M(x) \frac{dx}{d\xi} \right)^2 d\xi$. Discretizing and solving gives the physical mesh locations directly, however the Euler-Lagrange (EL) equations are nonlinear, and a system of nonlinear algebraic equations must be solved upon discretization.

As an example, consider constructing an equidistributing grid for the function $u(x) = (e^{\lambda x} - 1)/(e^{\lambda} - 1)$ for large λ. A uniform grid in the physical co-ordinate x would require a large number of mesh points to resolve the boundary layer near $x = 1$. Instead we solve the nonlinear BVP above on a uniform grid $\xi_i = \frac{i}{N}$

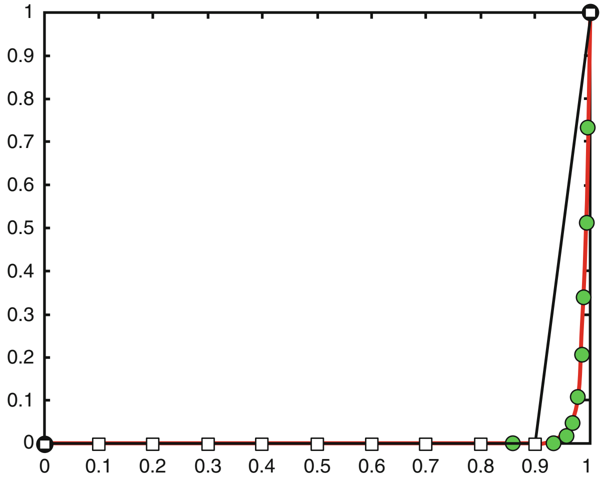

Fig. 1 An example of an equidistributing grid for a boundary layer function

with $M(x, u) \sim \sqrt{1 + |u_{xx}|^2}$ and we obtain the grid locations corresponding to the abscissa of the green circles in Fig. 1. The solution on a uniform grid (white squares) is shown for comparison.

Alternatively, we can solve for the the inverse transformation, $\xi(x)$, as the solution of

$$\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d\xi}{dx}\right) = 0, \qquad \xi(0) = 0, \quad \xi(1) = 1,$$

or as the minimizer of the functional $I[\xi] = \frac{1}{2}\int_0^1 \frac{1}{M(x)}\left(\frac{d\xi}{dx}\right)^2 dx$.

The EL equations are now linear, and discretizing on a uniform grid in x gives a linear system for the now non-uniform points in the computational co-ordinate ξ. We have to invert the transformation to find the required physical mesh locations. It is easier to ensure well-posedness in higher dimensions ($d \geq 2$) for this formulation.

In two dimensions, solution independent, but boundary fitted meshes, can be found by generalizing the formulations above, but setting the mesh density to be the identity function. The mesh transformation $\boldsymbol{x} = [x(\xi, \eta), y(\xi, \eta)] : \Omega_c \to \Omega_p$ can be found by minimizing

$$I[x, y] = \frac{1}{2}\int_{\Omega_c}\left[\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right] d\xi d\eta.$$

The EL equations are

$$\frac{\partial^2 x}{\partial \xi^2} + \frac{\partial^2 x}{\partial \eta^2} = 0, \qquad \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \eta^2} = 0.$$

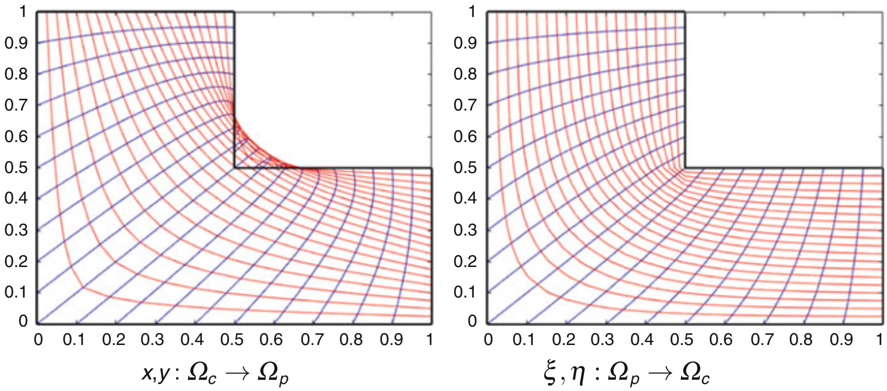

Fig. 2 PDE generated physical grid lines on L-shaped domains

Solving the EL equations subject to boundary conditions, which ensure mesh points on the boundary of Ω_p, gives a boundary fitted co-ordinate system. Care is required, however, as folded meshes may result if Ω_p is concave (see the left of Fig. 2 where Ω_p is L-shaped and $\Omega_c = [0, 1]^2$).

If instead we solve for the inverse mesh transformation $\boldsymbol{\xi} = [\xi(x, y), \eta(x, y)] : \Omega_p \to \Omega_c$ by minimizing

$$I[\xi, \eta] = \frac{1}{2} \int_{\Omega_p} \left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 \right] dxdy,$$

or solving the EL equations

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0, \qquad \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0,$$

subject to appropriate boundary conditions, we obtain the mesh on the right of Fig. 2. This is the *equipotential* mesh generation method of Crowley [6]. The physical grid lines are obtained as level curves $\xi = C$, $\eta = K$. This approach is more robust—well-posed if the domain Ω_c (which we get to choose) is convex, see [8]. But as mentioned previously it is more complicated to get the physical mesh.

Solution dependent meshes in higher dimensions can be constructed by specifying a scalar mesh density function $M = M(u, \boldsymbol{x}) > 0$, characterizing where additional mesh resolution is needed, and minimizing $I[x] = \frac{1}{2} \int_{\Omega_p} \frac{1}{M} \sum_i (\nabla \xi_i)^T \nabla \xi_i dx$. The EL equations give the variable diffusion mesh generator of Winslow [26], which requires the solution of the elliptic PDEs $-\nabla \cdot \left(\frac{1}{M} \nabla \xi_i\right) = 0, \quad i = 1, 2, \ldots, d$. This gives an *isotropic* mesh generator. Godunov and Prokopov [10], Thompson et al. [25] and Anderson [2], for example, add terms to the mesh PDEs to better control the mesh distribution and quality.

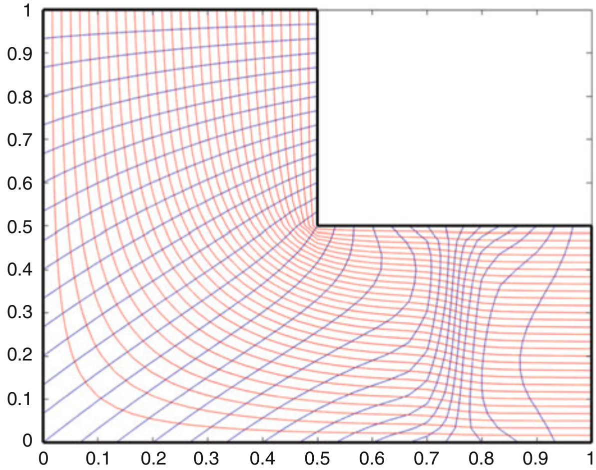

Fig. 3 A mesh generated using a Winslow generator on an L-shaped domain

As an example, in Fig. 3, we illustrate the mesh obtained by adapting a mesh for a solution with a rapid transition at $x = 3/4$ and using an arc-length based M.

If the physical solution has strong *anisotropic* behaviour, corresponding mesh adaptation is desired. This can be achieved by using a matrix-valued diffusion coefficient [5] and minimizing $I[\boldsymbol{x}] = \frac{1}{2}\int_{\Omega_p}\sum_i(\nabla\xi_i)^T\boldsymbol{M}^{-1}(\nabla\xi_i)d\boldsymbol{x}$ where $\boldsymbol{M}$ is a spd matrix.

These approaches can be extended to the time dependent situation, where $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, t)$ or $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}, t)$; we obtain moving mesh PDEs as the modified gradient flow equations for the adaptation functionals.

In addition to the variational approach to derive the mesh PDEs mentioned above, there are other PDE based approaches including harmonic maps, Monge–Ampère, and geometric conservation laws, see [15] for a recent extensive overview.

3 Domain Decomposition Approaches and Analysis for Nonlinear Mesh Generation

We wish to design and analyze parallel approaches to solve the continuous (and discrete forms) of the PDE mesh generators discussed above. Our research goal is to systematically analyze DD based implementations to solve mesh PDEs and coupled mesh-physical PDE systems.

3.1 Mesh/Physical PDE Solution Strategies

There are several approaches to introduce parallelism, by domain decomposition, while solving PDEs which require or benefit from a PDE based mesh generator. As an example we consider generating a time dependent mesh for a moving interior layer problem. In [14] we apply DD in the physical co-ordinates by partitioning Ω_p, and use an adaptive, moving mesh solver in each physical domain. This is illustrated in the left of Fig. 4 for two overlapping subdomains; the solver tracks a front which develops and moves to the right. In each physical subdomain, the mesh points react and follow the incoming front. In general, this approach needs hr-refinement to predict the number of mesh points in each subdomain and could result in a severe load balancing issue. Alternatively, one could fix the total number of mesh points and apply DD in the fixed, typically uniform, computational co-ordinates, by partitioning Ω_c. This gives rise to time dependent or moving subdomains, as viewed in the physical co-ordinate system, as shown in the right of Fig. 4 for a similar moving front. The subdomains are shaded dark and light gray, with the overlap in between.

In Fig. 5 we illustrate a two dimensional mesh computed using a classical Schwarz iteration applied in Ω_c, on two overlapping subdomains (the overlap is shown in green). DD is applied to the two dimensional nonlinear mesh generator of [16]. Here the mesh is adapted to the physical solution given by

$$u = \tanh(R(\frac{1}{16} - (x - \frac{1}{2})^2 - (y - \frac{1}{2})^2))$$

and

$$M = \frac{a^2 \nabla u \cdot \nabla u^T}{1 + b \nabla u^T \nabla u} + I,$$

where $a = 0.2$ and $b = 0$.

3.2 PDE Based Mesh Generation Using Schwarz Methods

Here we will focus on the analysis of DD methods for the mesh PDE applied in the computational co-ordinates, assuming an approximation to the solution of the physical PDE is given. To generate the physical mesh locations directly, we are interested in the solution of the nonlinear BVP (1).

A general parallel Schwarz approach would partition $\xi \in \Omega_c$ into two subdomains $\Omega_1 = (0, \beta)$ and $\Omega_2 = (\alpha, 1)$ with $\alpha \leq \beta$. Let x_1^n and x_2^n solve

$$\begin{aligned}
&\frac{d}{d\xi}\left(M(x_1^n)\frac{dx_1^n}{d\xi}\right) = 0 \quad \text{on } \Omega_1 &\qquad &\frac{d}{d\xi}\left(M(x_2^n)\frac{dx_2^n}{d\xi}\right) = 0 \quad \text{on } \Omega_2\\
&x_1^n(0) = 0 & &B_2(x_2^n(\alpha)) = B_2(x_1^{n-1}(\alpha))\\
&B_1(x_1^n(\beta)) = B_1(x_2^{n-1}(\beta)) & &x_2^n(1) = 1,
\end{aligned}$$

where $B_{1,2}$ are transmission operators between the subdomains.

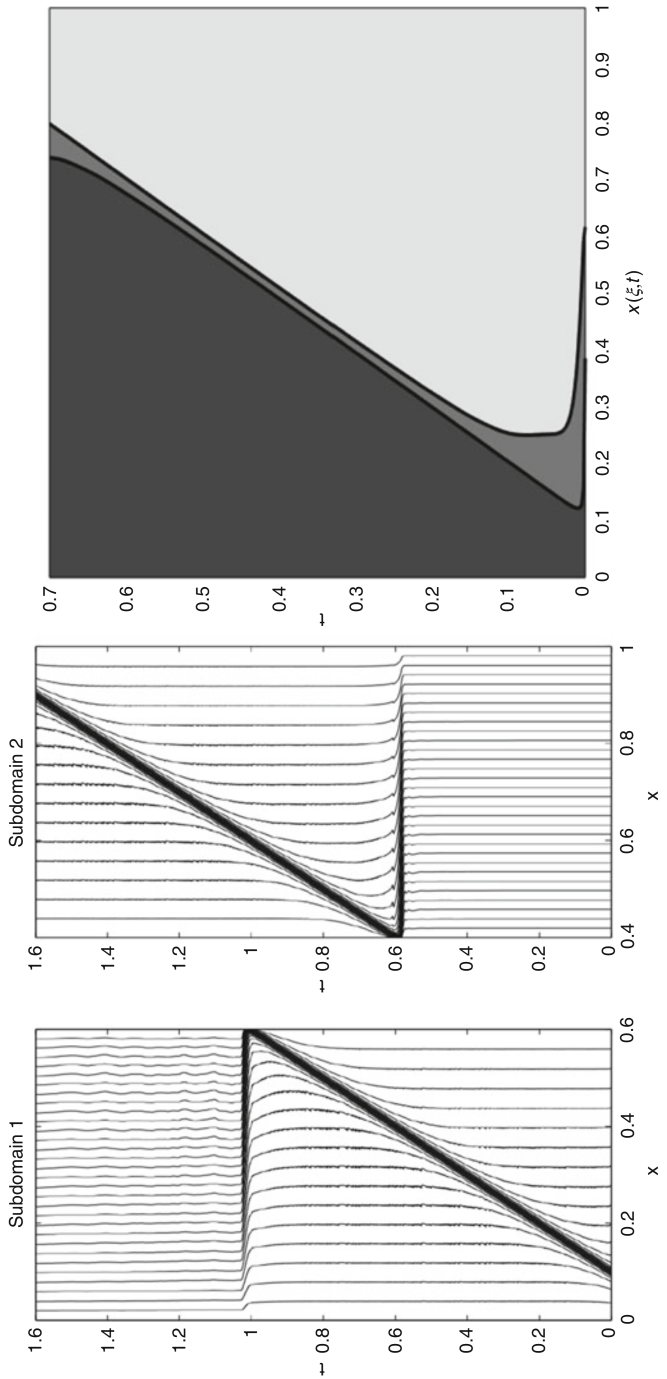

Fig. 4 DD in Ω_p (left) and DD in Ω_c (right)

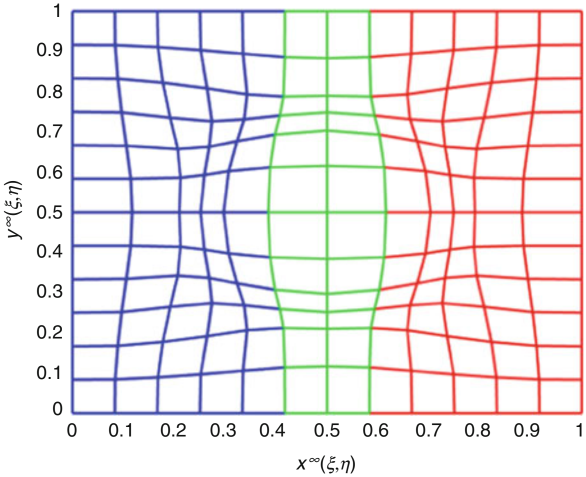

Fig. 5 DD solution of two-dimensional mesh generator

If $0 < \check{m} \leq M(x) \leq \hat{m} < \infty$, we show in [9] the overlapping ($\beta > \alpha$) parallel classical Schwarz iteration ($B_{1,2} = I$) converges for any initial guess $x_1^0(\alpha)$, $x_2^0(\beta)$, with a contraction factor $\rho := \frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha} < 1$ which improves with the size of the overlap. As expected $\alpha < \beta$ is needed for convergence. A multidomain result is also given in [9] with a contraction rate that deteriorates as the number of subdomains increases. This result motivates the need for a coarse correction (see Sect. 4). Optimal Schwarz methods using non-local transmission conditions (TCs) giving finite convergence have been proposed and analyzed in [9, 12]. This comes at a cost as nonlocal TCs are expensive!

We can recover a local algorithm, an OSM, on two subdomains by approximating the non-local TCs. We decompose $\xi \in [0, 1]$ into two non-overlapping subdomains $\Omega_1 = [0, \alpha]$ and $\Omega_2 = [\alpha, 1]$ and approximate the optimal TCs with nonlinear Robin TCs. Using the notation above, we choose $B_1(\cdot) = M(\cdot)\partial_\xi(\cdot) + p(\cdot)$ and $B_2(\cdot) = M(\cdot)\partial_\xi(\cdot) - p(\cdot)$, where p is a constant chosen to improve the convergence rate. The OSM is equivalent to a nonlinear Peaceman–Rachford interface iteration for the interface values

$$\begin{aligned}(pI - R_2)x_2^{n+1}(\alpha) &= (pI - R_1)x_1^n(\alpha),\\ (pI + R_1)x_1^{n+1}(\alpha) &= (pI + R_2)x_2^n(\alpha),\end{aligned} \tag{2}$$

where the operators R_1 and R_2, given by $R_1(x) = \frac{1}{\beta}\int_0^x M\,d\tilde{x}$ and $R_2(x) = \frac{1}{1-\beta}\int_x^1 M\,d\tilde{x}$, are strictly monotonic (increasing and decreasing respectively). This type of iteration has been analyzed by Kellogg and Caspar [17] and Ortega and Rheinboldt [18]. In [9] we show convergence for all $p > 0$ and the contraction rate can be minimized by an appropriate choice of p.

An analysis of the classical Schwarz algorithm at the discrete level has been provided in [13] in the steady and time dependent cases using a θ method to discretize in time. Using the notion of M-functions, which we will revisit in the next section, we have shown convergence of nonlinear Jacobi and Gauss–Seidel (and block versions) starting from super and sub solutions or from a uniform initial guess.

A dramatically different parallel technique for PDE mesh generation has been considered by Haynes and Bihlo in [3]. Motivated by the possible lower accuracy requirements for mesh generation we have investigated stochastic domain decomposition (SDD) methods, proposed by Acebrón et al. [1], Spigler [24], and Peirano and Talay [19]. These methods use the Feynmac-Kac formula (and Monte-Carlo) to approximate the linear mesh generator in 2D/3D along artificial interfaces. These interface solutions then provide boundary conditions for the deterministic solves in the subdomains. No iteration is required, and the method is fully parallel. The method may be expensive in the relatively rare situation that the mesh is needed with high-accuracy due to the slow convergence of the Monte Carlo evaluations.

4 Some Extensions

In this section, we provide previews of two extensions of the work described above.

4.1 Optimized Schwarz on Many Subdomains

Her we show an alternate approach to obtain a sufficient condition for convergence of the OSM for the grid generation problem. This approach, which guarantees a monotonic convergence result, is generalizable to an arbitrary number of subdomains. Here we will give a flavour of the analysis on two subdomains. The general result was studied by Sarker [22] and will be published elsewhere.

To demonstrate the difficulty of generalizing the OSM analysis to an arbitrary number of subdomains, consider partitioning Ω_c into three non-overlapping subdomains, $[0, \alpha_1]$, $[\alpha_1, \alpha_2]$ and $[\alpha_2, 1]$. The analysis of the parallel OSM to generate equidistributing grids requires us to study the interface iteration

$$\begin{aligned} py_1^n + R_1(x_1^n, y_1^n) &= px_2^{n-1} + R_2(x_2^{n-1}, y_2^{n-1}), \\ px_2^n - R_2(x_2^n, y_2^n) &= py_1^{n-1} - R_1(x_1^{n-1}, y_1^{n-1}), \\ py_2^n + R_2(x_2^n, y_2^n) &= px_3^{n-1} + R_3(x_3^{n-1}, y_3^{n-1}), \\ px_3^n - R_3(x_3^n, y_3^n) &= py_2^{n-1} - R_2(x_2^{n-1}, y_2^{n-1}), \end{aligned} \tag{3}$$

where $x_1^n = 0$ and $y_3^n = 1$, $R_i(x_i, y_i) = \frac{1}{\alpha_i - \alpha_{i-1}} \int_{x_i}^{y_i} M(\sigma)\, d\sigma$, and we define $\alpha_0 \equiv 0$ and $\alpha_3 \equiv 1$.

The Peaceman–Rachford analysis relies on the monotonicity of the operators which define the subdomain solutions. The difficulty in the analysis of (3) lies in the coupled system of equations which arise from the middle subdomain. This coupled system involves the operator $pI + H$. The operator $H = (-R_2, R_2)^T$ is not monotonic and hence the two subdomain analysis can not be repeated, at least not in a straightforward way.

We pursue an alternate tack to obtain a sufficient condition for convergence. It is well known that for linear systems, $Ax = b$, Gauss–Seidel and Jacobi will converge for any initial vector if A is symmetric positive definite, or if A is an M-matrix (for example if $a_{ij} \leq 0, i \neq j$, $a_{ii} > 0$ and A is strictly diagonally dominant). Analogous results for nonlinear systems, $Fx = b$, where

$$Fx \equiv (f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))^T \quad \text{and}$$
$$b = (b_1, b_2, \dots, b_n)^T,$$

were obtained by Schechter [23] who showed if F has a continuous, symmetric, and uniformly positive definite (Frechet) derivative then nonlinear Gauss–Seidel converges. The analogous M-matrix condition for convergence was extended to the nonlinear case by Rheinboldt [21], with the introduction of M-functions. To be an M-function requires F to have certain monotonicity, sign and diagonal dominance properties. Rheinboldt gives the following sufficient condition to guarantee a nonlinear map F is an M-function.

Theorem 1 *Let $\mathbb{D}$ be a convex and open subset of $\mathbb{R}^n$. Assume $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is off-diagonally non-increasing, and that for any $x \in \mathbb{D}$, the functions $q_i : S_i \subset \mathbb{R} \to \mathbb{R}^n$ defined as*

$$q_i(\tau) = \sum_{j=1}^{n} f_j(x + \tau e^i), \; i = 1, \dots, n, \quad \text{with} \quad S_i = \{\tau : x + \tau e^i \in \mathbb{D}\},$$

are strictly increasing. Then F is an M-function.

If F is an M-function and if $Fx = b$ has a solution then it is unique. Moreoever, Ortega and Rheinboldt [18] show that if F is a continuous, surjective M-function then for any initial vector the nonlinear Jacobi and Gauss–Seidel processes will converge to the unique solution. Results for the convergence of block versions of these iterations exist [20]. This result generalizes the classical result of Varga for M-matrices. We note that the parallel OSM (3) is a nonlinear block Jacobi iteration.

As an application of this theory we reconsider the two subdomain iteration (2). The technique generalizes to an arbitrary number of subdomains. The iteration (2) is well-posed. Existence and uniqueness for a given right hand side is trivial since the functions are uniformly monotone and tend to $\pm\infty$ as $x_{1,2} \to \pm\infty$. The two subdomain interface solution would solve the system $F = (f_1, f_2)^T = 0$ where

$f_1(x, y) = R_1(x) - R_2(y) + px - py = 0$ and $f_2(x, y) = -R_2(y) + R_1(x) + py - px = 0$. In [22] Sarker obtains the following result.

Theorem 2 *The function $F = (f_1, f_2)^T$ above is a surjective M-function if $p >$* $\max\{1/\alpha, 1/(1-\alpha)\}\hat{m}$. *Hence, the iteration (2) will converge to the unique solution of $F = 0$ for any initial vector. The convergence will be monotone if we start from a super or sub solution.*

Proof Clearly the function F is continuous. By direct calculation and the bounds on M we have $\frac{\partial f_1}{\partial x} = \frac{1}{\alpha}M(x) + p > 0$ and $\frac{\partial f_2}{\partial y} = \frac{1}{1-\alpha}M(y) + p > 0$, for all $p > 0$. Hence f_1 and f_2 are strictly increasing. Therefore, F is strictly diagonally increasing. Furthermore, $\frac{\partial f_1}{\partial y} = \frac{1}{1-\alpha}M(y) - p$ and $\frac{\partial f_2}{\partial x} = \frac{1}{\alpha}M(x) - p$. Hence, if $p > \{\frac{\hat{m}}{\alpha}, \frac{\hat{m}}{1-\alpha}\}$ then F is off-diagonally decreasing. A super (sub) solution, a vector $(\hat{x}, \hat{y})$ satisfying $F(\hat{x}, \hat{y}) \geq 0\,(\leq 0)$, can easily be constructed [22]. Monotone convergence from $(\hat{x}, \hat{y})$ follows from Theorem 13.5.2 of [18].

To show that F is an M-function, we now consider the functions $q_i(t) = \sum_{j=1}^{2} f_j(X + te^i)$ where $e^i \in R^2$ is the i-th standard basis vector, for $i = 1, 2$. The functions $q_1(t)$ and $q_2(t)$ are given by $q_1(t) = f_1(x + t, y) + f_2(x + t, y) = 2R_1(x+t) - 2R_2(y)$ and $q_2(t) = f_1(x, y+t) + f_2(x, y+t) = 2R_1(x) - 2R_2(y+t)$. Hence $\frac{dq_1}{dt} = \frac{2}{\alpha}M(t) > 0$ and $\frac{dq_2}{dt} = \frac{2}{1-\alpha}M(t) > 0$ and we conclude that q_i is strictly increasing, for $i = 1, 2$. Hence F is an M-function from Theorem 1. Surjectivity requires a super and sub solution for $Fx = b$ for a general b, see [22]. The convergence from any initial vector then follows from Theorem 13.5.9 of [18].

In Fig. 6 we see monotonic convergence (consistent with the M-function theory) if p is large enough and non-monotonic convergence for small p (consistent with the Peaceman–Rachford theory).

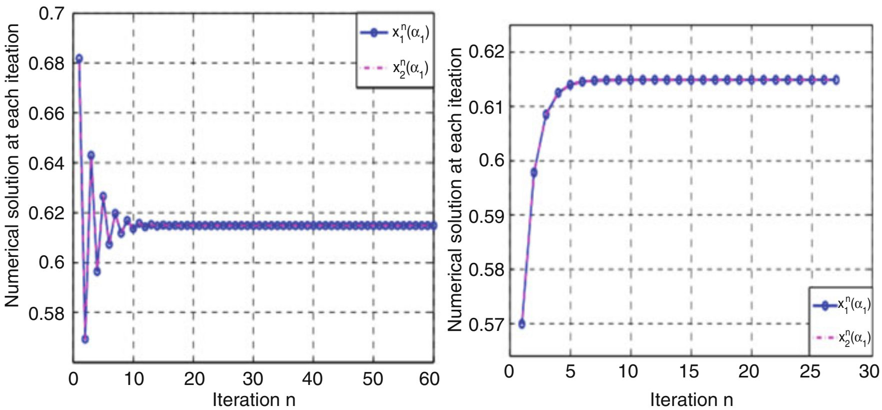

Fig. 6 Convergence history of the interface iteration for small and large p values

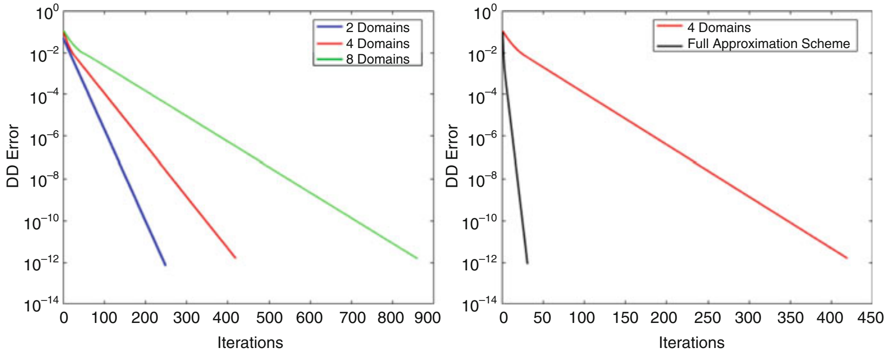

Fig. 7 Schwarz convergence results on multidomains and multidomains with a coarse correction

4.2 A Coarse Correction

The convergence rate of Schwarz methods suffers as the number of subdomains increases, see the left plot in Fig. 7. A coarse correction is able to improve the situation dramatically by providing a global transfer of solution information. Here we propose a coarse correction for the (nonlinear) PDE based mesh generation problem by using a two-grid method with a full approximation scheme (FAS) correction applied in the computational co-ordinates. This work was completed by Grant in [11] and will be published in full elsewhere.

FAS [4] provides a solution strategy for nonlinear PDEs. FAS restricts an approximation (and corresponding residual) of the PDE, obtained on a fine grid, to a coarse grid. The error in the approximation is found by solving a coarse problem. This error is then interpolated back to the fine grid and used to update the solution approximation.

FAS may be combined with a DD approach in a very natural way. We perform one classical Schwarz iteration to obtain approximate subdomain solutions on a fine grid. FAS is then applied to update the subdomain solutions before proceeding with the next Schwarz iteration. As shown in the right plot of Fig. 7, the effect is dramatic. This promising result for the nonlinear PDE mesh generator suggests the possibility of a two-grid FAS DD approach for the coupled mesh and physical PDEs.

5 Conclusions

PDE based mesh generators can be useful for problems which would benefit from automatically adaptive spatial grids. It is possible to analyze DD approaches for nonlinear mesh generators which directly give the physical mesh locations. We can then incorporate DD, within the coupled physical PDE/mesh PDE solution frameworks in a theoretically sound way.

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