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# Preface

In the last two decades adaptive moving mesh methods have received considerable attention from researchers and practitioners. It has been amply demonstrated that these methods, either used as stand-alone methods or combined with other adaptive mesh methods, are capable of producing meshes with desired adaptivity and good quality (particularly smoothness and alignment). While significant progress has been made, there has been a continuing effort to improve existing methods and develop more efficient adaptive moving mesh methods. To reflect research frontiers with adaptive moving mesh methods, this special issue presents five original research papers authored by long time researchers in the area. For a survey or literature review of the area, the interested reader is referred to the recent review articles by M. J. Baines, M. E. Hubbard, and P. K. Jimack (Velocity-based moving mesh methods for nonlinear partial differential equations. *Commun. Comput. Phys.*, 10:509–576, 2011) and C. J. Budd, W. Huang, and R. D. Russell (Adaptivity with moving grids. *Acta Numerica*, 18:111–241, 2009) and the monograph by W. Huang and R. D. Russell (Adaptive Moving Mesh Methods. Springer, New York, 2011).

Adaptive moving mesh methods are a special type of adaptive mesh method. Mesh adaptation has become an indispensable tool for use to improve computational efficiency in the numerical solution of partial differential equations and other mesh-related applications. Many problems arising in science and engineering contain local structures such as shock waves and sharp layers and interfaces and small mesh spacings are required to resolve them numerically. The use of uniform meshes is inefficient since a fine mesh has to be used and the number of degrees of freedom can become large. The basic idea of mesh adaptation is to distribute mesh points in a more efficient way so that more points are put in regions containing the local structures and less points are placed in other portions of the spatial domain. Typically a higher accuracy (as compared to a uniform mesh with the same number of points) can be attained with an adaptive mesh. Furthermore, an adaptive mesh, with a fewer number of points than with a uniform mesh, can be used to achieve a specified level of accuracy.

Loosely speaking, there are three types of adaptive mesh methods, $h$-, $p$-, and $r$-adaptive methods. $h$-adaptive methods achieve adaptivity by adding and deleting mesh points and swapping mesh edges/faces while $p$-adaptive methods do so by adjusting the order of solution approximation over mesh elements. On the other hand, $r$-adaptive methods, also called adaptive moving mesh methods or more simply, moving mesh methods, achieve desired adaptivity by relocating or moving mesh points. The mesh connectivity is kept fixed during mesh movement; nevertheless, the mesh points can be reconnected between time steps or iterations. Adaptive moving mesh methods can be used alone or combined with $h$- and $p$-adaptive methods. It should be pointed out that Lagrangian methods and arbitrary Lagrangian-Eulerian (ALE) methods in computational fluid dynamics are special types of adaptive moving mesh methods. Moreover,

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mesh smoothing methods such as Laplacian smoothing and optimization-based smoothing methods employed in mesh generation and mesh refinement can be viewed as a type of moving mesh method although their goal is to improve mesh quality.

The papers of this special issue are listed in the alphabetical order of the first authors. The first paper is *Explicit time-stepping for moving meshes* [pp. 93-105] by M. J. Baines. Velocity based or Lagrangian adaptive strategies obtain the mesh locations by integrating a velocity field. Lagrangian methods are able to maintain sharp interfaces in the solution but may suffer from severe mesh skewness or mesh tangling. Here the author presents a simple explicit time stepping scheme in 1D which ensures that the mesh is order–preserving. The scheme is shown to have the same accuracy as explicit Euler and the accuracy may be increased by a higher order quadrature. The method, in conjunction with the Lagrangian conservation method, is applied to the inviscid Burgers' equation and the porous medium equation. An extension to multiple dimensions is provided along with a relaxation strategy for implementation.

The second is *Stochastic domain decomposition for time dependent adaptive mesh generation* [pp. 106-124] by A. Bihlo, R. D. Haynes, and E. J. Walsh. The authors consider parallel, (linear) time dependent mesh generation in two spatial dimensions suitable for multi- or many–core environments. The generation of periodic meshes is also demonstrated. Typically, PDE based mesh generation methods solve for the mesh on the whole domain simultaneously. Here the authors provide a divide and conquer approach - dividing the spatial domain into subdomains, generating the mesh on the artificial subdomain boundaries using an embarrassingly parallel Monte-Carlo evaluation, and then solving the mesh PDE simultaneously on all the subdomains using the probabilistically computed solutions as boundary conditions. The approach is demonstrated using four test examples - a classic Burgers' equation example with Dirichlet boundary conditions, mesh generation on a periodic domain with a prescribed mesh density function, Burgers' equation on a periodic domain, and the shallow water equations on a periodic domain. The computed meshes are compared to the (global) meshes obtained on a single domain.

The third paper is *R-adaptive reconnection-based arbitrary Lagrangian Eulerian method - R-ReALE* [pp. 125-167] by W. Bo and M. Shashkov. Typically, an ALE method has three main phases, an explicit Lagrangian phase, a rezone phase, and a remapping phase. ReALE methods differ from more traditional ALE methods mainly in the Lagrangian and rezone phases and in the use of polygonal meshes. ReALE moves the seeds (of a Voronoi tessellation) using the Lagrangian method, and generates a polygonal mesh using centroidal Voronoi tessellation (CVT) in the rezone phase. The current work proposes to take mesh adaptation into consideration in ReALE in two spatial dimensions. This is achieved by defining a monitor function based on the $L^1$ norm error of a linear interpolation and using it as a weight function in CVT, which leads to an adaptive polygonal mesh. Numerical examples show that the new method can gain significant improvements in accuracy and convergence order over ReALE.

The fourth paper is *A comparative numerical study of meshing functionals for variational mesh adaptation* [pp. 168-186] by W. Huang, L. Kamenski, and R. D. Russell. Variational mesh adaptation serves as the base of a number of commonly used adaptive moving mesh methods. Understanding variational methods is crucial to the understanding of those variational based adaptive moving mesh methods. The paper presents a comparative study in both two and three dimensions for three of the most appealing meshing functionals, a generalization of Winslow's variable diffusion functional and two functionals based on the so-called equidistribution and alignment conditions. Their performance is investigated numerically in terms of equidistribution and alignment mesh quality measures. Critical for the study is to perform the substantial computations using a newly developed efficient implementation of the variational methods.

The last paper is *An adaptive grid method for a non-equilibrium PDE model from porous media* [pp. 187-198] by P. A. Zegeling. Flows in a porous media may develop non-monotone waves which are not present in traditional models such as Richards' equation or the Buckley-Levertt equation. Here the author considers the numerical solution of a mixed higher-order model from hydrology which is capable of producing non–monotone waves. In 1D smoothed adaptive grids are used. The standard approach of finding an appropriate mesh transformation is complicated by the presence of the higher-order mixed derivative term. The numerical experiments presented show that the adaptive grid requires a factor of four fewer mesh points (when compared to a uniform grid) to resolve the appropriate dynamics. Moreover, uniform grids may predict incorrect non-monotone waves.

It is our hope that the reader could find this special issue useful in understanding recent advances in adaptive moving mesh methods. But we should point out that this is only a glimpse of the research of the area. It is certain that a few papers are insufficient to summarize the advances of an area with activities scattered in many disciplines in science and engineering.

Finally, we would like to thank the authors for their valuable contributions to this special issue. Our thanks also go to the editors-in-chief Bo Guan and Jie Shen and the editorial office for their effort to make the publication of the issue smoothly and timely.

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