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MEMORIAL UNIVERSITY

DEPARTMENT OF MATHEMATICS & STATISTICS

TEST 1 | Math 2050 | FALL 2018

Last Name: First name: **Student ID:**

[9] 1. Let $\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -4 \end{bmatrix}$.

(1) Find $||\vec{u}||$ and $||\vec{v}||$.

(2) Find the angle between $\vec{u}$ and $\vec{v}$.

(3) Find a vector of length 5 in the opposite direction of $\vec{u}$.

(1) $||\vec{u}|| = \sqrt{1^2+4^2+1^2} = \sqrt{1+16+1} = 3\sqrt{2}$

$||\vec{v}|| = \sqrt{(-\frac{1}{2})^2+1^2+(-4)^2} = \sqrt{\frac{1}{4}+1+16} = \sqrt{\boxed{\frac{69}{4}}} = \frac{\sqrt{69}}{2}$

(2) $\vec{u}\cdot\vec{v} = ||\vec{u}||\cdot||\vec{v}||\cos\theta$

$\Leftrightarrow \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{2} \\ 1 \\ -4 \end{pmatrix} = 3\sqrt{2}\cdot\frac{\sqrt{69}}{2}\cos\theta \Leftrightarrow -\frac{1}{2} = \frac{3\sqrt{138}}{2}\cos\theta$

$\Rightarrow \theta = \arccos\left(-\frac{1}{3\sqrt{138}}\right)$

(3) Let $\vec{w} = \frac{\vec{u}}{||\vec{u}||} = \frac{\vec{u}}{3\sqrt{2}}$, then $\vec{w}$ is the unit vector in the directio of $\vec{u}$. Thus $-5\vec{w} = -\frac{5}{3\sqrt{2}}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ is the vector.

[5] 2. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors of length 3,2 and 6 respectively. Suppose $\vec{u}\cdot\vec{v} = 3$, $\vec{u}\cdot\vec{w} = 5$ and $\vec{v}\cdot\vec{w} = 2$. Find $||u - v - w||^2$.

$||\vec{u}-\vec{v}-\vec{w}||^2 = (\vec{u}-\vec{v}-\vec{w})\cdot(\vec{u}-\vec{v}-\vec{w})$

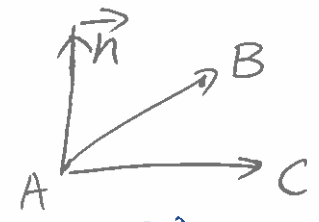
$= \vec{u}^2+\vec{v}^2+\vec{w}^2-2\vec{u}\cdot\vec{v}-2\vec{u}\cdot\vec{w}+2\vec{v}\cdot\vec{w}$

$= 3^2+2^2+6^2-2\cdot3-2\times5+2\times2$

$= 9+4+36-6-10+4 = 37$.

[10] 3. (a) Find the equation of the plane which contains the points $A(2,1,3), B(3,-1,5)$ and $C(0,2,-4)$.

(b) Find the area of the triangle ABC.



(a) $\vec{AB} = (1,-2,2)$, $\vec{AC} = (-2,1,-7)$

$$\vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ -2 & 1 & -7 \end{vmatrix} = 12\vec{i} + 3\vec{j} - 3\vec{k} = \begin{pmatrix} 12 \\ 3 \\ -3 \end{pmatrix}$$

Then choose A as a base point, the equation of a plane:

$$\vec{n}\cdot\begin{pmatrix} x-2 \\ y-1 \\ z-3 \end{pmatrix} = 0 \Leftrightarrow 12(x-2)+3(y-1)-3(z-3)=0$$

$$\Leftrightarrow 12x+3y-3z=18$$

(b) $S_{\triangle ABC} = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}\sqrt{12^2+3^2+(-3)^2} = \frac{1}{2}\sqrt{144+9+9}$

$$= \frac{9\sqrt{2}}{2} \left(= \frac{\sqrt{162}}{2}\right)$$

[6] 4. Does the line l_1 $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$ intersect the plane $4x+y-4z=14$? If yes, find the point of intersection.

The direction vector of l_1 is $\vec{u} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$, the normal vector of π is $\vec{v} = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$.

$$\vec{u}\cdot\vec{v} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}\cdot\begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix} = 8+1+16 = 25 \neq 0.$$

Thus, they will intersect.

From l_1, $x=-1+2t$, $y=1+t$, $z=2-4t$. Substituting them into the equation of the plane gives

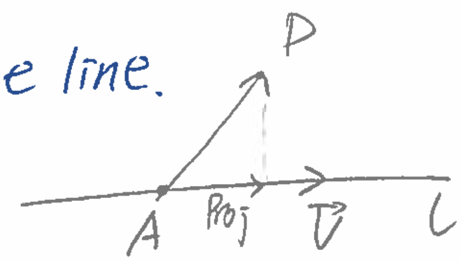
$$4(-1+2t)+(1+t)-4(2-4t)=14 \Rightarrow t=1$$

$\therefore$ $x=-1+2=1$, $y=2$, $z=-2$. The point is $(1,2,-2)$

[12] 5. (a) Find the distance from $P(-1,2,1)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) Find the distance from $P(1,-1,1)$ to the plane $2x - y + z = 2$.

(a) Choose a point $A(1,2,3)$ on the line.



$\vec{AP} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$, the direction vector is

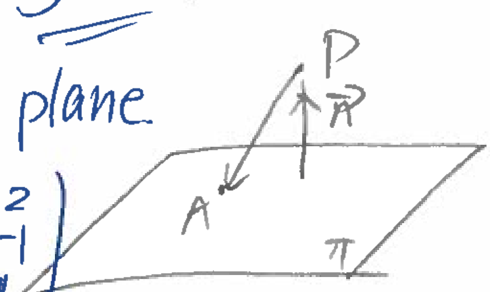
$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. $\text{Proj}_{\vec{v}} \vec{AP} = \frac{\vec{AP} \cdot \vec{v}}{\|\vec{v}\|^2} \cdot \vec{v} = \frac{\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{1^2+1^2+1^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$= \frac{-4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{AP} - \text{Proj}_{\vec{v}} \vec{AP} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} - \left(-\frac{4}{3}\right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 4/3 \\ -2/3 \end{pmatrix}$

$d = \|\vec{AP} - \text{Proj}_{\vec{v}} \vec{AP}\| = \frac{2}{3} \cdot \sqrt{1+4+1} = \frac{2\sqrt{6}}{3}$.

(b) Choose a point $A(1,0,0)$ on the plane.



$\vec{PA} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, the normal vector is $\vec{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\text{Proj}_{\vec{n}} \vec{PA} = \frac{\vec{PA} \cdot \vec{n}}{\|\vec{n}\|^2} \cdot \vec{n} = \frac{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{4+1+1} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \frac{-2}{6} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$d = \|\text{Proj}_{\vec{n}} \vec{PA}\| = \frac{1}{3} \sqrt{4+1+1} = \frac{\sqrt{6}}{3}$

[8] 6. Are vectors $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ linearly independent?

[50] Suppose $\exists\, k_1, k_2, k_3, k_4$ such that

$$k_1\vec{u} + k_2\vec{v} + k_3\vec{w} + k_4\vec{x} = 0$$

Then
$$\begin{cases} k_2 + k_3 - k_4 = 0 \\ k_1 + k_2 + 2k_4 = 0 \\ k_1 + k_2 + k_3 + k_4 = 0 \end{cases}$$

Let $k_4 = t$, then

$$\begin{cases} k_2 + k_3 = t \\ k_1 + k_2 = -2t \\ k_1 + k_3 = -t \end{cases} \Rightarrow k_1 + k_2 + k_3 = -t$$

~~Thus~~ $\Rightarrow$
$$\begin{cases} k_1 = -2t \\ k_2 = 0 \\ k_3 = t \\ k_4 = t \end{cases}$$

In conclusion, the solution for the above equations are $k_2 = 0$ and $k_3 = k_4 = -\frac{k_1}{2}$. In particular, one can let $k_1 = 2$, and we get $2\vec{u} - \vec{w} - \vec{x} = 0$, and hence vectors $\vec{u}, \vec{v}, \vec{w}$ and $\vec{x}$ are linearly dependent.