Linear and Nonlinear Speed Selection for Mono-stable Wave Propagations

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Abstract

In this paper, we study the selection mechanism of the minimal wave speed for traveling waves to an abstract monotone semiflow. A necessary and sufficient condition for the nonlinear selection of the minimal wave speed is established. Based on this result, we then derive conditions under which the linear or nonlinear selection is realized by way of comparison principle. Our results on nonlinear selection are new and novel, and they can be viewed as breakthroughs in this topic; and for the linear selection, we successfully improve previous conventional results that always require the monotone semiflow is dominated by its linear map. The applications to various biological models are also successful. We establish a series of new results to reaction-diffusion models with delay interactions, a lattice system, a scalar integro-difference equation and a cooperative system, which completely solve some open problems and conjectures in the related references.

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1 Introduction

Traveling waves and front propagations for reaction-diffusion equations have been extensively studied in biological, chemical and physical sciences, since the pioneering work of Fisher [8] and KPP [17] on the scalar equation

$$u_t = u_{xx} + f(u), \qquad u(0,x) = u_0(x), \tag{1.1}$$

where $u(t,x)$ is a real-valued function of spatial variable x and time t, and $u_0(x)$ is a non-negative initial function. The nonlinear function f satisfies $f(0) = f(1) = 0$, $f'(0) > 0$ and $f'(1) < 0$ with $f(x) > 0$ for $x \in (0,1)$. Of biological reasons, one is particularly interested in the invasion onto the unstable equilibrium and thus considers traveling wave solutions in the forms $u(t,x) = U(z)$, $z = x - ct$, moving with a speed c, where the profile function U satisfies the ordinary differential equation

$$U_{zz} + cU_z + f(U) = 0 \tag{1.2}$$

subject to

$$U(-\infty) = 1, \;\; U(+\infty) = 0. \tag{1.3}$$

It is well-known that (1.2)–(1.3) has non-negative traveling wave fronts U if and only if $c \geq c_{\min}$, where $c_{\min}$ is a constant that is related to the nonlinear function f and usually has no explicit formula. Linear stability analysis of (1.2)–(1.3) around zero easily indicates that the wave speed c must satisfy $c \geq c_0 = 2\sqrt{f'(0)}$ in order to have a nonnegative wave profile. This provides an estimate of the speed for the waves, while the determinacy of the minimal speed $c_{\min}$, depending on the nonlinearity of the function f, is usually non-trivial. When $c_{\min} = c_0$, the minimal wave speed is linearly selected and the corresponding wave profile is called a pulled front; otherwise, when $c_{\min} > c_0$, the wave speed is non-linearly selected and the corresponding wave profile is called a pushed front, where the speed $c_{\min}$ cannot be determined by the linearization at the leading edge of the front, but is controlled by the entire wave structure.

The problem of the minimal speed selection for (1.1) was also discussed in the physical literature [3–5, 32–34]. Among these papers, Van Saarloos' formal analysis argued that the nonlinear speed selection is realized on the existence of certain type of traveling wave solution with a faster decay coefficient chosen from the two possible decay rates. However, Van Saarloos's nonlinear marginal stability analysis has no mathematical rigor, awaiting further development. The first rigorous study on the minimal wave speed selection for (1.1) goes back to Aronson and Weinberger [1, 2] where they established the definition of the asymptotic spreading speed c^* and also showed that the speed c^* is indeed the minimal speed of all traveling wave fronts. In other words, they significantly proved that a unique (up to translation) traveling wave U exists for all given $c \geq c^*$, where c^* is the spreading speed in the sense that

$$\lim_{|c|>c^*,\;\; t\to\infty} u(t, x+ct) = 0, \qquad \lim_{|c|<c^*,\;\; t\to\infty} u(t, x+ct) = 1, \tag{1.4}$$

uniformly for x in any bounded domain, with $u(t,x)$ being the solution for (1.1) under some properly-chosen initial data $u_0(x)$ that have a compact support in $(-\infty, \infty)$. They also showed for $c < c^*$, there are no nonnegative traveling wave solutions subject to (1.3). Furthermore, they provided the asymptotic behavior for the wave profile near positive infinity with

$$\ln U \sim \lambda_+(c)\, z, \quad \text{as } z \to \infty, \quad \text{for } c > c^* \tag{1.5}$$

and

$$\ln U \sim \lambda_-(c)\, z, \ \text{ as } z \to \infty, \ \text{ for } c = c^*, \tag{1.6}$$

where

$$\lambda_\pm(c) = \frac{1}{2}\left(-c \pm \sqrt{c^2 - 4f'(0)}\right). \tag{1.7}$$

This is the most important feature for the linear selection vs. nonlinear selection, which agrees to Van Saarloos' argument [32–34] that the wave front with the minimal speed decays exponentially to zero at infinity with the faster rate $\lambda = \lambda_-$ instead of $\lambda = \lambda_+$. However, this argument didn't say when the case $c^* = c_0$ is chosen and when the case $c^* > c_0$ is realized. Therefore, they didn't provide an effective method to determine when the linear or nonlinear selection is realized. Recently Lucia, Muratov and Novaga [24], based on Aronson and Weinberger's result (1.5)-(1.6), developed a variational characterization for the wave front with the minimal speed $c^* > c_0$ and established a necessary and sufficient condition for the nonlinear selection mechanism. The following easy-to-apply results are directly from [24].

Lemma 1.1. *If $\int_0^u f(s)ds \le \frac{1}{2}f'(0)u^2$, $u \in (0,1)$, then the linear selection is realized.*

Lemma 1.2. *If $f'(0)u \le \frac{1}{2}\int_0^1 f(u)du$, then the nonlinear selection is realized.*

Remark 1.1. *Lemma 1.1 covers the classical result that the linear selection is realized for (1.1) if $f(u) \le f'(0)u$ for $u \ge 0$; Lemma 1.2 is new and has not been established before.*

For traveling waves to general reaction-diffusion systems and integro-difference equations, we refer to the recent advancements of Weinberger and his collaborators, see e.g., [19, 20, 39]. They extended the idea of the spreading speed and monotone traveling waves in [23, 37] to study a monotone map

$$u_{n+1} = Q[u_n], \quad n = 0, 1, 2, \cdots,$$

where $u_n = (u_n^1(x), u_n^2(x), \cdots, u_n^k(x))$ in $\mathcal{C} = BC(\mathcal{H}, \mathbb{R}^k)$ is the solution at time $n\tau$, with τ being a fixed time. For reaction-diffusion systems, $\mathcal{H}$ is the real line or the integer points, i.e., $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. Under some proper assumptions, they established the existence of the spreading speed c^* as well as the existence of traveling wave fronts. Linear selection of the minimal wave speed is realized under some further condition (i.e., the map is dominated by its linear part on a particular direction), but no nonlinear selection mechanism has been established.

In 2007, Liang and Zhao developed the idea of Weinberger and his collaborators [19, 20, 39] to consider

$$u_{n+1} = Q[u_n], \quad n \ge 0, \quad u_0 \in \mathcal{C}([-\tau, 0] \times \mathcal{H}, \mathbb{R}^k), \tag{1.8}$$

where $\mathcal{C}([-\tau, 0] \times \mathcal{H}, \mathbb{R}^k)$ is the set of bounded and continuous functions from $[-\tau, 0] \times \mathcal{H}$ to $\mathbb{R}^k$, $\mathcal{H}$ is the spatial habitat and τ is the time delay. They intended to establish the theory of wave propagation for monotone semiflows that cover autonomous time-delayed reaction-diffusion equations and lattice systems. With a further condition that Q admits only two fixed points 0 and β, they proved a series of results such as:

(a) the existence of spreading speed c^*;

(b) the existence of the minimal speed c_{min} such that the map Q has traveling waves $w(\theta, x)$ satisfying $Q^n[w] = w(\theta, x - nc)$, $w(\cdot, -\infty) = \beta$ and $w(\cdot, +\infty) = 0$ if and only if $c \ge c_{min}$;

(c) the equality $c^* = c_{min}$;

(d) the estimate of the spreading speed with $c^* \geq c_0$ where c_0 is the linear speed;

(e) the linear selection $c^* = c_0$ if $Q[u] \leq M[u]$ for all $u \in \mathcal{C}_\beta$, where the operator M is the linearization of Q at zero (Fréchet derivative), i.e.,

$$M[u] = \lim_{\rho \to 0} \frac{Q[\rho u]}{\rho}. \tag{1.9}$$

The more details can be founded in [21].

The research contributions from [19–21,39] are substantial extensions of the classical KPP-Fisher equation onto various scientific models. However, the problem on the speed selection is not deeply touched in their investigations. *Current situation is that they only provided the linear selection mechanism under a strong condition $Q[u] \leq M[u]$, and no nonlinear selection mechanism has been established.* In our opinion, the main challenges and difficulties are due to the following items:

(1) The classical phase plane analysis plus the construction of an invariant region definitely fails for the construction of traveling waves to the abstract map Q;

(2) The variational principle in Lucia, Murator and Novaga [24] CANNOT be applied to general monotone semiflows;

(3) Van Saarloos's nonlinear marginal stability analysis has no mathematical rigor and requires further development.

(4) For the general map in (1.8), as to the linear selection, one can compare the map Q with its linearized rival M. Once the map Q is dominated by its linear part on a particular direction, it is concluded that the linear selection is realized. However, for the study of nonlinear selection, usually we don't know which system (or map) to be compared with because no information about the minimal speed as well as its corresponding wavefront is known in advance. The method of geometrical singular perturbation is useful for finding the pushed front in [9], but it only works for finite dimensional space and small parameter regions.

Due to the significance of the spreading speed (the minimal speed) in the invasion of multiple biological species model or chemical reactions, it is particularly important to overcome the above difficulties in establishing the speed selection mechanism for general models including various reaction-diffusion systems. The purpose of this paper is on this direction, and we aim to work out the speed selection mechanism for the abstract monotone semiflow in (1.8). Based on a perturbation argument coupled with a comparison principle, we first establish a necessary and sufficient condition for the nonlinear selection. In view of this result, some easy-to-apply theorems on linear and nonlinear selection are provided respectively. Direct applications of our theorems and corollaries to reaction-diffusion equations with delayed interaction, lattice systems, a scalar integro-difference model and a cooperative system give some encouraging results and solve some important conjectures in literature. For instance, for the time-delayed reaction-diffusion equation

$$u_t = u_{xx} + f(u, u(t-\tau, x)), \tag{1.10}$$

where f satisfies

$$f(r,s) \geq 0, \quad \partial f_2(r,s) \geq 0, \quad \text{for } \ 0 \leq r, s \leq 1,$$

$$f(0,0)=f(1,1)=0, \quad f(r,r)>0 \text{ for } 0<r<1,$$

and

$$\partial f_1(0,0)+\partial f_2(0,0)>0.$$

Schaaf's Theorem 2.7 in [35] indicated that the minimal speed is always linearly selected without any extra condition. This is incorrect because the nonlinear selection can happen for some nonlinear functions f in (1.10), see Remark 3.3 in Section 3. Therefore, our result gives a new understanding of this topic, that is, we not only establish the nonlinear selection mechanism, but also provide a new result on the linear selection with a mild condition on f that does not necessarily require the nonlinearity is dominated by its linear part. In particular, for the delayed reaction diffusion equation (3.13) with the Hadeler and Rothe nonlinearity, we establish a necessary and sufficient condition for the selection mechanism.

The paper is organized as follows. The next section is devoted to summarizing the preliminaries and presenting our main results and their proofs. Some significant applications are shown in Sections 3-5. Finally, in Section 6 we summarize our results and discuss possible avenues for further research.

2 Linear and Nonlinear Selection

In this section, we first incorporate the setting of phase space, the definition of spreading speed and the existence of traveling waves that were originated from [23, 37, 39] and further extended in [21]. Based on this setting, our breakthroughs on the speed selection mechanism are successfully established in Section 2.1.

Now we proceed to present the setting of phase space for a monotone dynamical system.

Let τ be a nonnegative real number and $\mathcal{C}=C([-\tau,0]\times\mathcal{H},\mathbb{R}^k)$ be the set of bounded and continuous functions from $[-\tau,0]\times\mathcal{H}$ to $\mathbb{R}^k$ with $|\cdot|_{\mathcal{C}}=||\cdot||_\infty$, where $\mathcal{H}=\mathbb{R}$ or $\mathbb{Z}$. Any vectors in $\mathbb{R}^k$ and any element in $\bar{\mathcal{C}}=C([-\tau,0],\mathbb{R}^k)$ can be regarded as a function in $\mathcal{C}$. $\bar{\mathcal{C}}$ is equipped with the maximum norm and the positive cone is defined by $\bar{\mathcal{C}}_+=\{\phi\in\bar{\mathcal{C}}:\phi(\theta)\ge 0,\theta\in[-\tau,0]\}$. The space $\mathcal{C}$ is also equipped with the compact open topology in the sense that $v^n\to v$ in $\mathcal{C}$ means that the sequence $v^n(\theta,x)$ converges to $v(\theta,x)$ uniformly for (θ,x) in every compact set. A bounded subset $\mathcal{C}_\beta$ is defined by $\mathcal{C}_\beta=\{\varphi\in\mathcal{C}:0\le\varphi\le\beta\}$ where $\beta\in\bar{\mathcal{C}}$ with $\beta=(\beta^1,\beta^2,\cdots,\beta^k)\gg 0$.

Here for the notations "$<,\le,\ll$" of vectors or vector-valued functions, we follow the definitions in [21, page 3]. For the division and multiplication of vectors or vector-valued functions, we mean component-wise.

In [21] Liang and Zhao developed the ideas in Weinberger [37] and Lui [23] to consider a map

$$u_{n+1}=Q[u_n],$$

where $u_n=(u_n^1,u_n^2,\cdots,u_n^k)\in\mathcal{C}$. Define the translation operator T_y as $T_y(u)(\theta,x)=u(\theta,x-y)$ for any given $y\in\mathcal{H}$. The map $Q:\mathcal{C}_\beta\to\mathcal{C}_\beta$ satisfies the following assumptions:

(A1) $T_y\circ Q=Q\circ T_y$.

(A2) Q is continuous with respect to the compact open topology.

(A3) One of the following two conditions holds:

(a) $Q[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.

(b) There is a nonnegative number $\zeta < \tau$ such that $Q[u](\theta, x) = u(\theta + \zeta, x)$, $-\tau \le \theta \le -\zeta$, and the operator

$$S[u](\theta, x) = \begin{cases} u(0, x), & -\tau \le \theta < -\zeta, \\ Q[u](\theta, x), & -\zeta \le \theta \le 0 \end{cases}$$

is continuous on $\mathcal{C}_\beta$, and $S[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.

(A4) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is monotonic (order-preserving) in the sense $Q[u] \ge Q[v]$ whenever $u \ge v$ in $\mathcal{C}_\beta$.

(A5) $Q : \bar{\mathcal{C}}_\beta \to \bar{\mathcal{C}}_\beta$ has exactly two fixed points 0 and β, and $\lim_{n\to\infty} Q^n(r) = \beta$ for any $r \in \bar{\mathcal{C}}_\beta$ with $0 \ll r \ll \beta$.

To define the spreading speed, let $\alpha \in \bar{\mathcal{C}}_\beta$, with $0 \ll \alpha \ll \beta$, and assume $\phi = (\phi^1, \phi^2, \cdots, \phi^k) \in \mathcal{C}_\beta$ has the properties that

(B1) $\phi^i(\theta, \cdot)$ is a non-increasing function for any fixed $\theta \in [-\tau, 0]$ and $1 \le i \le k$.

(B2) $\phi^i(\theta, x) = 0$ for any $\theta \in [-\tau, 0]$, $x \ge 0$, $1 \le i \le k$.

(B3) $\phi(\theta, -\infty) = \alpha(\theta)$ for any $\theta \in [-\tau, 0]$.

Given a real number c, define an operator R_c and a sequence of functions $\{a_n\}_{n=0}^{\infty}$ as

$$R_c[a](\theta, s) = \max\{\phi(\theta, s), T_{-c}[Q[a]](\theta, s)\}; \tag{2.1}$$

$$a_0(c; \theta, s) = \phi(\theta, s), \;\; a_{n+1}(c; \theta, s) = R_c[a_n(c; \cdot)](\theta, s). \tag{2.2}$$

Thus, following the idea of Lui [23], the results below can be proved:

(a) $a_n \le a_{n+1} \le \beta$;

(b) $\lim_{n\to\infty} a_n(c; \theta, s) = a(c; \theta, s)$ pointwise;

(c) $a(c; \theta, s)$ is non-increasing in s and c;

(d) $a(c; \cdot, -\infty) = \beta$, $a(c; \cdot, +\infty)$ exists in $\bar{\mathcal{C}}_\beta$ and is a fixed point of Q.

Using the above properties, the spreading speed c^* of the operator Q is defined by

$$c^* := \sup\{c : a(c; \cdot, +\infty) = \beta\}. \tag{2.3}$$

Remark 2.1. *Define* $c_+^* := \sup\{c : a(c; \cdot, +\infty) \ne 0\}$*. Since* Q *admits only two fixed points in* $\bar{\mathcal{C}}_\beta$*, it can be proved that* $c^* = c_+^*$*.*

By a traveling wave solution of the map Q, we mean that a particular function $W(\theta, x)$ satisfies the profile equation

$$Q[W](\theta, x) = W(\theta, x - c) \tag{2.4}$$

for some constant c and $Q^n[W](\theta, x) = W(\theta, x - nc)$. We say that the traveling wave solution $W(\theta, x)$ connects β to 0 if $W(\theta, -\infty) = \beta$ and $W(\theta, \infty) = 0$. Now the main results in [21] on traveling waves can be summarized into the following lemmas.

Lemma 2.1 ([21]). *Let Q satisfy (A1)-(A5). Then the following statements are true.*

(a) For any $c \geq c^*$, Q has a traveling wave $W(\theta, x - nc)$ connecting β to 0 such that $W(\theta, x)$ is nonincreasing in x. Moreover, if $\mathcal{H} = \mathbb{R}$, then $W(\theta, x)$ is continuous in (θ, x).

(b) For any $c < c^*$, Q has no non-negative traveling wave $W(\theta, x - nc)$ connecting β and 0.

A natural extension to time-continuous semiflow $\{Q_t\}_{t=0}^{\infty}$ was also given in [21]. Let $\{Q_t\}_{t=0}^{\infty}$ be a semiflow on $\mathcal{C}_\beta$ with $Q_t(0) = 0$ and $Q_t(\beta) = \beta$. By a traveling wave solution of the map Q_t, we mean that a particular function $W(\theta; x)$ satisfies the equation

$$Q_t[W](\theta, x) = W(\theta, x - ct) \tag{2.5}$$

for some constant c. Likewise, it is said that $W(\theta, x)$ connects β to 0 if $W(\theta, -\infty) = \beta$ and $W(\theta, \infty) = 0$. Similarly, there is the following lemma.

Lemma 2.2 ([21]). *Let $\{Q_t\}_{t=0}^{\infty}$ be a monotone semiflow on $\mathcal{C}_\beta$ with $Q_t(0) = 0$ and $Q_t(\beta) = \beta$ for all $t \geq 0$. Suppose that $Q = Q_t$ satisfies all the hypotheses (A1)-(A5) for each fixed t, and let c^* be the spreading speed of Q_1.*

(a) For any $c \geq c^*$, $\{Q_t\}_{t=0}^{\infty}$ has a traveling wave $W(\theta, x - ct)$ satisfying $Q_t[W](\theta, x) = W(\theta, x - ct)$, $W(\theta, -\infty) = \beta$, $W(\theta, +\infty) = 0$, and $W(\theta, x)$ is non-increasing in x.

(b) For any $c < c^*$, $\{Q_t\}_{t=0}^{\infty}$ has no traveling wave $W(\theta, x)$ connecting β and 0.

We now define the minimal speed $c_{\min}$ as

$$c_{\min} = \inf\{c : \ Q \text{ has non-increasing traveling waves connecting } \beta \text{ and } 0 \text{ with speed } c\}. \tag{2.6}$$

Then the above two lemmas imply that $c^* = c_{\min}$, i.e., the asymptotic spreading speed is equal to the minimal speed of traveling waves.

2.1 Main results on the speed selection

In this subsection we concentrate on the study of speed selection of wave propagation for the abstract monotone semi-flow Q. Assume that

(A6) Q is Fréchet-differentiable around any $\varphi \in [0, \beta]$.

Let M be the linearization operator of Q around zero in the sense of (1.9). Suppose that the following hypotheses are true:

(M1) $T_y \circ M = M \circ T_y$.

(M2) M is continuous with respect to the compact open topology.

(M3) One of the following two conditions holds:

(a) $M[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.

(b) There is a nonnegative number $\zeta < \tau$ such that $M[u](\theta, x) = u(\theta + \zeta, x)$ for $-\tau \le \theta \le -\zeta$, and the operator $S : \mathcal{C}_\beta \to \mathcal{C}_\beta$ defined by

$$S[u](\theta, x) = \begin{cases} u(0, x), & -\tau \le \theta < -\zeta, \\ M[u](\theta, x), & -\zeta \le \theta \le 0 \end{cases}$$

is continuous on $\mathcal{C}_\beta$, and $S[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.

(M4) $M : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is monotonic (order-preserving) in the sense $M[u] \ge M[v]$ whenever $u \ge v$ in $\mathcal{C}_\beta$.

(M5) Define a linear map $B_\mu : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ as

$$B_\mu[\alpha](\theta) = M[\alpha e^{-\mu x}](\theta, x = 0), \ \ \theta \in [-\tau, 0],$$

where μ is a positive real number and α is a real vector. We assume that B_μ is compact and positive on $\bar{\mathcal{C}}$ with the existence of strongly positive eigenvector corresponding to a simple principal eigenvalue. The principal eigenvalue of B_0 is assumed to be greater than 1.

(M6) Let M_φ be the linearization of Q at φ where φ satisfies $0 \le \varphi \le \beta$. We assume that M_φ is compact and positive. In particular, define the linear map $B_{\gamma,\beta} : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ as

$$B_{\gamma,\beta}[\alpha](\theta) = M_\beta[\alpha e^{\gamma x}](\theta, x = 0), \ \ \theta \in [-\tau, 0].$$

We assume that $B_{\gamma,\beta}$ is compact and positive on $\bar{\mathcal{C}}$ with the existence of strongly positive eigenvector corresponding to a simple principal eigenvalue. Let $\bar{\lambda}(\gamma)$ be the principal eigenvalue of $B_{\gamma,\beta}$ and ζ_γ be the corresponding eigenvector so that $\bar{\lambda}(0) < 1$.

Remark 2.2. *Here by the Riesz representation theorem, it follows that*

$$B_\mu[\alpha](\theta) = \frac{M[\alpha e^{-\mu x}](\theta, x)}{e^{-\mu x}}.$$

Now we want to provide a formula for the linear spreading speed of the operator M. Let $\lambda(\mu)$ be the principal eigenvalue of B_μ and $\zeta_\mu(\cdot)$ be the corresponding strongly positive eigenvector. Define the linear speed c_0 as

$$c_0 = \inf_{\mu > 0} \frac{\ln \lambda(\mu)}{\mu} = \frac{\ln \lambda(\bar{\mu})}{\bar{\mu}}. \tag{2.7}$$

Here we assume that $\lambda(0) > 1$ and c_0 is attained at a finite value $\mu = \bar{\mu} > 0$. Since $\lambda(\mu)$ is log convex (see [21]), it is easy to know that for any $c > c_0$, there exist *positive* numbers $\mu_1(c)$ and $\mu_2(c)$, $\mu_1(c) < \mu_2(c)$, such that

$$\frac{\ln \lambda(\mu_1)}{\mu_1} = \frac{\ln \lambda(\mu_2)}{\mu_2} = c \tag{2.8}$$

and

$$\frac{\ln \lambda(\mu)}{\mu} < c \quad \text{for any} \quad \mu \in (\mu_1, \mu_2).$$

When $c = c_0$, we have $\mu_1 = \mu_2 = \bar{\mu}$. Furthermore, it can be derived that $\mu_1(c)$ is a decreasing function and $\mu_2(c)$ is an increasing function in $c \in [c_0, \infty)$.

Remark 2.3. *Alternatively, in the case when B_μ in (M5) is reducible, we can follow the idea in [39] to define the linear speed. Let the matrix B_μ have finite entries for all μ and be in Frobenius form. The principal eigenvalue of its σth diagonal block is $\lambda_\sigma(\mu)$. We assume that $\lambda_1(0) > 1 > \lambda_\sigma(0)$ for all $\sigma > 1$ and define*

$$c_0 = \inf_{\mu>0} \frac{\ln \lambda_1(\mu)}{\mu} = \frac{\ln \lambda_1(\bar{\mu})}{\bar{\mu}}, \tag{2.9}$$

where $\bar{\mu}$ is assumed to be a finite value. Further assume that $\lambda_1(\bar{\mu}) > \lambda_\sigma(\bar{\mu})$ for $\sigma > 1$. Then c_0 is called the linear speed of the linear map M, see Theorem 3.1 in [39]. The definition of "reducible" and "Frobenius" can be found in [39]

From [21], it is clearly shown that $c^* \geq c_0$ as long as Q is differentiable around zero in the Fréchet sense. Now we give the definition of linear selection and nonlinear selection of the spreading speed (or the minimal speed). This definition is consistent with what we have mentioned in the Introduction section.

Definition 2.3. *The spreading (minimal) speed is linearly selected if $c^* = c_0$, and nonlinearly selected if $c^* > c_0$.*

To obtain the speed selection mechanism, we now point out the exponential decay behavior of traveling waves. Assume that $W(\theta, x - c)$ is a traveling wave of Q, satisfying $Q[W](\theta, x) = W(\theta, x - c), W(\theta, -\infty) = \beta$ and $W(\theta, \infty) = 0$. Also suppose that

$$Q[W](\theta, x) - M[W](\theta, x) = o(W(\theta, x - c)) \tag{2.10}$$

as $x \to \infty$. For $c > c_0$, a straightforward derivation of the characteristics of the linear part of the wave profile equation implies either

$$W(\theta, x) \sim C_1 \zeta_{\mu_1}(\theta) e^{-\mu_1(c)x}, \; C_1 > 0, \tag{2.11}$$

or

$$W(\theta, x) \sim C_2 \zeta_{\mu_2}(\theta) e^{-\mu_2(c)x}, \; C_2 > 0 \tag{2.12}$$

as $x \to \infty$.

Remark 2.4. *Rigorous proof of (2.11)-(2.12) can be carried out by the method of successive approximation. Other methods such as two sides Laplace transform can also be applied to derive them, see e.g., [7]. The condition (2.10) is natural and can be easily verified for all reaction-diffusion models.*

Our main results are the following theorems and corollaries.

Theorem 2.4 (Necessary and sufficient condition)**.** *Let Q satisfy (A1)-(A6) and c^* be its spreading speed with the wavefront $W_{c^*}(\theta; x)$ having continuous derivative W'_{c^*} with respect to the second variable. Assume that (M1)-(M6) are true with the linear speed c_0 defined in (2.7). Furthermore, suppose that $M(W_{c^*})$ is strongly positive. Then the spreading speed c^* is nonlinearly selected if and only if there exists a speed $c = \bar{c} > c_0$ so that Q has a non-increasing traveling wave solution $U_{\bar{c}}(\theta, x)$ connecting β to 0 with the following behavior*

$$U_{\bar{c}}(\theta, x) = C \, \zeta_{\mu_2(\bar{c})}(\theta) e^{-\mu_2(\bar{c})\, x} \quad \text{as} \quad x \to \infty \tag{2.13}$$

for some positive constant C, where μ_2 is defined in (2.8). Furthermore, we have $c^* = \bar{c}$.

Proof. We first prove the sufficiency. Suppose that there exists $\bar{c} > c_0$ such that the traveling wave $U_{\bar{c}}(\theta, x)$ has the behavior $U_{\bar{c}}(\theta, x) = C\ \zeta_{\mu_2(\bar{c})}(\theta) e^{-\mu_2(\bar{c})\,x}$ as $x \to \infty$ for some positive number C. We first want to show that Q has no traveling waves for any c in $(c_0, \bar{c})$. Assume to the contrary that for some $c \in (c_0, \bar{c})$, we do have a traveling wave $W_c(\theta, x)$ satisfying

$$W_c(\theta, x - c) = Q[W_c](\theta, x) = M[W_c](\theta, x) + [Q[W_c](\theta, x) - M[W_c](\theta, x)]. \tag{2.14}$$

Since $W_c(\cdot, x) \to 0$ as $x \to \infty$, it easily follows that $[Q[W_c](\theta, x) - M[W_c](\theta, x)] = o(W_c(\theta, x))$. Then near infinity, the leading term of W_c, say W_0, must satisfy the equation $W_0(\theta, x - c) = M[W_0](\theta, x)$. Here by the leading term of W_c, we mean that $W_c = W_0 + O(e^{-\eta x})$ as $x \to \infty$ for some $\eta > \mu_1(c)$. For $c > c_0$, this linear equation implies either

$$W_c(\theta, x) \sim W_0 \sim C_1 \zeta_{\mu_1}(\theta) e^{-\mu_1(c)x},\ C_1 > 0, \tag{2.15}$$

or

$$W_c(\theta, x) \sim W_0 \sim C_2 \zeta_{\mu_2}(\theta) e^{-\mu_2(c)x},\ C_2 > 0 \tag{2.16}$$

as $x \to \infty$. Due to the monotonicity of $\mu_1(c)$ and $\mu_2(c)$ in c, this means $W_c(\theta, x) \gg U_{\bar{c}}$ for x near the positive infinity. Near the negative infinity, similarly W_c has the following behavior

$$W_c \sim \beta - \zeta_\gamma e^{\gamma x} \tag{2.17}$$

for some positive γ and vector ζ_γ. To derive a formula for γ, recall that M_β is the linearization operator of Q at β and $\bar{\lambda}(\gamma)$ is the principal eigenvalue of $B_{\gamma,\beta}$. From the equation $Q[W_c] = W_c(\cdot, x - c)$, we can derive that $\gamma c + \ln \bar{\lambda}(\gamma) = 0$. Based on the convexity of $\ln \bar{\lambda}(\gamma)$, it follows that there exists a unique γ solving this equation and γ is a decreasing function in c for $c \geq c_0$. This means further that $U_{\bar{c}}(\theta, x) \ll W_c(\theta, x)$ for x near $-\infty$. Therefore, we can make a shift of distance ξ_0 for the variable x in $W_c(\theta, x)$ to satisfy

$$\bar{W}_c(\theta, x) = W_c(\theta, x + \xi_0) \gg U_{\bar{c}}(\theta, x).$$

Applying the monotonicity of the map Q, we have that

$$\bar{W}_c(\theta, x - nc) = Q^n(\bar{W}_c(\theta, x)) \geq Q^n(U_{\bar{c}}(\theta, x)) = U_{\bar{c}}(\theta, x - n\bar{c}) \tag{2.18}$$

for $x \in (-\infty, \infty)$. However, on the line $x - n\bar{c} = z_0$ for some fixed value z_0, we have $U_{\bar{c}}(\theta, x - n\bar{c}) = U_{\bar{c}}(\theta, z_0) \gg 0$ and

$$\bar{W}_c(\theta, x - nc) = \bar{W}_c(\theta, z_0 + n(\bar{c} - c)) \to 0, \text{ as } n \to \infty,$$

which is a contradiction to (2.18). Therefore, Q has no traveling waves for $c \in (c_0, \bar{c})$. By Lemma 2.2, it is impossible to have traveling waves with speed $c = c_0$. Thus, the sufficiency is proved.

For the necessity, we assume that the minimal speed c^* is nonlinearly selected, i.e., $c^* > c_0$. We want to prove that at the speed $c = c^*$, the traveling wave $W_{c^*}(\theta, x)$ satisfies the following behavior

$$W_{c^*}(\theta, x) \sim C \zeta_{\mu_2(c^*)}(\theta) e^{-\mu_2(c^*)x} \quad \text{as} \quad x \to \infty \tag{2.19}$$

for some constant C. This can be shown by way of contradiction. Assume, to the contrary, that

$$W_{c^*}(\theta, x) \sim C_3 \zeta_{\mu_1(c^*)}(\theta) e^{-\mu_1(c^*)x} \quad \text{as} \quad x \to \infty \tag{2.20}$$

for some positive constant C_3 and eigenvector $\zeta_{\mu_1(c^*)}$. We shall prove that this assumption will imply that the operator Q has a traveling wave $W_c(\theta, x)$ satisfying

$$Q(W_c) = W_c(\cdot, x - c), \text{ or } T_{-c}Q(W_c) = W_c \tag{2.21}$$

for some speed $c = c^* - \delta$, where δ is a sufficiently small and positive number such that c^* is not the minimal speed, thus giving a contradiction. Indeed, under the assumption (2.20), we define

$$\bar{W} = W_{c^*}(\theta, x)\omega(x), \quad \omega(x) = \frac{1}{1 + \frac{\zeta_{\mu_1(c)(\theta)}}{\zeta_{\mu_1(c^*)(\theta)}}\delta e^{(\mu_1(c) - \mu_1(c^*))x}}. \tag{2.22}$$

In the above formulas, the division is component-wise so that the weight function ω is a vector. The multiplication defining $\bar{W}$ is also component-wise. Since δ is sufficiently small, $\bar{W}$ is close to W_{c^*} but with a different decaying rate at infinity. We shall apply a perturbation argument to show the existence of solution to (2.21) when δ is small. In (2.21), set

$$W_c = \bar{W} + V \tag{2.23}$$

and we have an equation for V as

$$T_{-c}Q(\bar{W} + V) = \bar{W} + V, \tag{2.24}$$

where $V = V(\theta, x)$ is a function to be determined. It follows that

$$V = T_{-c^*}M(W_{c^*})V + F_0 + M_\delta V + F_{high}(V), \tag{2.25}$$

where

$$F_0 = T_{-c}Q(\bar{W}) - \bar{W}, \tag{2.26}$$

$$M_\delta V = \left[T_{-c}M(\bar{W}) - T_{-c^*}M(W_{c^*})\right] V \tag{2.27}$$

and

$$F_{high}(V) = T_{-c}Q(\bar{W} + V) - T_{-c}Q(\bar{W}) - T_{-c}M(\bar{W})V. \tag{2.28}$$

Here $M(\bar{W})$ is the Frećhet derivative of Q around the function $\bar{W}$. After a simple estimate, we have $M_\delta V = O(\delta)V$, $F_0 = O(\delta)$, satisfying

$$F_0 = o(e^{-\mu_1(c^*)x}) \text{ as } x \to \infty.$$

To find a solution to (2.25), first we recall that for V in the space C_0, where

$$C_0 = \{u \in C([-\tau, 0] \times \mathcal{H}, \mathbb{R}^k) : u(\theta, \pm\infty) = 0\},$$

$M(W_{c^*})$ is defined by

$$M(W_{c^*})[V] = \lim_{\rho \to 0} \frac{Q[W_{c^*} + \rho V] - Q[W_{c^*}]}{\rho}.$$

It can be seen that the operator $T_{-c^*}M(W_{c^*})$ is compact and strongly positive with a principal eigenvalue $\lambda = 1$ and its corresponding eigenvector $\bar{v} = W'_{c^*}$. Here it is easy to know that W'_{c^*} shares the same decaying behavior as that of W_{c^*}, i.e.,

$$W'_{c^*} \sim \vec{C} e^{-\mu_1(c^*)x} \quad \text{as} \quad x \to \infty \tag{2.29}$$

for some vector $\vec{C}$, where W'_{c^*} represents the first derivative of $W_{c^*}(\theta, x)$ with respect to x.

To omit this eigenvector $\bar{v}$, we define a weighted space $\mathcal{V}$ as

$$\mathcal{V} = \{v \in C_0 : ve^{\mu_1(c)x} = o(1) \text{ as } x \to \infty, \}$$

where $c = c^* - \delta$. Therefore, it follows that the eigenvector $\bar{v} = W'_{c^*}$ is not in $\mathcal{V}$, which implies that $T_{-c^*}M(W_{c^*})$ has no eigenvalue $\lambda = 1$ in $\mathcal{V}$. Since the operator $T_{-c^*}M(W_{c^*})$ is compact and strongly positive in $\mathcal{V}$, we know that $T_{-c^*}M(W_{c^*}) - I$ has a bounded inverse in $\mathcal{V}$, where I is the identity operator. By the well-known inverse function theorem in the abstract space $\mathcal{V}$, we conclude that there exists a small positive number δ_0 so that the problem (2.25) has a solution V for any $\delta \in [0, \delta_0)$. Returning to (2.23), it follows that we have a solution W_c for $c = c^* - \delta$. The positivity of W_c can be guaranteed by the choice of a sufficiently small δ. The proof is complete. ∎

The above theorem presents the essential feature of the wavefront with the nonlinearly selected minimal speed. For the technique of the above perturbation argument, we have assumed that $M(W_{c^*})$ is strongly positive. Next we shall provide two easy-to-use theorems for the linear selection or the nonlinear selection without this assumption.

Theorem 2.5 (Linear Selection). *Let Q satisfy (A1)-(A6) with c^* being its spreading speed and M satisfy (M1)-(M6) with the linear speed c_0 defined in (2.7). Further assume that there exists a continuous and positive function $U(\theta, x)$ satisfying*

$$\liminf_{x \to -\infty} U(\theta, x) \gg 0, \quad \lim_{x \to \infty} U(\theta, x) = 0, \tag{2.30}$$

and

$$Q(U) \le U(\theta, x - c_0). \tag{2.31}$$

Then the linear selection is realized.

Proof. Recall from (2.3) that $c^* := \sup\{c : a(c; \theta, +\infty) = \beta\}$ where

$$a(c; \theta, s) = \lim_{n \to \infty} a_n(c; \theta, s),$$

with $a_0(c; \theta, s) = \phi(\theta, s)$, $a_{n+1}(c; \theta, s) = R_c[a_n(c; \cdot)](\theta, s)$, see (2.1) and (2.2). Here c^* is independent of the choice of $a_0(c; \theta, -\infty) = \alpha(\theta)$, see [21, 38]. Therefore, we can let $\alpha(\theta)$ be small so that the upper solution U (a shift of U if required) satisfies

$$a_0(c_0; \theta, s) = \phi \le U(\theta, s) \tag{2.32}$$

for all $s \in (-\infty, \infty)$. From (2.1), (2.2), (2.30) and (2.31), by induction it follows that

$$a_{n+1}(c_0; \theta, s) \le U(\theta, s), \quad n \ge 0$$

and $a(c_0; \theta, \infty) = 0$. By (2.3), we have $c^* \le c_0$. From the fact that the operator Q is Fréchet-differentiable at zero, it can be derived that $c^* \ge c_0$ in terms of Theorem 3.10(ii) in [21] (by replacing M by $M - \delta$ for any small constant δ). Therefore, we arrive at $c^* = c_0$, and then the linear selection is realized. ∎

Corollary 2.6. *Let Q satisfy (A1)-(A6) with c^* being its spreading speed and M satisfy (M1)-(M6) with the linear speed c_0 defined in (2.7). Suppose that $U = e^{-\bar{\mu}x}\zeta_{\bar{\mu}}(\theta)$ is an upper solution of the wave profile equation, i.e.,*

$$Q\big(e^{-\bar{\mu}x}\zeta_{\bar{\mu}}(\theta)\big) \leq e^{-\bar{\mu}(x-c_0)}\zeta_{\bar{\mu}}(\theta), \tag{2.33}$$

where $\bar{\mu}$ is defined in (2.7). Then the linear selection is realized.

Corollary 2.7. *Let Q satisfy (A1)-(A6) with c^* being its spreading speed and M satisfy (M1)-(M6) with the linear speed c_0 defined in (2.7). Suppose that $U = \frac{\beta}{1+\zeta_{\bar{\mu}}(\theta)e^{\bar{\mu}x}} := (\frac{\beta^1}{1+\zeta^1_{\bar{\mu}}(\theta)e^{\bar{\mu}x}}, \cdots, \frac{\beta^k}{1+\zeta^k_{\bar{\mu}}(\theta)e^{\bar{\mu}x}})$ is an upper solution of the wave profile equation, i.e., $Q(U) \leq U(\theta, x - c_0)$. Then the linear selection is realized.*

Now we proceed to provide a sufficient condition for the nonlinear selection.

Theorem 2.8 (Nonlinear selection)**.** *Let Q satisfy (A1)-(A6) with c^* being its spreading speed and M satisfy (M1)-(M6) with the linear speed c_0 defined in (2.7). For $c_1 > c_0$, suppose that there exists a function $V(\theta, x)$ satisfying*

$$0 \ll V(\theta,x) \ll \beta, \ \limsup_{x\to-\infty} V(\theta,x) \ll \beta, \quad V(\theta,x) = \zeta_{\mu_2(c_1)}(\theta)e^{-\mu_2(c_1)x} \quad \textit{as } x\to\infty \tag{2.34}$$

and

$$Q(V(\theta,x)) \geq V(\theta, x - c_1), \tag{2.35}$$

where $\mu_2(c_1)$ is defined in (2.8). Then $c^* \geq c_1$ and NO traveling waves exist for $c \in [c_0, c_1)$. In other words, the nonlinear selection is realized.

Proof. To the contrary, first assume that for $c \in (c_0, c_1)$ there exists a traveling wave $W(\theta, x)$ satisfying $Q^n(W) = W(\cdot, x - nc)$. By (2.8), we understand that the asymptotic behavior of W at infinity is either $\sim C_1e^{-\mu_1(c)x}$ or $\sim C_2e^{-\mu_2(c)x}$ for some positive vectors C_1 and C_2. Since $\mu_1(c)$ is an decreasing function and $\mu_2(c)$ is a increasing function in c for $c \geq c_0$, we conclude (after a shift of the variable x in W) that $W \geq V$ holds. In view of the monotonicity, we have $Q^n(W) \geq Q^n(V)$. By (2.35), it follows that

$$W(\theta, x - nc) \geq V(\theta, x - nc_1). \tag{2.36}$$

Fix $\xi_1 = x - nc_1$ so that $V(\theta, x - nc_1) = V(\theta, \xi_1) \gg 0$. However,

$$W(\theta, x - cn) = W(\theta, \xi_1 + (c_1 - c)n) \sim W(\theta, \infty) = 0 \text{ as } n \to \infty.$$

By the comparison in (2.36), it implies a contradiction. Therefore, no traveling waves exist for $c \in (c_0, c_1)$, which also implies that there is no traveling wave for $c = c_0$ by Lemma 2.2. Hence, the statement of the theorem is true. The proof is complete. ∎

Corollary 2.9. *Let Q satisfy (A1)-(A6) with c^* being its spreading speed and M satisfy (M1)-(M6) with the linear speed c_0 defined in (2.7). Assume that $V = \frac{\beta}{1+\zeta_{\bar{\mu}}(\theta)e^{\bar{\mu}x}}$ is a strongly-strict lower solution of the wave profile equation, i.e.,*

$$Q(V) \gg V(\theta, x - c_0).$$

Then nonlinear selection is realized.

Proof. By the continuity, it is easy to see there exists a constant number c_1 sufficiently close to c_0 and a vector β_0 sufficiently close to β so that $V = \frac{\beta_0}{1+\zeta_{\mu_2(c_1)}(\theta)e^{\mu_2(c_1)x}}$ is a lower solution. Hence, the result directly follows from Theorem 2.8. ∎

Now we directly extend the above results to time-continuous semiflow. Let Q_t satisfy all the assumptions (A1)-(A6) and M_t, satisfying (M1)-(M6), be the linearization of Q_t at zero in the sense of (1.9). Denote $Q = Q_1$ and $M = M_1$. Let c^* be the spreading speed of Q and c_0 be the linear speed of M defined by (2.7). Similarly, we have the following results.

Theorem 2.10 (Linear Selection)**.** *Assume that there exists a function $U(\theta, x) \geq 0$ satisfying*

$$\liminf_{x\to-\infty} U(\theta, x) \gg 0, \quad \lim_{x\to\infty} U(\theta, x) = 0$$

and

$$Q_t(U) \leq U(\theta, x - c_0 t). \tag{2.37}$$

Then the linear selection is realized.

Corollary 2.11. *Suppose that $U = e^{-\bar{\mu}x}\zeta_{\bar{\mu}}(\theta)$ is an upper solution of the wave profile equation $Q_t(U) = U(\theta, x - c_0 t)$, i.e.,*

$$Q_t\big(e^{-\bar{\mu}x}\zeta_{\bar{\mu}}(\theta)\big) \leq e^{-\bar{\mu}(x-c_0 t)}\zeta_{\bar{\mu}}(\theta). \tag{2.38}$$

Then the linear selection is realized.

Corollary 2.12. *Suppose that $U = \frac{\beta}{1+\zeta_{\bar{\mu}}(\theta)e^{\bar{\mu}x}}$ is an upper solution of the wave profile equation, i.e., $Q_t(U) \leq U(\theta, x - c_0 t)$. Then the linear selection is realized.*

Theorem 2.13 (Nonlinear selection)**.** *For $c_1 > c_0$, if there exists a function $V(\theta, x)$ satisfying*

$$0 \ll V(\theta, x) \ll \beta, \quad \limsup_{x\to-\infty} V(\theta, x) \ll \beta, \quad V(\theta, x) = \zeta_{\mu_2(c_1)}(\theta)e^{-\mu_2(c_1)x} \text{ as } x \to \infty \tag{2.39}$$

and

$$Q_t(V(\theta, x)) \geq V(\theta, x - c_1 t), \tag{2.40}$$

then $c^* \geq c_1$ and NO traveling wave exists for $c \in [c_0, c_1)$. In other words, the nonlinear selection is realized.

Corollary 2.14. *Assume that $V = \frac{\beta}{1+\zeta_{\bar{\mu}}(\theta)e^{\bar{\mu}x}}$ is a strongly-strict lower solution of the wave profile equation, i.e.,*

$$Q_t(V) \gg V(\theta, x - c_0 t).$$

Then the nonlinear selection is realized.

3 Application to delayed reaction-diffusion equations

3.1 A reaction-diffusion equation with discrete delay

Consider a time-delayed reaction-diffusion equation

$$u_t(t,x) = u_{xx}(t,x) + f(u(t,x), u(t-\tau,x)), \tag{3.1}$$

where $\tau > 0$ and f satisfies

$$f(r,s) \geq 0, \quad \partial f_2(r,s) \geq 0, \quad \text{for} \;\; 0 \leq r,s \leq 1, \tag{3.2}$$

$$f(0,0) = f(1,1) = 0, \quad f(r,r) > 0 \;\; \text{for } 0 < r < 1, \tag{3.3}$$

and

$$\partial f_1(0,0) + \partial f_2(0,0) > 0. \tag{3.4}$$

This system was originally studied by Schaaf in [35] where asymptotical behavior and traveling wave solutions for a general KPP or a bistable nonlinear system were investigated.

To apply our results, we first verify that all the conditions (A1)-(A6), (M1)-(M6) are satisfied. Indeed, (3.1) is a special form of (5.2) in [21]. Under the conditions (3.2)-(3.4), we can see from section 5.1 in [21] that all the conditions are satisfied. In particular, the linear operator M is strongly positive due to the property of the heat kernel.

Based on the monotonicity of system (3.1), by the main result in [21], it follows that there exists a constant $c_{\min}$ such that for $c \geq c_{\min}$, (3.1) has a traveling wave solution $u(t,x) = U(x-ct)$ satisfying

$$U'' + cU' + f(U, U(z+c\tau)) = 0, \tag{3.5}$$

and

$$U(+\infty) = 1, \quad U(-\infty) = 0. \tag{3.6}$$

However, no information on $c_{\min}$ was mentioned if the nonlinear function f is not dominated by its linear part.

To further understand the mechanism of the speed selection on the minimal speed, we linearize (3.5) at zero solution to get

$$U'' + cU' + AU + BU(z+c\tau) = 0, \; A = \partial f_1(0,0), \; B = \partial f_2(0,0). \tag{3.7}$$

Then the linear speed c_0 is determined by

$$c_0 := \inf\{c : h(\mu) = \mu^2 - c\mu + A + Be^{-\mu c\tau} = 0 \;\; \text{has a positive real solution}\}. \tag{3.8}$$

It is known that $h(\mu)$ has two positive zero points $\mu_1(c)$ and $\mu_2(c)$, with $\mu_1(c) < \mu_2(c)$, if $c > c_0$. When $c = c_0$, $h(\mu)$ has a unique positive zero point $\bar{\mu}$ so that $\mu_1(c) = \mu_2(c) = \bar{\mu}$.

Now we want to obtain a linear selection mechanism of the minimal wave speed of (3.1) by applying Theorem 2.10.

Theorem 3.1. *Let* $U(z) = \frac{1}{1+e^{\bar{\mu}z}}$. *If*

$$-2\bar{\mu}^2 + \frac{f(U, U(z+c_0\tau)) - AU(1-U) - Be^{-\bar{\mu}c_0\tau}U(1-U)}{U^2(1-U)} \leq 0, \tag{3.9}$$

then the linear selection of the minimal wave speed of (3.1) is realized.

Proof. Making use of the fact that

$$U' = -\bar{\mu}U(1-U), \;\; U'' = \bar{\mu}^2 U(1-U)(1-2U), \tag{3.10}$$

at $c = c_0$ we obtain

$$\begin{aligned} & U'' + c_0 U' + f(U, U(z+c_0\tau)) \\ &= \bar{\mu}^2 U(1-U)(1-2U) - \bar{\mu}U(1-U) + f(U, U(z+c_0\tau)) \\ &= U^2(1-U)\big(-2\bar{\mu}^2 + \tfrac{f(U,U(z+c_0\tau)) - AU(1-U) - Be^{-\bar{\mu}c_0\tau}U(1-U)}{U^2(1-U)}\big). \end{aligned} \tag{3.11}$$

From (3.9), it follows that U is an upper solution of (3.5). Thus $u_t(\theta, x) = u(t+\theta, x) = U(x - c_0(t+\theta))$ becomes an upper solution of the original equation (3.1) with initial condition $u(\theta, x) = U(x - c_0\theta)$. Let

$$Q_t(\phi)(\theta, x) = u_t(\theta, x, \phi)$$

be the solution semi-flow of (3.1). $Q_t(U)$ is the solution of (3.1) with $\phi = U(x - c_0\theta)$. By the comparison principle, this gives that for $t > 0$,

$$Q_t(U) \leq U(x - c_0(t+\theta)).$$

Therefore, (2.37) in Theorem 2.10 is satisfied. By Theorem 2.10, the desired result follows. ∎

Remark 3.1. *In the above proof, we show that the upper solution of the wave profile equation (3.5) always implies the existence of the upper solution of the solution semiflow in (2.37). This fact is also true for lower solutions. Later, for convenience, we will directly work on the finding of the upper solution/lower solutions of the wave profile equation, instead of on the finding of those solutions in (2.37) or (2.40) in terms of the semiflow.*

Remark 3.2. *When* $c = c_0$, *the function* $U = e^{-\bar{\mu}z}$ *is an upper solution of (3.5) provided that*

$$f(u, u(t-\tau, x)) \leq f_1(0,0)u + f_2(0,0)u(t-\tau, x), \; \textit{for } u \geq 0, \; u(t-\tau, x) \geq 0, \tag{3.12}$$

holds. By Theorem 2.10, the linear selection is realized. In other words, when f *is dominated by its linear part, the linear selection is realized.*

By Theorem 2.13, we can obtain the nonlinear selection mechanism of the minimal wave speed of (3.1).

Theorem 3.2. *Assume* $U = \frac{1}{1+e^{\mu_2(c_1)z}}$ *and*

$$-2\mu_2^2(c_1) + \min_{z\in(-\infty,+\infty)} \frac{f(U, U(z+c_1\tau)) - AU(1-U) - Be^{-\mu_2(c_1)c_1\tau}U(1-U)}{U^2(1-U)} > 0$$

for some $c_1 > c_0$. *Then the nonlinear selection of the minimal wave speed of (3.1) is realized. Furthermore, there exist no traveling waves for* $c \in [c_0, c_1)$.

Obviously, the delayed Hadeler and Rothe model

$$u_t = u_{xx} + u(t-\tau, x)(1-u(t,x))(1+au(t,x)), a \in [0,\infty) \tag{3.13}$$

is a particular form of (3.1). We now apply Theorems 3.1 and 3.2 to obtain its linear and nonlinear selection results of the minimal wave speed.

For $U = \frac{1}{1+e^{\mu z}}$, note that

$$\frac{e^{\mu c\tau}-1}{e^{2\mu c\tau}}U^2 \leq U(z+c\tau) - e^{-\mu c\tau}U \leq \frac{e^{\mu c\tau}-1}{e^{\mu c\tau}}U^2.$$

By using Theorems 3.1 and 3.2, we know that if the following condition

$$-2\bar{\mu}^2 + 2a + \frac{e^{\bar{\mu}c_0\tau}-1}{e^{\bar{\mu}c_0\tau}} \leq 0, \ i.e., \ a \leq \bar{\mu}^2 - \frac{e^{\bar{\mu}c_0\tau}-1}{2e^{\bar{\mu}c_0\tau}}$$

is satisfied, then the linear selection of the minimal wave speed is realized. Similarly, we can obtain that the nonlinear selection is realized if $a > \bar{\mu}^2 - \frac{e^{\bar{\mu}c_0\tau}-1}{2e^{2\bar{\mu}c_0\tau}}$.

Since the system is monotonic in a in the sense that if the linear selection is realized for the system (3.13) when $a = a_\beta$, then the linear selection is realized for all $a \leq a_\beta$, we can have a threshold value $a = a_c$ so that the following theorem is true.

Theorem 3.3. *There exists a critical value a_c such that the minimal wave speed of (3.13) is linearly selected for $a \leq a_c$ and nonlinearly selected for $a > a_c$. Furthermore, we have the estimate*

$$\bar{\mu}^2 - \frac{e^{\bar{\mu}c_0\tau}-1}{2e^{\bar{\mu}c_0\tau}} \leq a_c \leq \bar{\mu}^2 - \frac{e^{\bar{\mu}c_0\tau}-1}{2e^{2\bar{\mu}c_0\tau}}.$$

Remark 3.3. *We now find that Theorem 2.7 in [35] is misleading with a statement that the minimal speed is always linearly selected for its model. Indeed, when the nonlinearity term $f(w(t,x), w(t-\tau,x))$ in (2.1) of [35] is taken as $w(t-\tau,x)(1-w(t,x))(1+aw(t,x)), a \in [0,\infty)$, all the conditions in section 2.1 of [35] are satisfied. Theorem 2.7 claimed that the minimal speed is always equal to the linear speed for all a. However, this is not true in view of our Theorem 3.3.*

3.2 A reaction-diffusion system with a distributed delay

Consider a nonlocal reaction diffusion model

$$u_t = Du_{xx} - d_1 u + e^{-\int_0^\tau d_2(a)da} \int_{-\infty}^{+\infty} f(x-y)b(u(t-\tau,y))dy, \ \ \tau > 0, \tag{3.14}$$

where $f(x) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{x^2}{4\alpha}}$ and we set $k = e^{-\int_0^\tau d_2(a)da}$. As an illustration, we assume that $d_1 u = kb(u)$ has two non-negative zeros $u_1 = 0, u_2 = K > 0$ and $b'(\eta) \geq 0$ for $\eta \in [0, K]$. The applications of this model can be seen in [26, 28, 29] and references therein.

This is a special model of (5.3) in [21]. Therefore, all the conditions (A1)-(A6) and (M1)-(M6) are satisfied.

To study the traveling wave solutions, we let $u(t,x) = U(z), z = x - ct$. Then, by (3.14), we have the following equation for U

$$\begin{aligned} &DU'' + cU' - dU + k\textstyle\int_{-\infty}^{+\infty} f(z-s)b(U(s+c\tau))ds = 0, \\ &U(-\infty) = K, \ U(+\infty) = 0. \end{aligned} \tag{3.15}$$

Furthermore, in view of the linearized system around the equilibrium 0, we define

$$h(\mu) = D\mu^2 + c\mu - d + kb'(0)e^{-\mu c\tau + \alpha\mu^2}.$$

Then, the linear speed c_0 is determined by

$$c_0 := \inf\{c : h(\mu) \text{ has a positive zero point}\}. \tag{3.16}$$

Similarly, it is easily known that for $c > c_0$, $h(\mu)$ has two positive zeros denoted by $\mu_1(c)$ and $\mu_2(c)$ with $\mu_1(c) < \mu_2(c)$. When $c = c_0$, we have $\mu_1(c_0) = \mu_2(c_0)$ and we set $\overline{\mu} = \mu_1(c_0) = \mu_2(c_0)$.

Taking $U = \frac{K}{1+e^{\mu z}}$, at $\mu = \overline{\mu}$ we have

$$U' = -\bar{\mu}U(1 - \frac{U}{K}) \text{ and } U'' = \bar{\mu}^2(1 - \frac{2U}{K})U(1 - \frac{U}{K}).$$

Thus, by (3.15), we have

$$DU^{''} + cU^{'} - dU + k\int_{-\infty}^{+\infty} f(z-s)b(U(s+c\tau))ds = (1 - \frac{U}{K})\frac{U^2}{K}\Phi(\bar{\mu}, z), \tag{3.17}$$

where

$$\Phi(\bar{\mu}, z) = -2D\bar{\mu}^2$$
$$+\frac{k}{U(1-\frac{U}{K})\frac{U}{K}} \int_{-\infty}^{+\infty} f(s)\left[b(U(z-s+c\tau) - \frac{d}{K}U^2(z)) - b'(0)e^{-\bar{\mu}(-s+c\tau)}U(z)(1 - \frac{U(z)}{K})\right]ds.$$

Then by applying Theorems 2.10 and 2.13, we have the following result.

Theorem 3.4. *If $\Phi(\overline{\mu}, z) \le 0$ for $z \in (-\infty + \infty)$, then the linear selection of the minimal wave speed for (3.14) is realized. If $\Phi(\mu_2(c_1), z) > 0$, $z \in (-\infty + \infty)$ for some $c_1 > c_0$, then the minimal wave speed for (3.14) is nonlinearly selected.*

4 Application to discrete systems

4.1 A lattice system

Consider a lattice system

$$x_i'(t) = d(x_{i-1} - 2x_i + x_{i+1}) + f(x_i), \ \ i = 0, \pm 1, \pm 2, \cdots, \tag{4.1}$$

where f is a differentiable function satisfying $f(0) = f(1) = 0$, $f(u) > 0$ for $u \in (0, 1)$ and $f'(0) > 0$. This system can be also thought as a space-discretized Fisher-KPP equation.

Due to the lack of compactness of the solution semiflow of (4.1), we first consider the followed perturbed model

$$x_i'(t) = d(x_{i-1} - 2x_i + x_{i+1}) + f(x_i) + \frac{\epsilon}{2\pi}\sum_{k=-\infty}^{\infty} \beta_\alpha(i-k)f(x_i), \ \ i = 0, \pm 1, \pm 2, \cdots, \tag{4.2}$$

where

$$\beta_\alpha(l) = 2e^{-v}\int_0^\pi \cos(lw)e^{v\cos w}dw, \tag{4.3}$$

ϵ is small and $v = 2\alpha$. This is a similar model of (5.7) in [21]. Following the same argument as in Proposition 5.2 of [21], we can verify that the semi-flow Q_t^ϵ satisfies all the condition of (A1)-(A6) and (M1)-(M6). In particular, the solution semi-flow Q_t^ϵ is also continuous in ϵ with the speeds $c_{\min}^\epsilon$ and c_0^ϵ to be uniformly bounded for $\epsilon \in [0,1]$. By way of limiting argument coupled with the dominated convergence theorem, it can be easy to verify

$$\lim_{\epsilon\to 0} c_{\min}^\epsilon = c_{\min}, \quad \text{and } \lim_{\epsilon\to 0} c_0^\epsilon = c_0,$$

where $c_{\min}$ is the minimal speed of traveling waves of the model (4.1) and c_0 is the linear speed of it. All the results in our theorem hold when we put $\epsilon \to 0$.

To be exact, for a traveling wave of (4.1), we set $x_i = U(z), z = i - ct$; then it follows from (4.1) that

$$d\left[U(z-1) - 2U(z) + U(z+1)\right] + cU' + f(U) = 0 \tag{4.4}$$

subject to

$$U(-\infty) = 1, U(+\infty) = 0.$$

The linearized system of (4.4) at 0 gives a characteristic equation as

$$d(e^\mu - 2 + e^{-\mu}) - c\mu + f'(0) = 0. \tag{4.5}$$

Let

$$F(\mu) = d(e^\mu - 2 + e^{-\mu}) - c\mu + f'(0).$$

It is easy to see that

$$F(0) = f'(0) > 0, \qquad F''(\mu) > 0$$

and

$$F'(\mu) = d(e^\mu - e^{-\mu}) - c.$$

The linear speed c_0 is determined by the system of $F(\mu) = 0$ and $F'(\mu) = 0$. Indeed, there exist c_0 and $\bar\mu$ so that $F(\bar\mu) = 0$ for $c = c_0$; for $c > c_0$, there exist two positive zeros $\mu_1(c)$ and $\mu_2(c)$, $\mu_1(c) < \mu_2(c)$, satisfying $F(\mu_1) = F(\mu_2) = 0$. When $c = c_0$, it also follows that $\mu_1(c) = \mu_2(c) = \bar\mu$.

Assume $c_{\min}$ is the minimal speed for (4.4). Similarly as before, the function $U = e^{-\bar\mu z}$ is an upper solution of (4.4), if $f(x) \le f'(0)x, x \ge 0$. Thus, we have $c_{\min} = c_0$. This is the conventional result on the linear selection.

To have a better and sharp result, now we choose $U = \frac{1}{1+e^{\mu z}}$, where μ satisfies $F(\mu) = 0$. In view of $U' = -\mu U(1-U)$, we get

$$\begin{aligned} &d(U(z-1) - 2U(z) + U(z+1))) + cU' + f(U) \\ &= dU^2(1-U)\left\{\tfrac{U(z-1)-2U(z)+U(z+1)}{U^2(1-U)} - \tfrac{(e^\mu-2+e^{-\mu})}{U} + \tfrac{f(U)-f'(0)U(1-U)}{dU^2(1-U)}\right\}. \end{aligned} \tag{4.6}$$

Set

$$G(z) = -\left[\frac{U(z-1) - 2U(z) + U(z+1)}{U^2(1-U)} - \frac{(e^{\bar\mu} - 2 + e^{-\bar\mu})}{U}\right],$$

where $U = \frac{1}{1+e^{\bar{\mu} z}}$. Assume m_1 and m_2 are the minimal and maximal values of $G(z)$ in the interval $(-\infty, \infty)$, i.e.,

$$m_1 = \min_{z\in(-\infty,\infty)} G(z),\ m_2 = \max_{z\in(-\infty,\infty)} G(z). \tag{4.7}$$

By Corollaries 2.11 and 2.14, we have the following theorem.

Theorem 4.1. *The minimal wave speed of the lattice system (4.4) is linearly realized provided that*

$$\frac{f(U) - f'(0)U(1-U)}{(1-U)U^2} \leq dm_1, \quad U \in (0,1).$$

On the other side, the minimal wave speed of (4.4) is nonlinearly realized provided that

$$\frac{f(U) - f'(0)U(1-U)}{(1-U)U^2} > dm_2, \quad U \in (0,1).$$

4.2 A scalar Integro-difference equation

We consider an integro-difference equation

$$u_{n+1}(x) = \int_{\mathbb{R}} f(u_n(y))k(x-y)dy,\ \ x \in \mathbb{R},\ n \geq 0, \tag{4.8}$$

where $u_0(x)$ is a bounded and continuous function from $\mathbb{R}$ to $\mathbb{R}$. $k(x)$ is a positive Lebesque measurable function satisfying $\int_{\mathbb{R}} k(y)dy = 1$ and $\int_{\mathbb{R}} e^{\mu y}k(y)dy < \infty$ for positive constant μ. For the application of this model, see [16]. Here we assume that $u = f(u)$ has two fixed point $u = 0$ and $u = K$, and that $f(u)$ is positive and monotonic for $u \in (0, K)$. The kernel function k is assumed to satisfy

$$\lim_{s\to 0} \int_{-\infty}^{\infty} |k(x+s) - k(x)|dx = 0$$

which implies that solution semi-flow Q[u] defined by (4.8) is equi-continuous. Based on the monotonicity of f, it is straightforward to verify that all the conditions of (A1)-(A6) and (M1)-(M6) are satisfied and we omit the details here. Therefore, it is known that there exists a constant $c_{\min}$ such that for $c \geq c_{min}$, the equation (4.8) has a traveling wave solution $u(n, x) = U(x - cn)$ where U satisfies

$$U(x - c(n+1)) = \int_{\mathbb{R}} f(U(y-cn))k(x-y)dy, \qquad x - cn = z \tag{4.9}$$

or

$$\begin{aligned} &U(z-c) = \textstyle\int_{\mathbb{R}} f(U(y))k(z-y)dy = \int_{\mathbb{R}} f(U(z-y))k(y)dy, \\ &U(-\infty) = K,\ U(+\infty) = 0. \end{aligned} \tag{4.10}$$

Let $z - c = s$. We can also get

$$U(s) = \int_{\mathbb{R}} f(U(y))k(s+c-y)dy. \tag{4.11}$$

For the derivation of the linear speed, we may replace the above formula by inserting $f(U) = f'(0)U$ and work out the characteristic equation as

$$F(\mu) = \int_{\mathbb{R}} f'(0)e^{\mu(y-c)}k(y)dy - 1 = 0. \tag{4.12}$$

Assume that

$$F(0) = \int_{\mathbb{R}} f'(0)k(y)dy - 1 = f'(0) - 1 > 0.$$

It is easy to derive that there exists a constant $c = c_0$ so that the equation (4.12) has two positive solutions $\mu_1(c)$ and $\mu_2(c)$ (we might as well set $\mu_1 < \mu_2$) for $c > c_0$, one solution $\bar{\mu} = \mu_1(c_0) = \mu_2(c_0)$ for $c = c_0$ and no real solution for $c < c_0$. Indeed, the constant c_0 can be determined by the system $F(\mu) = 0$ and $F'(\mu) = 0$.

Now we concentrate on the speed selection for the nonlinear equation (4.10). Similarly as before, it is easy to have the linear selection if $f(u) \le f'(0)u, u \in (0, 1)$. To improve this result, we choose a testing function as $U = \frac{K}{1+e^{\mu z}}$. Then we have

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}} f(U(y))k(z-y)dy - U(z-c)\\
&= \textstyle\int_{\mathbb{R}} f(\frac{K}{1+e^{\mu(y)}})k(z-y)dy - \frac{K}{1+e^{\mu(z-c)}}\\
&= \textstyle\int_{\mathbb{R}} [f(\frac{K}{1+e^{\mu(y)}}) - \frac{K}{1+e^{\mu(z-c)}}]k(z-y)dy\\
&= \textstyle\int_{\mathbb{R}} [f'(0)\frac{K}{1+e^{\mu(y)}} - \frac{K}{1+e^{\mu(z-c)}}]k(z-y)dy + \int_{\mathbb{R}} [f(\frac{K}{1+e^{\mu(y)}}) - f'(0)\frac{K}{1+e^{\mu(z-y)}}]k(z-y)dy.
\end{aligned} \tag{4.13}$$

Using (4.12), the first term can be estimated as

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}} [f'(0)\frac{K}{1+e^{\mu y}} - \frac{K}{1+e^{\mu(z-c)}}]k(z-y)dy\\
&= -K\textstyle\int_{\mathbb{R}} \frac{f'(0)e^{-2\mu y}}{e^{-\mu y}+1}k(z-y)dy + K\int_{\mathbb{R}} \frac{e^{-2\mu(z-c)}}{e^{-\mu(z-c)}+1}k(z-y)dy.
\end{aligned} \tag{4.14}$$

Therefore, we have

$$\int_{\mathbb{R}} f(U)k(z-y)dy - U(z-c) = \Psi(\mu, z), \tag{4.15}$$

where

$$\begin{aligned}
&\Psi(\mu, z)\\
&= K\textstyle\int_{\mathbb{R}} [\frac{e^{-2\mu(z-c)}}{e^{-\mu(z-c)}+1} - \frac{f'(0)e^{-2\mu y}}{e^{-\mu y}+1}]k(z-y)dy + \int_{\mathbb{R}} [f(\frac{K}{1+e^{\mu y}}) - f'(0)\frac{K}{1+e^{\mu y}}]k(z-y)dy.
\end{aligned}$$

Therefore, the application of Corollaries 2.7 and 2.9 leads to the following theorem.

Theorem 4.2. *The following statements are true:* (1) *If $\Psi(\overline{\mu}, z) \le 0$ for $z \in (-\infty, \infty)$, then the minimal wave speed for (4.8) is linearly selective.* (2) *If $\Psi(\overline{\mu}, z) > 0$ for $z \in (-\infty, \infty)$, then the minimal wave speed for (4.8) is nonlinearly selective.*

5 Application to a cooperative system

Our previous applications are all on scalar equations. In this section, we demonstrate our application to a monotonic coupled system with two equilibria.

Consider the following cooperative system

$$\begin{cases} u_t = u_{xx} + u(1-u) + auv(1-u), \\ v_t = dv_{xx} + r(u-v), \end{cases} \tag{5.1}$$

where u and v represent population density of two species, or two stages of one species, a, d, r are positive constants. It is easy to verify that all the conditions (A1)-(A6) and (M1)-(M6) are satisfied when $d \leq 1$, see also section 4 in [39]. As in [39], by substituting $u = \alpha_1 e^{-\mu x}, v = \alpha_2 e^{-\mu x}$ into the right-hand side of the linearization of (5.1) at zero and setting $x = 0$, we can define a matrix C_μ as

$$C_\mu = \begin{pmatrix} \mu^2 + 1 & 0 \\ r & d\mu^2 - r \end{pmatrix}. \tag{5.2}$$

This gives $B_\mu = e^{C_\mu}$. Thus the principal eigenvalue of B_μ becomes e^{μ^2+1} and the relation between c and μ is given by

$$c = \frac{\mu^2 + 1}{\mu}. \tag{5.3}$$

Therefore, the linear speed is $c_0 = 2$ with $\bar{\mu} = 1$. When $c > c_0$, equation (5.3) has two solutions

$$\mu_1(c) = \frac{c - \sqrt{c^2 - 4}}{2}, \text{ and } \mu_2(c) = \frac{c + \sqrt{c^2 - 4}}{2}. \tag{5.4}$$

By a traveling wavefront to (5.1), we mean a special solution $u(t, x) = U(z), v(t, x) = V(z)$, $z = x - ct$, where c is the wave speed. After substitution, we obtain the wave profile equation given by

$$\begin{cases} U'' + cU' + U(1 - U) + aUV(1 - U) = 0, \\ dV'' + cU' + r(U - V) = 0. \end{cases} \tag{5.5}$$

To obtain the speed selection, we want to find suitable upper/lower solutions for the wave profile system (5.5). To proceed, we have the following theorem.

Theorem 5.1. *Assume that $d \leq 1$. The minimal speed of traveling wavefronts to (5.1) is linearly selected if $a < 2$, and non-linearly selected if*

$$\frac{2}{a} < \frac{r}{r + d + 2}. \tag{5.6}$$

Proof. For $c = c_0 = 2$, we first construct the upper solution as

$$\bar{U}(z) = \frac{1}{1 + e^z}, \text{ and } \bar{V}(z) = \bar{U}(z). \tag{5.7}$$

By a straightforward calculation with the use of $\bar{U}'(z) = -\bar{U}(1-\bar{U})$ and $\bar{U}''(z) = \bar{U}(1-\bar{U})(1-2\bar{U}(z))$, we can verify that (5.7) is an upper solution of (5.5) if $a < 2$.

On the other hand, for $c = c_0 + \varepsilon$, where ε is a small positive number, set a lower solution as

$$\underline{U}(z) = \frac{1}{1 + e^{\mu_2(c)z}}, \text{ and } \underline{V}(z) = k\underline{U}(z) \tag{5.8}$$

where k is a constant satisfying

$$\frac{2}{a} < k < \frac{r}{r + d + 2}. \tag{5.9}$$

Again using the fact that $\underline{U}'(z) = -\mu_2(c)\underline{U}(1 - \underline{U})$, we can verify that (5.8) is a lower solution to the wavefront profile (5.5) under the condition (5.9), as long as ε is sufficiently small.

Thus, by applying Theorem 2.5 and Theorem 2.8, we obatin our result. The proof is complete. ∎

Remark 5.1. *When $d > 1$, the spreading speed of this model is well defined. However, the matrix C_μ (also B_μ) has no strongly positive eigenvector for some μ. Alternatively we still can follow the idea of Remark 2.3 to define the linear speed c_0 by the first block of C_μ. The speed selection problem could in principle be similarly studied. This will leave to interested readers.*

6 Conclusion and discussion

In this work, we investigate the selection mechanism of the minimal wave speed for traveling waves to an abstract monotone semiflow. We successfully improve the known conventional results on the linear selection and make a breakthrough at the study of nonlinear selection. A necessary and sufficient condition for the nonlinear selection of the minimal wave speed is established. Using it as a milestone, we derive explicit conditions for the realization of linear or nonlinear selection by applying the comparison principle. Our results on nonlinear selection are new and novel, and for the linear selection, we improve the condition that the monotone semiflow is dominated by its linear map.

We also apply our results to some classical biological models including reaction-diffusion models with delay interactions, a lattice system, a scalar integro-difference equation and a cooperative system. Compared with the references, a series of new results are obtained. Definitely, the idea, method and results should in principle be easily extended and applied to various biological systems that model the spread of infectious diseases such as Lyme and Malaria, see [22] and [40].

We should mention that in our paper, we assume that there are only two fixed points 0 and β for the map Q. This is not essential in our method and idea, and it can be interestingly and non-trivially developed to study the case when the map Q admits more than two fixed points with at least one or more fixed points on the boundary of the positive cone. In this case, due to the existence of other equilibria, say $\alpha_i, i = 1, 2, \cdots, n$, between 0 and β, the classical upper/lower solution defined in $\mathcal{C}_\beta$ cannot guarantee the existence of traveling wave front with the connection just on 0 and β. In other words, we also need to figure out all possible connections between 0 and α_i, α_i and $\alpha_j, i, j = 1, 2, \cdots, n$, and between α_i and β. Even for the spreading speed, we have the following definitions

$$c^* := \sup\{c : a(c; \cdot, +\infty) = \beta\} \text{ and } c^*_+ := \sup\{c : a(c; \cdot, +\infty) \neq 0\} \tag{6.1}$$

with $c^*_+ \geq c^*$, see the reference [39]. If $c^*_+ = c^*$, the system is called with a single speed.

For example, let us consider the well-known Lotka-Volterra model

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = dv_{xx} + r(1 - v)(a_2 u - v), \end{cases} \tag{6.2}$$

where a_1, a_2 and d are positive constants. Under the condition

$$0 < a_1 < 1 < a_2, \tag{6.3}$$

the system (6.2) has three equilibrium solutions $e_0 := (0, 0)$, $e_1 := (1, 1)$, and $e_2 := (0, 1)$. Researchers are interested in the existence of traveling waves

$$(u, v)(t, x) = (U, V)(z), \ z = x - ct$$

connecting e_1 and e_0. In other words, one is particularly concerned with the existence of $U(z)$ and $V(z)$ that satisfy

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) = 0, \\ dV'' + cV' + r(1 - V)(a_2 U - V) = 0 \end{cases} \tag{6.4}$$

subject to

$$(U, V)(-\infty) = e_1, \quad (U, V)(\infty) = e_0. \tag{6.5}$$

The linear speed of this model is given by

$$c_0 = 2\sqrt{1 - a_1},$$

see [13]. Let $c_{\min}$ be the minimal speed so that (6.4)-(6.5) has a solution. Based on his numerical simulations, in 1998, Hosono raised the following conjecture.

Conjecture 6.1. *If $a_1 a_2 \le 1$, then $c_{\min} = c_0$ for all $r > 0$. If $a_1 a_2 > 1$, then there exits a positive number r_c such that $c_{\min} = c_0$ for $0 < r \le r_c$, and $c_{\min} > c_0$ for $r > r_c$.*

This conjecture remains outstanding since 1998, except that some partial results were provided in [19] and [14, 15], see also [9, 10, 12]. In [9], the authors addressed Conjecture 4 of Hosono [13] by way of upper/lower solutions coupled with geometric singular perturbation arguments. It gave an example of a pushed front which invades slower than the linearized dynamics predict. In [10], Holzer and Scheel rigorously derived the linear spreading speed c_0. The reference [12] demonstrated how coupling may or may not increase the invasion speed of one of the components of the reaction diffusion systems. We will develop our method to work on the Hosono's conjecture in a separate paper.

Similarly as in above discussion, the method in this work could also be developed to study the multi-stage invasion, see e.g., [11, 30] and recent work of Bayliss and Volpert [36].

In the derivation of the spreading speed in our paper by following the idea of [23] and [37], the initial data are assumed to be non-increasing and have a compact support near infinity. It will be interesting to study the invasion speed dependence/convergence on the decay behavior of the initial data. For instance, for the classical one-dimensional reaction diffusion equation

$$u_t = du_{xx} + f(u), \tag{6.6}$$

where $d > 0$, $f(0) = f(1) = 0$, there have been some studies in this direction. In the KPP-Fisher case when $d = 1$ and $f(u) = u(1 - u)$, the minimal speed is linearly selected with $c^* = c_0 = 2$. Consider the initial data as

$$u(0, x) = Ae^{-ax}, \tag{6.7}$$

where $a > 0$ and $A > 0$. For $d = 1$, the asymptotic speed c (or spreading speed) of (6.6) is given by

$$c = a + \frac{1}{a},\ 0 < a < 1;\ \ c = 2,\ a \ge 1, \tag{6.8}$$

see e.g. the analysis in [27] and the proof in [25] and [18]. In the case when the nonlinear selection is realized, the situation is a bit different. In [31], Rothe studied the convergence to the traveling wave with the minimal speed to be nonlinearly selective. His result showed that if the initial data has the above behavior (6.7), the solution $u(x, t)$ will stabilize exponentially to the traveling wave with the minimal speed $c = c^*$ if

$$a > \mu_2(c^*) = \frac{c^* + \sqrt{(c^*)^2 - 4}}{2}.$$

No result was shown in [31] if $a \le \mu_2(c^*)$. For both above results (in the linear and nonlinear selection cases), it will be extremely interesting to make an extension study to our abstract map Q. Alternatively, for $d = \frac{1}{2}$ and the KPP nonlinearity $f(u) = u(1-u)$ in (6.6) with a step initial function, Bramson in [6] studied the asymptotic formula of the level set $X_\epsilon(t)$ with a nice result

$$X_\epsilon(t) = 2^{\frac{1}{2}} t - 3 \cdot 2^{-\frac{3}{2}} \log t + O(1),$$

where $X_\epsilon(t)$ is defined by $u(t, X_\epsilon(t)) = \epsilon$. We wonder if this result can be extended to our abstract model and it will be part of our future study.

Finally, it is also of particular interest to extend the present study to general cases such as periodic monotone semiflow, semiflow with periodic habitat, semiflow with weak compactness and semiflow with unbounded delay or distributed delay, by using tools similar to the ones proposed herein. Efforts along these directions are currently in progress and will be presented in future publications.

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