Solution.

Last Name: **First name:** **Student ID:**

[10] 1. Find the Cosine series for the function $f(x) = x,\ 0 \le x < L$. Use the formula to evaluate $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$.

$$f(x) = x \sim \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}, \quad 0 \le x \le L$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{-4L}{2\pi^2}\frac{1}{n^2}\left(1-(-1)^n\right).$$

$$\therefore \quad x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{n\pi x}{L} + \cos\frac{3\pi x}{L}\Big/3^2 + \frac{\cos 5\pi x/L}{5^2} + \cdots\right)$$

for $0 \le x \le L$.

Let $x=0 \Rightarrow$ $$0 = \frac{L}{2} - \frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

2. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0, \qquad \frac{d\phi}{dx}(1) = 0,\ \frac{d\phi}{dx}(2) = 0.$$

[6] (a) Multiplying $1/x$ puts this into the Sturm-Liouville form. Show that $\lambda \geq 0$.

[6] (b) Determine all eigenvalues. Is $\lambda = 0$ an eigenvalue?

(a) $\times\frac{1}{x}$: $\quad x\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} + \lambda\frac{1}{x}\phi = 0, \quad 1<x<2$

$\frac{d}{d}\left(x\frac{d\phi}{dx}\right) + \lambda\frac{1}{x}\phi = 0.$ S-L system with

$p=x,\ q=0,\ \sigma=\frac{1}{x}$

$$\lambda = \frac{-p\phi'\phi\big|_1^2 + \int_1^2 x(\phi')^2 dx}{\int_1^2 \phi^2\cdot\frac{1}{x}dx} = \frac{\int_1^2 x(\phi')^2 dx}{\int_1^2 \phi^2\cdot\frac{1}{x}dx} \geq 0.$$

(b) Set $\phi = x^r$. then $r(r-1)x^r + rx^r + \lambda x^r = 0$

$r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}\, i$

$\phi(x) = x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln x} \Rightarrow \phi = c_1\cos(\sqrt{\lambda}\ln x) + c_2\sin(\sqrt{\lambda}\ln x)$

$\frac{d\phi}{dx} = -c_1\sin(\sqrt{\lambda}\ln x)\cdot\frac{\sqrt{\lambda}}{x} + c_2\cos(\sqrt{\lambda}\ln x)\cdot\frac{\sqrt{\lambda}}{x}$

$\frac{d\phi}{dx}(1) = 0 \Rightarrow c_2 = 0,\quad \frac{d\phi}{dx}(2) = 0 \Rightarrow \ln 2\sqrt{\lambda} = n\pi,\ n=1,2,\cdots$

$\lambda = \left(\frac{n\pi}{\ln 2}\right)^2,\ n=1,2,3,\cdots,$

if $\lambda = 0 \Rightarrow \phi' = 0.$ we get $\phi = c_1\ (\neq 0)$
is an eigenfunction

$\therefore\ \lambda = 0$ is an eigenvalue.

[11] 3. Solve

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \; 0 < x < L, 0 < y < H$$

subject to $u(0,y,t) = 0, u(L,y,t) = g(y)$, $\frac{\partial u}{\partial y}(x,0,t) = 0$, $u(x,H,t) = 0$ and the initial condition $u(x,y,0) = f(x,y)$. Solve the problem and find the limit as $t \to \infty$.

Solve the equilibrium solution first:

$$\begin{cases} 0 = k\left(\frac{\partial^2 u_E}{\partial x^2} + \frac{\partial^2 u_E}{\partial y^2}\right) \\ u_E(0,y) = 0, \quad u_E(L,y) = g(y) \\ \frac{\partial u_E}{\partial y}(x,0) = 0, \quad u_E(x,H) = 0 \end{cases}$$

$$u_E = \sum_{n=1}^{\infty} C_n \cos\frac{(n-\frac{1}{2})\pi y}{H} \sinh\frac{(n-\frac{1}{2})\pi x}{H}$$

$$u_E(L,y) = g(y) \Rightarrow g(y) = \sum_{n=1}^{\infty} C_n \cos(n-\tfrac{1}{2})\pi y/H \, \sinh\frac{(n-\frac{1}{2})\pi L}{H}$$

$$C_n = \frac{\int_0^H g(y)\cos\frac{(n-\frac{1}{2})\pi y}{H}\,dy}{\frac{H}{2}\sinh\frac{(n-\frac{1}{2})\pi L}{H}}$$

Denote $V = u - u_E$. Then

$$\begin{cases} \frac{\partial V}{\partial t} = k(V_{xx} + V_{yy}) \\ V(0,y,t) = 0, \quad V(L,y,t) = 0 \\ \frac{\partial V}{\partial y}(x,0,t) = 0, \quad V(x,H,t) = 0 \end{cases}$$

$$V(x,y,0) = f(x,y) - u_E := h(x,y)$$

$$V = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \cos\frac{(m-\frac{1}{2})\pi y}{H} a_{nm} e^{-\lambda_{nm} kt}$$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{(m-\frac{1}{2})\pi}{H}\right)^2, \quad n = 1, 2, \cdots, \quad m = 1, 2, \cdots$$

$$a_{nm} = \frac{4}{LH}\int_0^H\int_0^L h(x,y)\sin\frac{n\pi x}{L}\cos\frac{(m-\frac{1}{2})\pi y}{H}\,dx\,dy$$

when $t \to \infty$, $e^{-\lambda kt} \to 0$, we have

$$\lim_{t\to\infty} u = u_E$$

[12] 4. Solve the problem $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \sin\frac{\pi x}{L}$ subject to $u(0,t) = A$, $u(L,t) = B$ and $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$, where A, B are constants.

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Solve $\begin{cases} 0 = c^2 \frac{d^2 u_E}{dx^2} + \sin\frac{\pi x}{L} \\ u_E(0) = A, \quad u_E(L) = B \end{cases}$

$$u_E = \frac{L^2}{c^2\pi^2}\sin\frac{\pi x}{L} + \frac{B-A}{L}x + A$$

$V = u - u_E$, $u = V + u_E$

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \\ V(0,t) = V(L,t) = 0 \\ V(x,0) = f(x) - u_E(x). \end{cases}$$

$$\frac{\partial v}{\partial t}(x,0) = g(x) - \left(\frac{L}{c^2\pi}\cos\frac{\pi x}{L} + \frac{B-A}{L}\right).$$

$V = h(t)\phi(x)$ $\quad \begin{cases} h'' = -\lambda c^2 h \\ \phi'' + \lambda\phi = 0 \end{cases}$

$$V = \sum_{n=1}^{\infty} \left[\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}\right.$$

$$V(x,0) = f(x) - u_E(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}.$$

$$A_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\frac{n\pi x}{L}\,dx$$

$$B_n = \frac{2}{n\pi c}\int_0^L \left(g(x) - \frac{du_E(x)}{dt}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$u = V + u_E$ is the solution.