MEMORIAL UNIVERSITY

DEPARTMENT OF MATHEMATICS

Solution

ST 1 | Math 4160

Last Name: | First name: | **Student ID:**

1. Determine the equilibrium temperature distribution for one dimensional rod with constant thermal properties with the following source and boundary conditions:

[8] (a) $Q = 0$, $u(0) = T$, $\frac{\partial u}{\partial x}(L) = \alpha$

[8] (b) $\frac{Q}{K_0} = x^2$, $u(0) = T$, $\frac{\partial u}{\partial x}(L) = 0$

Heat equation $\frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + Q$

(a) $Q = 0$. equilibrium : $\begin{cases} \frac{d^2 u}{dx^2} + 0 = 0 \\ u(0) = T, \; u'(L) = \alpha \end{cases}$

$u = c_1 x + c_2$, $\quad u'(L) = \alpha \Rightarrow c_1 = \alpha$;

$\quad u(0) = T \Rightarrow c_2 = T$

$u = \alpha x + T$

(b) $\frac{Q}{K_0} = x^2 \Rightarrow \begin{cases} u'' + x^2 = 0 \\ u(0) = T, \; u'(L) = 0 \end{cases}$

$u' = -\frac{1}{3}x^3 + c_1 \qquad u = -\frac{1}{12}x^4 + c_1 x + c_2$

$u(0) = T \Rightarrow c_2 = T$,

$u'(L) = 0 \Rightarrow c_1 = \frac{1}{3}L^3$

$u(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + T$.

[12] 2. Consider $\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \end{cases}$ subject the initial condition $u(x,0) = \begin{cases} 0, & 0 < x \le L/2 \\ 1, & L/2 < x < L. \end{cases}$

Find the solution.

Set $u = \phi(x)G(t)$ substitute : $\frac{G'}{kG} = \frac{\phi''}{\phi} = -\lambda$

$\Rightarrow \begin{cases} \phi''(x) + \lambda\phi(x) = 0 \\ \phi'(0) = 0, \phi'(L) = 0 \end{cases}$ $\begin{cases} \lambda = (\frac{n\pi}{L})^2, & n = 0, 1, 2, \cdots \\ \phi = \cos(\frac{n\pi}{L}x), & n = 0, 1, 2, \cdots \end{cases}$

$G(t) = c_1 e^{-\lambda_n k t}$

Superposition : $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \cdot e^{-k\cdot(\frac{n\pi}{L})^2 \cdot t}$

$u(x,0) = f(x) \Rightarrow \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x) = \begin{cases} 0, & 0 < x < L/2 \\ 1, & \frac{L}{2} < x < L. \end{cases}$

$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\left[\int_0^{L/2} 0 + \int_{L/2}^{L} 1\cdot dx\right] = \frac{1}{2}.$

$A_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[\int_0^{L/2} 0\,dx + \int_{L/2}^{L} 1\cdot\cos\frac{n\pi x}{L}\,dx\right], \quad n \ge 1$

$= \frac{2}{n\pi}\left[\sin n\pi - \sin\frac{n\pi}{2}\right]$

$= -\frac{2}{n\pi}\cdot\sin\frac{n\pi}{2}.$

$u(x,t) = \sum_{n=0}^{\infty} -\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{L}\, e^{-k(\frac{n\pi}{L})^2\cdot t}.$

[12] 2. Solve the laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, 0 < y < \overset{H}{\cancel{b}}$, with the following boundary condition

$$u(0,y) = 0,\ u(L,y) = 0,\ u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0,\ u(x,H) = f(x)$$

$u_{xx} + u_{yy} = 0$. set $u(x,y) = h(x)\,\phi(y)$ substitute:

$$-\frac{h''(x)}{h(x)} = \frac{\phi''(y)}{\phi(y)} = \lambda.$$

$$\begin{cases} h'' + \lambda h = 0 \\ h(0) = 0,\ h(L) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2,\ n = 1, 2, \cdots$$

$$h_n(x) = \sin\frac{n\pi}{L}x,\quad n = 1, 2, \cdots$$

$$\begin{cases} \phi'' - \lambda\phi = 0, & \lambda = \left(\frac{n\pi}{L}\right)^2 \\ \phi(0) - \phi'(0) = 0 \end{cases}$$

$$\phi(y) = C_1 \sinh\frac{n\pi y}{L} + C_2 \cosh\frac{n\pi y}{L}$$

$$\phi'(y) = C_1 \frac{n\pi}{L} \cosh\frac{n\pi y}{L} + C_2 \frac{n\pi}{L} \sinh\frac{n\pi y}{L}$$

$$\phi(0) - \phi'(0) = 0 \Rightarrow : \quad C_2 - C_1 \frac{n\pi}{L} = 0 \Rightarrow C_2 = \frac{n\pi}{L} C_1$$

$$\phi_n(y) = C_1 \left[\sinh\frac{n\pi y}{L} + \frac{n\pi}{L}\cosh\frac{n\pi y}{L}\right],\quad n = 1, 2, 3, \cdots$$

superposition:

$$u(x,y) = \sum_{n=1}^{\infty} a_n h_n(x)\,\phi_n(y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x \left[\sinh\frac{n\pi y}{L} + \frac{n\pi}{L}\cosh\frac{n\pi y}{L}\right]$$

Finally $u(x,H) = f(x) \Rightarrow \sum_{n=1}^{\infty} \underbrace{a_n \left[\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}\right]}_{= A_n} \sin\frac{n\pi y}{L} = f(x)$

$$A_n = a_n \left[\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}\right] = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$a_n = \frac{1}{\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}} \cdot \frac{2}{L} \cdot \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$