Solution

9.2.2. $c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K_0 \frac{\partial u}{\partial x}) + Q(x,t)$.

devide both side by $c\rho$, we have.

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho}\frac{\partial}{\partial x}(K_0 \frac{\partial u}{\partial x}) + \frac{Q(x,t)}{c\rho}$$

consider eigenvalue Problem

$$\begin{cases} \frac{d}{dx}(K_0(x)\frac{d\phi}{dx}) + \lambda c \rho \phi = 0 \\ \phi(0) = \phi(L) \end{cases}$$ so that $\int_{-\infty}^{+\infty} \phi^2(x)\, c\rho\, dx = 1$

Assume we know: $\lambda_1 < \lambda_2 < \cdots < \cdots$

$\phi_1, \phi_2, \cdots, \phi_n(x), \cdots,$

Suppose $\frac{Q(x,t)}{c\rho} = \sum_{n=1}^{\infty} g_n(t)\phi_n(x)$, $g_n(t) = \int_0^L \frac{Q(x,t)}{c\rho} \cdot \phi_n(x) \cdot c \cdot \rho \, dx$

$= \int_0^L Q(x_0,t)\, \phi_n(x_0)\, dx_0$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

After substitution, we have

$$\sum_{n=1}^{\infty} a_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} a_n(t)\cdot(-\lambda)\cdot\phi_n(x) + \sum_{n=1}^{\infty} g_n(t)\phi_n(x)$$

i.e., $a_n'(t) = -\lambda_n a_n(t) + g_n(t)$, $n = 1, 2, \cdots$

$$a_n(t) = e^{-\lambda_n t} a_n(0) + \int_0^t e^{-\lambda_n(t-s)} g_n(s)\, ds$$

where $a_n(0) = \int_0^L g(x_0)\, \phi_n(x_0)\, c(x_0)\, \rho(x_0)\, dx_0$

$$u = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\, a_n(0) + \sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-s)} g_n(s)\phi_n(x)$$

$$= \int_0^L g(x_0) \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x_0)\phi_n(x)\, c(x_0)\rho(x_0)\, dx_0$$

$$+ \int_0^L\int_0^t Q(x_0,t_0) \sum_{n=1}^{\infty} e^{-\lambda_n(t-t_0)} \phi_n(x)\phi_n(x_0)\, dx_0.$$

9.2.3 $\quad u(x,t)=\sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L}$

Hence, $\quad \frac{d^2 a_n}{dt^2}+\left(\frac{n\pi c}{L}\right)^2 a_n = q_n(t)$

5' where $\quad q_n(t)=\frac{2}{L}\int_0^L Q(x,t)\sin\frac{n\pi x}{L}\,dx$

$$a_n(t)=A_n\cos\left(\frac{n\pi c}{L}t\right)+B_n\sin\left(\frac{n\pi c}{L}t\right)+a_n^P(t)$$

where $\quad a_n^P(t)=\frac{1}{n\pi c/L}\Big[-\cos\left(\frac{n\pi c}{L}t\right)\int_0^t\sin\left(\frac{n\pi c}{L}t_0\right)q_n(t_0)\,dt_0$

$$+\sin\left(\frac{n\pi c}{L}t\right)\int_0^t\cos\left(\frac{n\pi c}{L}t_0\right)q_n(t_0)\,dt_0\Big]$$

$$=\frac{1}{n\pi c/L}\int_0^t q_n(t_0)\sin\left(\frac{n\pi c}{L}(t-t_0)\right)dt_0$$

~~Thus,~~ $\cancel{u(x,t)=\sum_{n=1}^{\infty}A_n}$

$u(x,0)=f(x)\Rightarrow$ $\left.\begin{array}{l}\sum_{n=1}^{\infty}a_n(0)\sin\left(\frac{n\pi x}{L}\right)=f(x)\\ a_n(0)=A_n\end{array}\right\}\Rightarrow A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$\frac{\partial u}{\partial t}(x,0)=g(x)\Rightarrow$ $\left.\begin{array}{l}\sum_{n=1}^{\infty}a_n'(0)\sin\left(\frac{n\pi x}{L}\right)=g(x)\\ a_n'(0)=B_n\cdot\frac{n\pi c}{L}\end{array}\right\}\Rightarrow B_n=\frac{1}{\frac{n\pi c}{L}}\cdot\frac{2}{L}\int_0^2 g(x)\sin\frac{n\pi x}{L}\,dx$

Thus, $u(x,t)=\int_0^L f(x_0)\cdot\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\cos\left(\frac{n\pi c}{L}t\right)dx_0$

$$+\int_0^L g(x_0)\cdot\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\frac{\sin\left(\frac{n\pi c}{L}t\right)}{n\pi c/L}\,dx_0$$

$$+\int_0^t\sum_{n=1}^{\infty}q_n(t_0)\sin\frac{n\pi x_0}{L}\frac{\sin\left(\frac{n\pi c}{L}(t-t_0)\right)}{n\pi c/L}\,dt_0 \qquad \frac{\sin\left(\frac{n\pi c}{L}(t-t_0)\right)}{n\pi c/L}$$

$$=\int_0^L f(x_0)\frac{\partial G}{\partial t}(x,t;x_0,0)\,dx_0+\int_0^L g(x_0)\,G(x,t;x_0,0)\,dx_0$$

$+\int_0^L\int_0^t Q(x_0,t_0)\,G(x,t;x_0,t_0)\,dt_0\,dx_0\;;\; G(x,t;x_0,t_0)=\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\cancel{\sin}$

10.4.4 $\bar{U} = \mathcal{F}(u(x,t))$

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial t} = k\cdot(iw)^2 \cdot \bar{U} - r\,\bar{U} \\ \bar{U}(w,0) = F(w) = \mathcal{F}(f(x)) \end{cases}$$

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial t} = (-kw^2 - r)\,\bar{U} \\ \bar{U}(w,0) = F(w) \end{cases} \Rightarrow \bar{U} = F(w)\,e^{-rt-kw^2t}$$

$$u(x,t) = \mathcal{F}^{-1}(\bar{U}) = \mathcal{F}^{-1}\left(F(w)e^{-rt}e^{-kw^2t}\right)$$

$$= e^{-rt}\cdot \mathcal{F}^{-1}\left(F(w)\cdot e^{-kw^2t}\right)$$

$$= e^{-rt}\cdot f * \sqrt{\frac{\pi}{kt}}\,e^{-x^2/4kt}$$

$$= e^{-rt}\cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x_0)\sqrt{\frac{\pi}{kt}}\,e^{-(x-x_0)^2/4kt}\,dx_0$$

$$= e^{-rt}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}}\,e^{-(x-x_0)^2/4kt}\cdot f(x_0)\,dx_0$$

10.4.9 $\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & y > 0 \\ & -\infty < x < \infty \\ u(x,0) = f(x) \end{cases}$

$$\bar{U}(w,y) = \mathcal{F}(u(x,y)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,y)\, e^{iwx}\, dx$$

$$\frac{\partial^2 \bar{U}}{\partial y^2} - w^2 \bar{U} = 0$$

$$\Rightarrow \bar{U}(w,y) = c(w)\, e^{-|w|y},$$

$$c(w) = \mathcal{F}(f(x))$$

$$u(x,y) = f(x) * \mathcal{F}^{-1}\left(e^{-|w|y}\right)$$

$$= f(x) * \frac{2y}{x^2+y^2}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\, \frac{2y}{(x-\bar{x})^2+y^2}\, d\bar{x}$$

10.4.10. $$\bar{U}(w,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,t)\, e^{iwx}\, dx$$

$$\begin{cases} \frac{\partial^2 \bar{U}}{\partial t^2} = -c^2 w^2 \bar{U} \\ \bar{U}(w,0) = F(w) = \mathcal{F}(f(x)) \\ \frac{\partial \bar{U}}{\partial t}(w,0) = 0 \end{cases}$$

$$\Rightarrow \bar{U}(w,t) = A(w)\cos cwt + B(w)\sin cwt$$

$B(w) = 0$. $\quad A(w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{iwx}\, dx$

$$\bar{U}(w,t) = A(w)\cos cwt = \underline{\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{iwx}\, dx}\cdot \cos cwt$$

$$= F(w)\cos cwt$$

$$u(x,t) = \int_{-\infty}^{+\infty} F(w)\cos cwt\; e^{-iwx}\, dw$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} F(w)\left[e^{-iw(x-ct)} + e^{-iw(x+ct)}\right] dw$$

$$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$