10 1

20 5.3.9 (a). $\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \frac{1}{x}\lambda\Phi = 0$

$$\phi(1)=0, \quad \phi(b)=0$$

$$p(x)=x, \quad q(x)=0, \quad \sigma(x)=\frac{1}{x} \qquad a=1.$$

(b). $$\lambda = \frac{-p(x)\phi\phi'\Big|_a^b + \int_a^b \left(p(x)(\phi')^2 - q\phi^2\right)dx}{\int_a^b \phi^2(x)\sigma(x)\,dx}$$

$$= \frac{0 + \int_1^b x(\phi'(x))^2\,dx}{\int_1^b \phi^2(x)\cdot\frac{1}{x}\,dx} > 0$$

(c). Assume $\phi = x^m$

we have $m(m-1) + m + \lambda = 0$

$$m^2 + \lambda = 0 \qquad \lambda > 0.$$

$$m = \pm\sqrt{\lambda}\,i$$

$$x^m = x^{\pm\sqrt{\lambda}i} = e^{\pm i\sqrt{\lambda}\ln x}$$

$$\Phi = c_1\cos(\sqrt{\lambda}\ln x) + c_2\sin(\sqrt{\lambda}\ln x)$$

$\Phi(1)=0 \Rightarrow c_1=0 \quad \phi = c_2\sin(\sqrt{\lambda}\ln x)$

$\phi(b)=0 \Rightarrow \sin(\sqrt{\lambda}\ln b)=0 \Rightarrow \lambda = \left(\frac{n\pi}{\ln b}\right)^2, \quad n=1,2,3,\cdots$

$\lambda=0 \Rightarrow \quad x\phi'' + \phi' = 0$

$$\phi = c_1 + c_2\ln x$$

$\phi(1)=0 \Rightarrow c_1=0; \ \phi(b)=0 \Rightarrow c_2\ln b = 0. \Rightarrow c_2=0 \ (b>1)$

$\lambda=0$ is not an eigenvalue.

$\lambda_1 = \left(\frac{\pi}{\ln b}\right)^2$ is the smallest eigenvalue (d). $\sigma(x) = \frac{1}{x}$

$$\int_1^b \sin\left(\frac{n\pi}{\ln b}\ln x\right)\sin\left(\frac{m\pi}{\ln b}\ln x\right)\cdot\frac{1}{x}\,dx \quad ; \quad y = \frac{\ln x}{\ln b}.$$

$$\overset{\ln x/\ln b = y}{=\!=\!=} \int_0^1 \sin(n\pi y)\sin(m\pi y)\cdot \ln b \cdot dy$$

$$= \ln b \cdot \int_0^1 \sin(n\pi y)\sin(m\pi y)\,dy$$

$$= 0 \quad \text{if} \quad m \neq n.$$

(e). $\phi_n = \sin\left(\frac{n\pi}{1}\cdot\frac{\ln x}{\ln b}\right).$

$1 < x < b$, $0 < \ln x < \ln b \Rightarrow 0 < \frac{n\pi}{\ln b}\ln x < n\pi$

By the property of sine function, this function has $n-1$ zeros in $(1, b)$.

15 5.4.1. $\begin{cases} c\rho\, u_t = \frac{\partial}{\partial x}\left(k_0 \frac{\partial u}{\partial x}\right) + \alpha u \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x). \end{cases}$

(a). $u = \Phi(x)\cdot h(t)$, then $\Phi(x)$ satisfies.

$$\begin{cases} \frac{d}{dx}\left(k_0 \frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\cdot\rho\cdot\phi = 0 \\ \phi(0) = \phi(L) = 0 \end{cases}$$

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b (p(\phi')^2 - q\phi^2)\,dx}{\int_0^L \phi^2\cdot c\cdot\rho\, dx}$$

$$= \frac{\int_0^L (k_0(\phi')^2 - \alpha\phi^2)\,dx}{\int_0^L \phi^2\cdot c\cdot\rho\, dx}$$

> 0 if $\alpha < 0$, $c > 0$ $\rho > 0$. $k_0 > 0$.
and $\phi \neq 0$.

(b). $u(x,t) = \sum_{n=1}^{\infty} a_n \Phi_n(x)\, e^{-\lambda_n t}$

$u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x) = f(x)$.

$$a_m = \frac{\int_0^L f(x)\cdot c\cdot\rho\cdot\phi_m(x)\,dx}{\int_0^L \phi_m^2(x)\, c\,\rho\, dx}$$

(c). since $\lambda_n > 0$, $n = 1, 2, \cdots$,
we have $\lim_{t\to\infty} u(x,t) = 0$.

$\boxed{5.4.5}$ ~~$u = \phi(x)\, h(t)$~~

(b)

$$\frac{1}{T_0 h}\frac{d^2h}{dt^2} = \frac{1}{\rho\,\phi}\frac{d^2\phi}{dx^2} + \frac{\alpha}{T_0\rho} = -\lambda$$

$$\begin{cases} h'' + T_0\lambda h = 0 \\ \phi''(x) + \frac{\alpha}{T_0}\phi + \lambda\rho\phi = 0 \end{cases}$$

$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$

$\phi_1,\ \phi_2(x),\ \cdots \phi_n(x)$

$$\left(q = \frac{\alpha}{T_0},\quad \sigma = \rho \qquad p(x) = 1 \right)$$

$$h(t) = c_1 \cos(\sqrt{\lambda T_0}\, t) + c_2 \sin(\sqrt{\lambda T_0}\, t)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \phi_n \cos(\sqrt{\lambda T_0}\, t) + \sum_{n=1}^{\infty} B_n \phi_n \sin(\sqrt{\lambda T_0}\, t)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \phi_n = f(x) \Rightarrow A_n = \frac{\int_0^L f(x)\phi_n(x)\rho(x)\,dx}{\int_0^L \phi_n^2(x)\rho(x)\,dx}$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \phi_n \sqrt{\lambda T_0} = g(x)$$

$$\Longrightarrow B_n\sqrt{\lambda_n T_0} = \frac{\int_0^L g(x)\phi_n(x)\rho(x)\,dx}{\int_0^L \phi_n^2(x)\rho(x)\,dx}$$

7.3.2. 10

$$\begin{cases} u_t = k(u_{xx}+u_{yy}+u_{zz}) \\ u(x,y,z,0) = f(x,y,z). \end{cases}$$

(a). $u = \phi(x,y,z)\, h(t)$.

$$\begin{cases} h' = -\lambda k h. \\ \phi_{xx}+\phi_{yy}+\phi_{zz} = -\lambda\phi. \end{cases}$$

$$\lambda_{nml} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{-\frac{\pi}{2}+l\pi}{W}\right)^2$$

$n = 1,2,3,\dots$

$m = 0,1,2,3,\dots$

$l = 1,2,3,\dots,\dots$

$$\phi = \sin\left(\frac{n\pi}{L}x\right)\cdot\cos\frac{m\pi}{H}y\ \cos\left(\frac{-\frac{\pi}{2}+l\pi}{W}z\right)$$

$$u = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=1}^{\infty} A_{nml}\cdot\sin\left(\frac{n\pi x}{L}\right)\cos\frac{m\pi}{H}y\ \cos\frac{-\frac{\pi}{2}+l\pi}{W}z\cdot e^{-\lambda kt}$$

$u(x,y,z,0) = f(x,y,z)$

$$A_{nml} = \frac{8}{LHW}\int_0^L\int_0^H\int_0^W f(x,y,z)\sin\left(\frac{n\pi x}{L}\right)\cos\frac{m\pi y}{H}\cos\frac{-\frac{\pi}{2}+l\pi}{W}z\; dx\,dy\,dz$$

$n \geq 1\quad m \geq 1,\ l \geq 1.$

$$A_{nml} = \frac{4}{LHW}\int_0^L\int_0^H\int_0^W f(x,y,z)\sin\left(\frac{n\pi x}{L}\right)\cdot\cos\frac{-\frac{\pi}{2}+l\pi}{W}z\; dx\,dy\,dz$$

$(m=0)$.

As $t\to\infty$ $u(x,y,z,t)\to 0$.

(c) 7.3.4 $u = \phi(x,y)\,h(t)$.

$$\begin{cases} h''(t) = -c^2\lambda h. \\ \phi_{xx}+\phi_{yy} = -\lambda\phi. \end{cases} \qquad \begin{aligned}&\phi(0,y)=\phi(L,y)=0\\ &\phi_y(x,0)=\phi_y(x,H)=0\end{aligned}$$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2. \qquad n=1,2,3,\dots, \quad m=0,1,2,\dots$$

$$\phi_{nm} = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{H}y\right).$$

$n=1,2,3,\dots$

$m=0,1,2,3,\dots$

$$h(t) = c_1\sin(c\sqrt{\lambda}\,t) + c_2\cos(c\sqrt{\lambda_{nm}}\,t).$$

$$u = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{mn}\cos(c\sqrt{\lambda_{nm}}\,t)\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}$$

$$+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} B_{mn}\sin(c\sqrt{\lambda_{nm}}\,t)\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}$$

$$u(x,y,0)=0 \Rightarrow A_{mn}=0$$

$$\frac{\partial u}{\partial t}(x,y,0)=f(x,y) \Rightarrow \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} B_{nm}\,c\sqrt{\lambda_{nm}}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right) = f(x,y)$$

$$B_{nm} = \frac{1}{c\sqrt{\lambda_{nm}}}\,\frac{\int_0^L\int_0^H f(x,y)\cos\frac{m\pi y}{H}\sin\left(\frac{n\pi x}{L}\right)dx\,dy}{\int_0^L\int_0^H \cos^2\frac{m\pi y}{H}\sin^2\frac{n\pi x}{L}\,dx\,dy}$$

$n=1,2,3,\dots,\dots$

$m=0,1,2,\dots,\dots$

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$$\begin{cases} \phi_{xx}+\phi_{yy}+\lambda\phi=0 \\ \phi_x(0,y)=\phi_x(L,y)=0 \\ \phi(x,0)=\phi(x,H)=0 \end{cases}$$

(a). $\lambda_{nm}=\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2$

$n=0,1,2,3,\cdots$

$m=1,2,3,\cdots$)

$\phi=\cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)$

$n=0,1,2,3,\cdots$,

$m=1,2,3,\cdots$)

(b). $L=H$. $\lambda_{nm}=(n^2+m^2)\cdot\frac{\pi^2}{L^2}$

In most cases, we have $\lambda_{21}=\lambda_{12}$

$\lambda_{31}=\lambda_{13}$

$\lambda_{mn}=\lambda_{nm}$

$\phi_{nm}=\cos\frac{n\pi}{L}x\sin\left(\frac{m\pi}{H}y\right)$

$\phi_{mn}=\cos\frac{m\pi}{L}x\sin\left(\frac{n\pi}{H}y\right)$.

$\phi_{nm}\neq\phi_{mn}$. two eigenfunctions.

(c). $\int_0^H\int_0^L \phi_{n_1m_1}\phi_{n_2m_2}\,dx\,dy=\int_0^H\sin\frac{m_1\pi}{H}y\sin\left(\frac{m_2\pi}{H}y\right)\times$

$\int_0^L\left(\cos\frac{n_1\pi}{L}x\cos\frac{n_2\pi}{L}x\right)dx$

$=0$. if $(n_1,m_1)\neq(n_2,m_2)$.

(5) 7.4.2. $\lambda = \dfrac{0 + \iint |\phi_x^2 + \phi_y^2| \, dx\,dy}{\iint_R \phi^2 \, dx\,dy} \geq 0.$

8.2.4. Consider Equilibrium Problem first

(10)
$$\begin{cases} 0 = k(u_{xx} + u_{yy}) \\ u(0,y) = 0 \qquad u_y(x,0) = 0 \\ u(L,y,t) = 0 \qquad u(x,H) = g(x). \end{cases}$$

$$u_E = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cosh\frac{n\pi y}{L}.$$

$$u_E(\pi, H) = g(x) \Rightarrow A_n = \frac{1}{\cosh\frac{n\pi H}{L}} \frac{2}{L} \int_0^L g(x) \sin\frac{n\pi x}{L} \, dx$$

Let $v = u(x,y,t) - u_E(x,y)$

$$\begin{cases} \frac{\partial v}{\partial t} = k(v_{xx} + v_{yy}) \\ v(0,y,t) = 0 \qquad \frac{\partial}{\partial y} v(x,0,t) = 0 \\ v(L,y,t) = 0 \qquad v(x,H,t) = 0. \end{cases}$$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{(m-\frac{1}{2})\pi}{H}\right)^2, \quad n = 1, 2, 3, \cdots \quad m = 1, 2, 3, \cdots$$

$$\phi_{nm} = \sin\frac{n\pi}{L}x \cos\left[\frac{(m-\frac{1}{2})\pi}{H} y\right].$$

$$v(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) \cos\left[\frac{(m-\frac{1}{2})\pi}{H} y\right] e^{-\lambda_{nm} k t}$$

$$f(x,y) - u_E(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{(m-\frac{1}{2})\pi}{H} y\right).$$

$$A_{nm} = \frac{\int_0^L \int_0^H \left(f(x,y) - u_E\right) \sin\frac{n\pi}{L}x \cos\left[\frac{(m-\frac{1}{2})\pi}{H}y\right] dxdy}{\frac{L}{2}\cdot\frac{H}{2}}$$

$$u(x,y,t) = V(x,y,t) + u_E.$$

As $t \to \infty$ we have $u(x,y,t) \to u_E(x,y)$.

$\boxed{8.2.6}$ (b). (10)

$$\begin{cases} 0 = c^2 u'' + Q(x) \\ u(0) = u(L) = 0 \end{cases} \Rightarrow u_E = -\frac{1}{2c^2}x^2 + \frac{L}{2c^2}x$$

$$V(x,t) = u(x,t) - u_E(x,t).$$

$$\begin{cases} \frac{\partial^2}{\partial t^2}V = c^2\frac{\partial^2 V}{\partial x^2} \\ V(0,t) = V(L,t) = 0 \\ V(x,0) = f(x) - u_E(x) \end{cases}$$

$$\frac{\partial V}{\partial t}(x,0) = g(x) - 0 = g(x).$$

$$V = h(t)\phi(x). \Rightarrow h''(t) = -c^2\lambda h(t)$$

$$\phi''(x) + \lambda\phi = 0.$$

$$V(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{cn\pi}{L}t\right) + B_n\sin\left(\frac{cn\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

$V(x,0) = f(x) - u_E(x)$: $A_n = \frac{2}{L}\int_0^L (f(x) - u_E)\cdot\sin\left(\frac{n\pi x}{L}\right)dx$

$\frac{\partial V}{\partial t}(x,0) = g(x)$: $B_n = \frac{L}{cn\pi}\cdot\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$

(C). Equilibrium Problem.

$$\begin{cases} 0 = c^2 U'' + 1 \\ U(0) = A \quad U(L) = B \end{cases} \Rightarrow U_E = -\frac{1}{2c^2}x^2 + \left(\frac{B-A}{L} + \frac{L}{2c^2}\right)x + A$$

$V = U(x,t) - U_E$

$$\Rightarrow \begin{cases} V_{tt} = c^2 V_{xx} \\ V(0,t) = V(L,t) = 0 \end{cases}$$

$$\begin{cases} V(x,0) = f(x) - U_E(x) \\ V_t(x,0) = g(x) - 0 = g(x). \end{cases}$$

$$V(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\right] \sin\frac{n\pi}{L}x$$

$$V(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x = f(x) - U_E(x)$$

$$A_n = \frac{2}{L}\int_0^L (f(x) - U_E(x)) \sin\frac{n\pi x}{L}\, dx$$

$$\frac{\partial V}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} B_n \sin\frac{n\pi x}{L} = g(x)$$

$$B_n = \frac{1}{cn\pi/L} \cdot \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$