Assignment 2 solution

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2.2.4 (a) $\because$ $L(u_p) = f$

$L(u_1) = 0$, $L(u_2) = 0$.

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As such, $L(u_p + c_1 u_1 + c_2 u_2) = L(u_p) + c_1 L(u_1) + c_2 L(u_2)$

$= f + 0 + 0 = f$

Therefore $u = u_p + c_1 u_1 + c_2 u_2$ is a solution to the problem $L(u) = f$.

(b) $L(u_{p_1}) = f_1$ $\quad L(u_{p_2}) = f_2$

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then $L(u_{p_1} + u_{p_2}) = L(u_{p_1}) + L(u_{p_2})$

$= f_1 + f_2$

Therefore $u = u_{p_1} + u_{p_2}$ is a particular solution of $L(u) = f_1 + f_2$.

2.3.1 (b) $u_t = k\frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}$. $\quad u = G(t)\phi(x)$.

$G'(t)\phi(x) = kG(t)\phi'' - v_0 G(t)\phi'$

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$$\frac{G'}{kG(t)} = \frac{\phi'' - \frac{1}{k}v_0\phi'}{\phi} = -\lambda$$

$$\begin{cases} G' + \lambda k G(t) = 0 \\ \phi'' - \frac{1}{k}v_0\phi' + \lambda\phi = 0 \end{cases}$$

(d) $u = G(t)\phi(r)$.

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$G'\phi = \frac{k}{r^2}\frac{\partial}{\partial r}(r^2 G(t)\phi')$

$= G(t)\frac{k}{r^2}\frac{\partial}{\partial r}(r^2\phi')$

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$$\frac{G'}{kG} = \frac{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\phi')}{\phi} = -\lambda$$

$$\Rightarrow \begin{cases} G' + k\lambda G = 0 \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) + \lambda\phi = 0 \end{cases}$$

2.3.2. (a) $\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = 0 = \phi(\pi) \end{cases}$ $L = \pi$

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$\lambda = \left(\frac{n\pi}{L}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2$, $n = 1, 2, 3, \cdots$

$\phi = \sin\frac{n\pi}{L}x = \sin nx$, $n = 1, 2, 3, \cdots$

(c) $\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi'(0) = 0, \ \phi'(L) = 0 \end{cases}$

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$\lambda = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \cdots$

$\phi = \cos\frac{n\pi}{L}x$, $n = 0, 1, 2, \cdots$

(e) $\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi'(0) = 0, \ \phi(L) = 0 \end{cases}$

$\lambda > 0$. $\phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

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$\phi'(0) = 0 \Rightarrow c_2 = 0$

$\cos\sqrt{\lambda}\cdot L = 0$ $\sqrt{\lambda}L = \frac{\pi}{2} + n\pi$, $n = 0, 1, \cdots$

$\lambda = \left(\left(\frac{1}{2} + n\right)\pi / L\right)^2$, $n = 0, 1, 2, \cdots$

$\phi = \cos\left(\frac{1}{2} + n\right)\pi\frac{x}{L}$, $n = 0, 1, 2, \cdots$

Assignment 2 Solution. 44 + 32 = 76 ①

2.3.3 (b) 4

$$\begin{cases} u_t = k\, u_{xx} \\ u(0,t)=0 \quad u(L,t)=0 \end{cases} \qquad u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$$

$$u(x,t) = 3\, e^{-(\frac{\pi}{L})^2 k t} \cdot \sin\frac{\pi x}{L} - e^{-(\frac{3\pi}{L})^2 k t} \sin\frac{3\pi x}{L}.$$

(d). $f(x) = \begin{cases} 1 & 0<x<L/2 \\ 2 & L/2<x<L. \end{cases}$

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$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \frac{2}{L}\int_0^{L/2} 1\cdot\sin\frac{n\pi}{L}x\,dx + \frac{1}{L}\int_{L/2}^{L} 2\cdot\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) - \frac{4}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right).$$

2.3.5. 4

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L \frac{1}{2}\left[\cos\left(\frac{n\pi x - m\pi x}{L}\right) - \cos\left(\frac{n\pi x + m\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\int_0^L \cos\frac{(n-m)\pi x}{L}\,dx = \begin{cases} 0 & n\neq m \\ \frac{L}{2} & m=n. \end{cases}$$

2.3.6. 4

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} L & m=n=0 \\ L/2 & m=n\neq 0 \\ 0. & m\neq n \end{cases}$$

2.4.1

(a). $$u = \sum_{n=0}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 k t} \cos\frac{n\pi}{L}x.$$

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$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\left[\int_0^{L/2} 0 + \int_{L/2}^{L} 1\,dx\right] = \frac{1}{2}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi}{L}x\,dx \qquad n\geq 1$$

$$= \frac{2}{L}\int_{L/2}^{L} 1\cdot\cos\frac{n\pi}{L}x\,dx = \frac{2}{n\pi}\left(-\sin\frac{n\pi}{2}\right)$$

4 (c). $$A_0 = \frac{1}{L}\int_0^L -2\sin\frac{\pi x}{L}\,dx = \frac{-4}{\pi}.$$

$$A_n = \frac{2}{L}\int_0^L (-2)\sin\frac{\pi x}{L}\cdot\cos\frac{n\pi}{L}x\,dx = \frac{-2}{L}\left[\int_0^L \sin(1+n)\frac{\pi x}{L}dx - \int_0^L \sin(n-1)\frac{\pi x}{2}\cdot dx\right]$$

$$= \frac{4}{\pi(n^2-1)}\left[1-\cos(n-1)\pi\right]$$

2.4.3

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0)=\phi(2\pi) \quad \phi'(0)=\phi'(2\pi). \end{cases}$$

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$\lambda = n^2$, $\phi = \sin nx$, $n \ge 1$

and $\phi = \cos nx$, $n = 0, 1, 2, \dots$

2.4.4: If $\lambda < 0$ then $-\lambda > 0$.

C.E. $r^2 = -\lambda$ $\quad r_1 = \sqrt{-\lambda}$ $\quad r_2 = -\sqrt{-\lambda}$.

$$\phi = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}.$$

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$\phi'(0) = \phi'(L) = 0 \Rightarrow c_1 = 0, \; c_2 = 0.$

$\phi = 0$. No non-zero solution.

There are no negative eigenvalues!.

2.5 (b).

$$\begin{cases} \nabla u = 0 \\ u_x(0,y) = g(y), \quad u_x(L,y) = 0, \quad u(x,0) = 0, \quad u(x,H) = 0. \end{cases}$$

$u = h(x)\cdot\phi(y)$. $\qquad \dfrac{1}{h}\dfrac{d^2h}{dx^2} = -\dfrac{1}{\phi}\dfrac{d^2\phi}{dy^2} = \lambda.$

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$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = \phi(H) = 0. \end{cases}$$

$\lambda = \left(\frac{n\pi}{H}\right)^2$, $n = 1, 2, \dots$

$\phi = \sin\frac{n\pi}{H}y$, $n = 1, 2, \dots$

$$\begin{cases} \dfrac{d^2h}{dx^2} = \lambda h = \left(\frac{n\pi}{H}\right)^2 h. \\ h'(L) = 0. \end{cases}$$

$h = \cosh\left((x-L)\frac{n\pi}{H}\right)$

$$u = \sum_{n=1}^{\infty} B_n \cosh\left(\frac{n\pi}{H}(x-L)\right) \sin\frac{n\pi}{H}y$$

$u_x(0,y) = g(y)$: $\displaystyle\sum_{n=1}^{\infty} B_n \cdot \frac{n\pi}{H} \cdot \sinh\left(\frac{n\pi}{H}\cdot(-L)\right) \cdot \sin\frac{n\pi}{H}y = g(y)$

$$B_n = \frac{1}{\frac{n\pi}{H}\cdot\sinh\left(\frac{n\pi}{H}(-L)\right)} \cdot \frac{2}{H}\int_0^L g(y)\cdot\sin\frac{n\pi}{H}y \, dy$$

(d). $u = h(x)\phi(y)$

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$$\begin{cases} \phi''(y) + \lambda\phi(y) = 0 \\ \phi'(0) = 0,\ \phi(H) = 0 \end{cases}$$

$\phi = \cos\left((n-\frac{1}{2})\pi y/H\right)$, $\lambda = \left((n-\frac{1}{2})\frac{\pi}{H}\right)^2$

$h = \sinh\left((n-\frac{1}{2})\frac{\pi}{H}(x-L)\right)$

$u = \sum_{n=1}^{\infty} A_n \sinh\left((n-\frac{1}{2})\frac{\pi}{H}(x-L)\right)\cos\left((n-\frac{1}{2})\frac{\pi}{H}y\right)$

$g(y) = u(0,y) = \sum_{n=1}^{\infty} A_n \sinh\left((n-\frac{1}{2})\frac{\pi}{H}(-L)\right)\cos\left((n-\frac{1}{2})\frac{\pi}{H}y\right)$

$A_n = \frac{2}{H}\,\frac{\int_0^H g(y)\cos\left((n-\frac{1}{2})\frac{\pi}{H}y\right)}{\sinh\left((n-\frac{1}{2})\frac{\pi}{H}(-L)\right)}$.

6 (3) (4)

(4). $u = h(x)\phi(y)$. $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \lambda$

(C4) $\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi'(0) = \phi'(H) = 0. \end{cases}$ $\lambda = \frac{n\pi}{H}$, $n = 0, 1, 2 \cdots$

$\phi_n = \cos\frac{n\pi}{H}y$

$n \geq 1$ $\begin{cases} \frac{d^2h}{dx^2} = \lambda h = \left(\frac{n\pi}{H}\right)^2 h \\ h(L) = 0. \end{cases}$ $h_n = \sinh\left(\frac{n\pi}{H}\cdot(x-L)\right)$.

$n \geq 1$

$n = 0$. $\begin{cases} \frac{d^2h}{dx^2} = 0 \cdot h = 0 \\ h(L) = 0. \end{cases}$ $h_0 = (x-L)$.

$$u = A_0\cdot\phi_0\cdot h_0 + \sum_{n=1}^{\infty} A_n\cdot\phi_n\cdot h_n$$

$$= A_0\cdot(x-L) + \sum_{n=1}^{\infty} A_n\cdot\sinh\left(\frac{n\pi}{H}(x-L)\right)\cdot\cos\frac{n\pi y}{H}.$$

$u(0,y) = f(y) \Rightarrow$ $A_0(-L) + \sum_{n=1}^{\infty} A_n\cdot\sinh\left(\frac{n\pi}{H}(-L)\right)\cdot\cos\frac{n\pi y}{H} = f(y)$.

$$A_0(-L) = \frac{1}{H}\int_0^H f(y)dy \Rightarrow A_0 = \frac{1}{-LH}\int_0^H f(y)dy$$

$$A_n = \frac{1}{\sinh\left(\frac{n\pi}{H}(-L)\right)}\cdot\frac{2}{H}\int_0^H \cos\frac{n\pi y}{H} f(y)\,dy$$