BU Solution to Assignment 1. ~~82~~ 50 1

1 (a). The minus sign means: If $\frac{\partial \phi}{\partial x} > 0$ for $a \le x \le b$, ϕ is an increasing function of x. The heat energy in this interval must decrease ($\frac{\partial e}{\partial t} < 0$).

(2)

(b) The minus sign means: If the temperature increases as x increases ($\partial u/\partial x > 0$), then the heat energy flows to the left.

(2)

2. (a) The total energy between x and $x+\Delta x$ is

$$\int_x^{x+\Delta x} e\, A\, dx$$

(4) Since $Q = 0$, we have

$$\frac{d}{dt}\int_x^{x+\Delta x} e\cdot A\, dx = \phi(x,t)A - \phi(x+\Delta x,t)A$$

when Δx is small, $\int_x^{x+\Delta x} e\, A\, dx \doteq e\cdot A\cdot \Delta x$

so we have

$$\frac{\partial}{\partial t}\left(e(x,t)\cdot A\cdot \Delta x\right) = \phi(x,t)A - \phi(x+\Delta x,t)A.$$

let $\Delta x \to 0$:

$$\frac{\partial e}{\partial t} = \lim_{\Delta x \to 0}\frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x} = -\frac{\partial \phi}{\partial x}.$$

(b) $[a,b]$ The total energy is

(4)

$$\int_a^b e\cdot A\, dx$$

The conservation law gives

$$\frac{d}{dt}\int_a^b e\cdot A\, dx = A\phi(a,t) - \phi(b,t)\cdot A = A\int_a^b -\frac{\partial \phi}{\partial x}\, dx$$

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x}\right) dx = 0$$

3. The latter condition at $x=x_0$ is.

$$K_0(x_{0+})\cdot\frac{\partial u}{\partial x}(x_{0+},t)=K_0(x_{0-})\frac{\partial u}{\partial x}(x_{0-},t).$$

4

In the case $K_0(x_{0+})=K_0(x_{0-})$, we have

$$\frac{\partial u}{\partial x}(x_{0+},t)=\frac{\partial u}{\partial x}(x_{0-},t).$$

So in this case, $\frac{\partial u}{\partial x}$ is continuous at $x=x_0$.

4. (a) $\rho c\frac{\partial u}{\partial t}=K_0\frac{\partial^2 u}{\partial x^2}+Q$. $Q=0$

$$\begin{cases} \\ u(0)=T, \quad u(L)=0 \end{cases}$$

4

equilibrium: $\left(\frac{\partial u}{\partial t}\equiv 0\right)$
$$\begin{cases}\frac{d^2u}{dx^2}=0\\ u(0)=T, \quad u(L)=0\end{cases}$$

$$u=C_1x+C_2\ ;\qquad u(0)=T \Longleftrightarrow C_2=T.$$
$$u(L)=0 \Longleftrightarrow C_1=-\frac{T}{L}$$

$$\therefore u(x)=-\frac{T}{L}x+T.$$

(b). $\frac{\partial u}{\partial t}=0$, $\frac{Q}{K_0}=1$.

equilibrium
$$\begin{cases}\frac{d^2u}{dx^2}+1=0\\ u(0)=T_1, \quad u(L)=T_2.\end{cases}$$

4

$$u=-\frac{x^2}{2}+C_1x+C_2.$$

$$u(0)=T_1 \Rightarrow C_2=T_1$$

$$u(L)=T_2 \Rightarrow -\frac{L^2}{2}+C_1L+T_1=T_2 \Rightarrow C_1=\frac{1}{L}\left(T_2-T_1+\frac{L^2}{2}\right)$$

$$\therefore u(x)=-\frac{x^2}{2}+\frac{1}{L}\left(T_2-T_1+\frac{L^2}{2}\right)x+T_1$$

(c) $u_{xx}=0 \Rightarrow u=C_1x+C_2$, $u(0)=T \Rightarrow C_2=T.$

4

$u_x(L)+u(L)=0 \Rightarrow C_1+C_1L+C_2=0 \Rightarrow C_1=-T/(1+L)$.

$$u=-T/(1+L)\cdot x+T$$

5 Equation for $0<x<1$

$$\begin{cases} c\rho \dfrac{\partial u}{\partial t} = k_0 \dfrac{\partial^2 u}{\partial x^2} + Q \\ u(0)=0. \quad c\rho=1 \quad Q=1. \quad k_0=1 \end{cases}$$

Equation for $1<x<2$

$$\begin{cases} c\rho \dfrac{\partial u}{\partial t} = k_0 \dfrac{\partial^2 u}{\partial x^2} + Q \\ u(2)=0 \quad Q=0. \quad c\rho=1 \quad k_0=2 \end{cases}$$

at $x=1$ we have
$$\begin{cases} u(1-)=u(1+), \\ k_0(1-)\cdot \dfrac{\partial u}{\partial x}(1-) = k_0(1+)\cdot \dfrac{\partial u}{\partial x}(1+). \end{cases}$$

6 Equilibrium when $0<x<1$:
$$\begin{cases} \dfrac{d^2u}{dx^2}+1=0. \\ u(0)=0. \end{cases} \Rightarrow \quad u=-\frac{x^2}{2}+c_1x+c_2$$
$u(0)=0 \Rightarrow c_2=0.$

$$u=-\frac{x^2}{2}+c_1x, \qquad 0<x<1.$$

Equilibrium when $1<x<2$:
$$\begin{cases} \dfrac{d^2u}{dx^2}=0. \\ u(2)=0 \end{cases} : \quad u=c_3+c_4x \Rightarrow c_3=-2c_4$$

$$u=c_4(-2+x)$$

At $x=1$:
$$\begin{cases} -\dfrac{x^2}{2}+c_1x\Big|_{x=1_-} = c_4(-2+x)\Big|_{x=1_+} \\ 1\cdot\dfrac{\partial u}{\partial x}\Big|_{x=1_-} = 2\cdot\dfrac{\partial u}{\partial x}\Big|_{x=1_+} \end{cases}$$

i.e.
$$\begin{cases} -\dfrac{1}{2}+c_1=-c_4 \\ 1\cdot(-x+c_1)\Big|_{x=1_-} = 2\cdot c_4. \end{cases}$$

$$c_4=-\frac{1}{6} \qquad c_1=\frac{2}{3}$$

$$u=-\frac{1}{6}(-2+x), \qquad 1<x<2.$$

and
$$u=-\frac{x^2}{2}+\frac{2}{3}x, \quad 0<x<1$$

6 (a) Equilibrium equation

$$\begin{cases} k_0 u_{xx} + Q_0 = 0. \\ u_x(0)=0, \quad u_x(L)=0 \end{cases} \Rightarrow u = \frac{-Q_0}{2k_0}x^2 + c_1 x + c_2$$

$u_x(0)=0 \Rightarrow \quad c_1 = 0$

$u_x(L)=0 \Rightarrow \quad -\frac{Q_0}{k_0}\cdot L + c_1 = 0.$

So it require $Q_0 L/k_0 = 0$.

this is impossible. No existence of equilibrium temperature distribution

physically: since two ends of the rod are insulated, and there are a source term Q_0 inside the rod, the temperature of the rod will tend to infinity if $Q_0 > 0$.

(b).
$$\begin{cases} \rho c \cdot \frac{\partial u}{\partial t} = k_0 u_{xx} + Q_0. \\ u_x(0) = u_x(L) = 0. \end{cases}$$

Total energy $E = A\int_0^L \rho c \cdot u(x,t)\, dx.$

$$\frac{dE}{dt} = A\int_0^L \rho \cdot c \cdot u_t(x,t)\, dx$$

$$= A\int_0^L \rho \cdot c \cdot (k_0 u_{xx} + Q_0)\, dx \qquad \text{(using the equation)}$$

$$= 0 + A\int_0^L \rho c\, Q_0\, dx. \qquad \text{(using the BC)}$$

$$\therefore \frac{dE}{dt} = A\int_0^L \rho c\, Q_0\, dx = \begin{cases} >0, & \text{if } Q_0 > 0 \\ <0, & \text{if } Q_0 < 0. \end{cases}$$

$E(t) = E(0) + A \cdot t \int_0^L \rho \cdot c\, Q_0\, dx$

E is an increasing function if $Q_0 > 0$ and an decreasing function if $Q_0 < 0$.

7. (a) $\begin{cases} u_{xx}=0 \\ u_x(0)=1 \quad u_x(L)=\beta \end{cases} \Rightarrow u = c_1 x + c_2$.

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$u_x(0)=1 \Rightarrow c_1=1$.

$u_x(L)=\beta \Rightarrow c_1=\beta$.

When $\beta=1$, there are solutions.

(b) $\begin{cases} u''+x-\beta=0. \\ u_x(0)=0 \quad u'(L)=0. \end{cases}$ $\quad u=-\frac{1}{6}(x-\beta)^3+c_1x+c_2$.

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$u_x(0)=0 \Rightarrow -\frac{1}{2}(x-\beta)^2\big|_{x=0}+c_1=0 \Rightarrow -\frac{1}{2}\beta^2+c_1=0.$

$u'(L)=0 \Rightarrow -\frac{1}{2}(L-\beta)^2+c_1=0 \Rightarrow -\frac{1}{2}(L-\beta)^2+c_1=0$

$\therefore \beta^2=(L-\beta)^2 \Rightarrow \beta=\frac{L}{2}$.

when $\beta=\frac{L}{2}$, there are solutions.

8. $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=u_r\cdot(\sqrt{x^2+y^2})_x+u_\theta\cdot\left(\arctan\frac{y}{x}\right)_x$

$=u_r\cdot\frac{x}{(x^2+y^2)^{\frac{1}{2}}}+u_\theta\cdot\frac{-y}{x^2+y^2}$

4

$u_y=u_r\cdot\frac{y}{(x^2+y^2)^{\frac{1}{2}}}+u_\theta\cdot\frac{x}{x^2+y^2}$

Continue the process to have $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

Method 2: $\frac{\partial u}{\partial r}=u_x\cos\theta+u_y\sin\theta$.

$\frac{\partial u}{\partial \theta}=u_x(-r\sin\theta)+u_y(r\cos\theta)$.

Finally. $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=(\cos^2\theta+\sin^2\theta)u_{xx}+(\cos^2\theta+\sin^2\theta)u_{yy}$

$=u_{xx}+u_{yy}$.