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Speed selection for traveling waves of a reaction–diffusion–advection equation in a cylinder

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ABSTRACT

This paper is concerned with linear or nonlinear selection mechanism for the minimal speed of traveling wave solutions to a reaction–diffusion–advection equation in a cylindrical domain with Fisher–KPP-type nonlinearity. By using the method of upper and/or lower solutions, we establish the speed selection results. Precisely, we obtain sufficient conditions under which the linear or nonlinear selection is realized when the model is prescribed by Neumann boundary conditions and Dirichlet boundary conditions respectively.

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1. Introduction

In this paper, we investigate speed selection mechanism for traveling wave solutions to a reaction–diffusion–advection equation in an infinite cylindrical domain. The equation we consider is in the following form

$$\begin{cases} u_t = u_{xx} + \Delta_y u + \alpha(y)u_x + f(u), \ (x,y) \in \mathbb{R} \times \Omega, \ t > 0, \\ Bu = 0, \ (x,y) \in \mathbb{R} \times \partial\Omega, \\ u(x,y,0) = u_0(x,y), \ (x,y) \in \mathbb{R} \times \Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^{n-1}(n \geq 2)$ is a bounded smooth domain. The boundary condition "$Bu = 0$" denotes either the Neumann boundary condition, i.e., $\partial_\nu u(x,y,t) = 0$ for $(x,y) \in \mathbb{R} \times \partial\Omega$, which implies there is no flux of u across the wall of the cylinder, or the Dirichlet boundary condition, i.e., $u(x,y,t) = 0$ for $(x,y) \in \mathbb{R} \times \partial\Omega$, which means the value of u is fixed to be zero on the wall of the cylinder. The third term $\alpha(y)u_x$ on the right hand side is a predetermined transport term, or a driving flow, in the x-direction, and the function $\alpha(y)$ is always assumed to be bounded. The reaction term $f : \mathbb{R} \to \mathbb{R}$ is assumed to be a C^2 function with the properties: $f(0) = f(1) = 0$, $f'(0)$ and $f'(1)$ with $f'(1) < 0$.

There are three typical types of function f in applications:
(A1) $f > 0$ on $(0, 1)$;
(A2) for some $\theta \in (0, 1)$, $f = 0$ on $[0, \theta]$ and $f > 0$ on $(\theta, 1)$;
(A3) for some $\theta \in (0, 1)$, $f < 0$ on $(0, \theta)$, $f(\theta) = 0$, and $f > 0$ on $(\theta, 1)$.

Actually, when (A1) or (A3) occurs, these semilinear parabolic equations have many applications in biology, such as population dynamics, gene developments and so on. For more details and descriptions, please see [1–6]. When (A1) or (A2) occurs, such equations also arise in the study of flame propagation in a tube. For a detailed derivation and physical discussion, we refer readers to [2,4,7–12].

Here, we focus on the so-called traveling wave solutions. The traveling wave solutions are defined as solutions of the form

$$u(x,y,t) = U(\xi,y), \ \xi = x - ct. \tag{1.2}$$

Here, $U(\xi, y)$ is called the wave profile, and ξ is the wave variable, and $c \in \mathbb{R}$ is the speed of the wave, which is to be determined. After substituting the solution form (1.2) into Eq. (1.1), we find the equation for $U(\xi, y)$ as

$$U_{\xi\xi} + \Delta_y U + [\alpha(y) + c]U_\xi + f(U) = 0. \tag{1.3}$$

The traveling wave solutions are required to satisfy the limiting conditions

$$\lim_{\xi\to+\infty} U(\xi,y) = 0, \quad \lim_{\xi\to-\infty} U(\xi,y) = \beta(y) \not\equiv 0, \tag{1.4}$$

uniformly for $y \in \bar{\Omega}$, where the non-negative limiting state $\beta(y)$ is the solution of

$$\begin{cases} \Delta_y U + f(U) = 0, \ y \in \Omega, \\ BU = 0, \ y \in \partial\Omega. \end{cases} \tag{1.5}$$

Clearly, if the Neumann boundary condition occurs, it is easy to have $\beta(y) \equiv 1$. On the other hand, in the case of the Dirichlet

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boundary condition, we can have only one non-negative solution $\beta(y)$ with $0 < \beta(y) < 1$ for $y \in \Omega$ under some mild condition (i.e., the zero solution is linearly unstable) and this can be shown later.

Before stating our main results, we review relevant references on the traveling wave solutions of (1.3)–(1.4). There is a vast list of literature on the theory related to the existence of the traveling wave solutions in such an equation. For example, in [13–15], the authors studied the theory of asymptotic speeds of spreading in terms of abstract monotonic systems. In particular, in [4,12,16–19], the authors investigated the existence and uniqueness of the traveling wave solutions in a cylindrical domain.

The most related works to ours are [12] and [14]. In [12], Berestycki and Nirenberg considered Eqs. (1.3)–(1.4) prescribed by the Neumann boundary condition. When $f(U)$ satisfies (A2) or (A3) respectively, the authors proved the existence of a traveling wave solution (c, U) and then used the sliding method to further prove the uniqueness of such a solution. Here, the uniqueness is up to a translation, i.e., if there exist solutions (c, U) and (c', U'), then $c' = c$ and $U'(\xi, y) = U(\xi + \tau, y)$ for some real constant τ. For the case (A1), the authors proved that there exists a critical number (or the minimum number) $c^* \in \mathbb{R}$ such that the solution (c, U) exists for c being any value in $[c^*, +\infty)$ and also showed that, if $f(s) \le f'(0)s$ for $0 < s < 1$, this critical number c^* is explicitly determined by Ω, $\alpha(y)$ and the value of $f'(0)$.

In section 6 of [14], Liang and Zhao focused on investigating the theory of spreading speeds and traveling waves for abstract monostable evolution systems. They proved that the spreading speed c^* coincides with the minimal wave speed with a result that traveling wave solutions, connecting β and 0, exist for all $c \ge c^*$. When f satisfies the subhomogeneous condition in the sense that $f(\varrho s) \ge \varrho f(s)$ for all $\varrho \in [0, 1]$ and $0 \le s \le 1$, they obtained a formula for the speed c^*.

Based on the results in [12–14], in the case (A1), we know that there always exists a minimal wave speed $c_{\min}$ such that (1.1) has a traveling wave solution if $c \ge c_{\min}$ and no traveling wave solution exists if $c < c_{\min}$. To proceed, we only consider the case (A1) in this paper and denote the minimal wave speed $c_{\min}$ as

$$c_{\min} := \inf\{c : \text{the system (1.3)–(1.4) has a non-negative solution } U(\xi, y)\}.$$

With the understanding that the minimal wave speed is always the spreading speed of biological invasion for a model with two fixed points only, it is natural to ask how to determine the speed $c_{\min}$. To estimate it, first by the standard linearization analysis near the zero solution, we will obtain a linear system and the linear speed c_0 in the next section, where $c_{\min} \ge c_0$ will be shown. Furthermore, it was numerically observed that depending on the nonlinearity $f(u)$, the wave speed $c_{\min}$ is either equal to or greater than the linear speed c_0. Thus, to distinguish the two different cases, we give the following classification of the speed selection mechanism.

Definition 1.1. The speed selection mechanism for (1.3)–(1.4) is called a linear selection if $c_{\min} = c_0$; otherwise, it is called a nonlinear selection if $c_{\min} > c_0$.

When the space dimension is confined in one dimension, the speed selection can be found in [20–25] and the references therein. But in higher dimensions with a non-constant convection term, there are not many references on such a topic. In this paper, we shall focus on the speed selection of monotone traveling wave solutions connecting β to 0 under the condition when the zero solution is linearly unstable and β is linearly stable. To see the linear stability, we linearize (1.5) near one of the steady states (using ψ to denote either 0 or β) and consider the corresponding eigenvalue problem as

$$\begin{cases} \Delta_y \phi + f'(\psi)\phi = \mu_1(\psi)\phi, \; y \in \Omega, \\ B\phi = 0, \; y \in \partial\Omega, \end{cases}$$

where $\mu_1(\psi)$ is the principal eigenvalue. We say that ψ is linearly stable if $\mu_1(\psi) < 0$ and linearly unstable if $\mu_1(\psi) > 0$. Thus, to have such a monotone traveling wave solution, we further require f to satisfy the following conditions:

(A4) If (1.3)–(1.4) is prescribed by the Neumann boundary condition, then we require $f'(0) > 0$ and $f'(1) < 0$;

(A5) If (1.3)–(1.4) is prescribed by the Dirichlet boundary condition, then we require $\mu_1(0) > 0$, and $\mu_1(\beta) < 0$.

Under these conditions, we can confirm that there exists a unique solution $\beta(y)$ to (1.5) satisfying $0 < \beta(y) \le 1$.

With the application of the upper and lower solution method, we are able to establish the linear and/or nonlinear selection mechanism for our system. The detail is shown in Sections 3 and 4, which are valid for both Neumann and Dirichlet boundary conditions. We also find a sufficient condition for the nonlinear selection mechanism to our model under the Neumann boundary condition. We should emphasize that our investigations greatly extend the conclusions in [20,22,26].

The rest of this paper is organized as follows. In Section 2, we perform the local analysis near zero to find the linear speed c_0. In Section 3, we study the speed selection mechanism and present the main result. Then, we give two applications in Section 4, one with a cubic nonlinear term and the other with a subcritical quintic Ginzburg–Landau equation in a cylindrical domain. Finally, in Section 5, we summarize the obtained results and discuss some open problems. Appendix is to illustrate the upper and lower solutions method used in our model.

2. Local analysis near zero

Linearizing Eq. (1.3) near zero gives

$$\begin{cases} U_{\xi\xi} + \Delta_y U + (\alpha(y) + c)U_\xi + f'(0)U = 0, \xi \in (-\infty, \infty), \\ \quad y \in \Omega, \\ BU = 0, \; y \in \partial\Omega. \end{cases} \tag{2.1}$$

Then, letting $U = \varphi(y)e^{-\lambda\xi}$ for some non-negative function $\varphi(y)$ and a real constant λ, we obtain an eigenvalue problem

$$\begin{cases} \Delta_y \varphi + [\lambda^2 - \lambda(\alpha(y) + c) + f'(0)]\varphi = 0, \; y \in \Omega, \\ B\varphi = 0, \; y \in \partial\Omega. \end{cases} \tag{2.2}$$

To further discuss the above problem, we denote

$$L_\lambda = \Delta_y + \left[\lambda^2 - \lambda(\alpha(y) + c) + f'(0)\right]. \tag{2.3}$$

Then solving the problem (2.2) can be regarded as seeking the non-negative solution(s) of $L_\lambda\varphi = 0$ with the boundary condition $B\varphi = 0$. Let $\mu(\lambda)$ be the principal eigenvalue of the operator L_λ, and we consider the following eigenvalue problem

$$L_\lambda \psi = \mu(\lambda)\psi, \quad B\psi|_{y\in\partial\Omega} = 0, \tag{2.4}$$

for some non-negative non-zero function $\psi(y)$, $y \in \Omega$. It is clear that to find the solution of (2.2) is equivalent to find (c, λ) such that $\mu(\lambda) = 0$, with the corresponding eigenfunction $\psi(y)$ as the solution. For the eigenvalue problem (2.4), we have the following results.

(1) When $\lambda \to 0$, $L_\lambda\varphi \to \Delta_y\psi + f'(0)\psi$. From (A4) or (A5), we have $\mu(0) > 0$.

(2) When $\lambda \to +\infty$, we have $\lambda^2 - \lambda(\alpha(y)+c)+f'(0) > M$ for any large positive number M. In this case, by comparison, we have $\mu(+\infty) > 0$ for both boundary conditions.

Furthermore, due to the convexity of the function "$\lambda^2 - \lambda(\alpha(y)+c)+f'(0)$" with respect to λ, it is easy to have the following proposition.

Proposition 2.1. *The principal eigenvalue $\mu(\lambda)$ defined in (2.2) is convex with respect to $\lambda > 0$.*

From Eq. (2.4), it is clear to see that μ is decreasing in c. Thus, we can define

$$c_0 := \min\{c \mid c \in \mathbb{R} \text{ such that } \mu(\lambda) = 0 \text{ has a solution } \lambda \in (0, +\infty)\}.$$

Now, in view of the above proposition, we can arrive at the following theorem.

Theorem 2.2. *For the eigenvalue problem (2.4), there exists a critical number $c_0 \in \mathbb{R}$ such that*
(1) when $c < c_0$, there is no positive λ such that $\mu(\lambda) = 0$, and (2.2) has no non-negative non-zero solution;
(2) when $c = c_0$, there is only one positive λ_0 such that $\mu(\lambda) = 0$, and (2.2) has one solution $\varphi_0 = \psi_0$, where ψ_0 is the principal eigenfunction corresponding to $\lambda = \lambda_0$ in (2.4);
(3) when $c > c_0$, there exist $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_2(c) > \lambda_1(c) > 0$ such that $\mu(\lambda_i(c)) = 0, i = 1, 2$, and (2.2) has two solutions $\varphi_j = \psi_j$ when $\lambda = \lambda_j(c)$, where ψ_j is the principal eigenfunction corresponding to $\lambda = \lambda_j(c)$ in (2.4) for $j = 1, 2$.

Remark 2.3. Near $\xi = \infty$, Eq. (1.3) is approximated by the linear equation (2.1). From the above theorem, we can see that $c \geq c_0$ is a necessary condition for (1.3)–(1.4) to have a non-negative traveling wave solution. Therefore, $c_{\min} \geq c_0$. Moreover, $\lambda_2(c) > \lambda_0(c_0) > \lambda_1(c) > 0$ if $c > c_0$.

3. The speed selection

In this section, we study the speed selection mechanism for (1.3)–(1.4) through the upper and lower solutions method. The key point is to construct a pair of suitable upper and lower solutions. The definition of an upper (or a lower) solution and the details of this method are shown in the Appendix section. To begin with, we denote the left hand side of Eq. (1.3) as

$$\mathcal{L}(U) := U_{\xi\xi} + \Delta_y U + (\alpha(y)+c)U_\xi + f(U). \tag{3.1}$$

For any $c = c_0 + \epsilon_1$ with $\epsilon_1 > 0$, we have two pairs of solutions $(\lambda_1(c), \varphi_1)$ and $(\lambda_2(c), \varphi_2)$ with $\lambda_2(c) > \lambda_1(c) > 0$ for (2.2) by Theorem 2.2. Then we define a continuous function $\overline{U}(\xi, y)$ as the solution of the following equation

$$\overline{U}_\xi = -\lambda_1(c)\overline{U}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right), \tag{3.2}$$

where $\gamma > 0$ is a parameter to be determined. Considering the boundary conditions as $\overline{U}(\xi, y) \sim \beta(y)$ when $\xi \to -\infty$, and $\overline{U}(\xi, y) \sim \varphi_1(y)e^{-\lambda_1(c)\xi} \to 0$ when $\xi \to +\infty$, we obtain the formula for $\overline{U}$ as

$$\overline{U} = \frac{\beta\varphi_1}{\left[\beta^\gamma e^{\lambda_1(c)\gamma\xi} + \varphi_1^\gamma\right]^{\frac{1}{\gamma}}}. \tag{3.3}$$

It is easy to see that $0 \leq \overline{U} \leq \beta$ for all $(\xi, y) \in \mathbb{R} \times \Omega$ and

$$\overline{U}_{\xi\xi} = \lambda_1^2(c)\overline{U}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)\left(1 - (\gamma+1)\frac{\overline{U}^\gamma}{\beta^\gamma}\right). \tag{3.4}$$

By substituting the formulas of $\overline{U}, \overline{U}_\xi, \overline{U}_{\xi\xi}$ and $\Delta_y\overline{U}$ into (3.1), and after a tedious computation, we finally obtain

$$\mathcal{L}(\overline{U}) = \frac{\overline{U}^{(\gamma+1)}}{\beta^\gamma}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)\Bigg\{ -(\gamma+1)\,\lambda_1^2(c) - (\gamma+1) \times \frac{\varphi_1^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_1}\right)\right]^2 + G_1(\xi, y)\Bigg\}, \tag{3.5}$$

where

$$G_1(\xi, y) = \frac{[f(\overline{U}) - f'(0)\overline{U}] + \left(\frac{\overline{U}^{\gamma+1}}{\beta^\gamma}\right)\left[f'(0) - \frac{f(\beta)}{\beta}\right]}{\frac{\overline{U}^{\gamma+1}}{\beta^\gamma}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)}. \tag{3.6}$$

It is clear that if $\epsilon_1 \to 0$, then $c \to c_0$, $\lambda_1(c) \to \lambda_0(c_0)$ and $\varphi_1 \to \varphi_0$. Thus, for $\epsilon_1 \ll 1$, in the sense of Definition A.1 and Lemma A.2, the function $\overline{U}$ is an upper solution to (3.1) if

$$\max_{(\xi,y)\in\mathbb{R}\times\Omega} G_1(\xi, y) < (\gamma+1)\lambda_0^2(c_0) + (\gamma+1)\frac{\varphi_0^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_0}\right)\right]^2. \tag{3.7}$$

Consequently, we have the following lemma for an upper solution.

Lemma 3.1. *Suppose $c = c_0 + \epsilon_1$ with ϵ_1 being a sufficiently small positive number. If the inequality (3.7) holds, then the function $\overline{U}$, defined in (3.3), is an upper solution to system (1.3)–(1.4) with $\overline{U}(-\infty, y) = \beta(y)$ and $\overline{U}(+\infty, y) = 0$.*

Remark 3.2. To have the above lemma hold, we need the boundedness of G_1 (at least being bounded from above). Indeed, $G_1(\xi, y)$ is continuous on $(\xi, y) \in \mathbb{R} \times \Omega$. Thus it suffices to find $\lim_{\xi\pm\infty} G_1(\xi, y)$ and determine whether they are bounded. As $\xi \to -\infty$, i.e., $\overline{U} \to \beta$, we have

$$\begin{aligned}\lim_{\xi\to-\infty} G_1(\xi, y) &= \lim_{\overline{U}\to\beta}\left\{\frac{f(\overline{U}) - \frac{\overline{U}^{\gamma+1}}{\beta^{\gamma+1}}f(\beta)}{\frac{\overline{U}^{\gamma+1}}{\beta^\gamma}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)} - \frac{f'(0)}{\frac{\overline{U}^\gamma}{\beta^\gamma}}\right\} \\ &= -\frac{f'(\beta)}{\gamma} + \frac{\gamma+1}{\gamma\beta}f(\beta) - f'(0).\end{aligned} \tag{3.8}$$

The last equality is obtained by L'Hospital's rule. For $\xi \to +\infty$, i.e., $\overline{U} \to 0$, we have

$$\begin{aligned}\lim_{\xi\to+\infty} G_1(\xi, y) &= \lim_{\overline{U}\to 0}\left\{\frac{f(\overline{U}) - f'(0)\overline{U}}{\frac{\overline{U}^{\gamma+1}}{\beta^\gamma}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)} + \frac{f'(0) - \frac{f(\beta)}{\beta}}{1 - \frac{\overline{U}^\gamma}{\beta^\gamma}}\right\} \\ &= \lim_{\overline{U}\to 0}\frac{f''(U)}{\gamma(\gamma+1)\frac{\overline{U}^{\gamma-1}}{\beta^\gamma} - 2\gamma(2\gamma+1)\frac{\overline{U}^{2\gamma-1}}{\beta^{2\gamma}}} \\ &\quad + f'(0) - \frac{f(\beta)}{\beta}\end{aligned} \tag{3.9}$$

The boundedness of the above term depends on the choice of γ and the formula of $f(u)$. Actually, we give the following results.

(1) If $f''(0)$ exists, then by choosing $\frac{1}{2} \leq \gamma \leq 1$, we find that G_1 is bounded for all $-\infty < \xi < +\infty$.
(2) If $U = 0$ is a solution for $f''(U) = 0$ with multiplicity k, $k = 1, 2, \ldots$, then by choosing $\gamma = k+1$, we also find that G_1 is bounded for all $-\infty < \xi < +\infty$.

According to Theorem A.4 in the Appendix, to obtain the existence of traveling wave solution $U(\xi, y)$, we also need to find a lower solution to Eq. (1.3) when $c = c_0 + \epsilon_1$. For this purpose, define a continuous function $\underline{U}(\xi, y)$ as

$$\underline{U}(\xi, y) = \max\{0, \varphi_1(y)(1 - Me^{-\delta\xi})e^{-\lambda_1(c)\xi}\}. \tag{3.10}$$

Here, $(\lambda_1(c), \varphi_1)$ has the same meaning as in $\overline{U}$ from Lemma 3.1. We fix a small $\delta > 0$ such that $\lambda_1 + \delta < \lambda_2$ and the constant $M > 0$ is to be determined. Letting $\xi_0 = \frac{\ln M}{\delta}$, it is easy to see that $\underline{U}$ satisfies the following:

(1) When $\xi \le \xi_0$, $\underline{U} = 0$;
(2) When $\xi > \xi_0$, $\underline{U} = \varphi_1(1 - Me^{-\delta\xi})e^{-\lambda_1\xi}$.

Notice that $\max_{\xi\in\mathbb{R}} \underline{U}(\xi, y) = \frac{\delta\varphi_1(y)}{\lambda_1+\delta}\left[\frac{\lambda_1}{M(\lambda_1+\delta)}\right]^{\frac{\lambda_1}{\delta}} \ll 1$ when M is sufficiently large. Furthermore, we can obtain the following lemma.

Lemma 3.3. *When $c = c_0 + \epsilon_1$, the function defined in (3.10) is a lower solution to the system (1.3)–(1.4).*

Proof. If $\xi \le \xi_0$ (i.e., $\underline{U} = 0$), a direct computation gives $\mathcal{L}(\underline{U}) = 0$. If $\xi > \xi_0$, by substituting the formula of $\underline{U}$, we obtain the following:

$$\begin{aligned}\mathcal{L}(\underline{U}) &= -e^{-(\delta+\lambda_1)\xi} ML_{(\lambda_1+\delta)}\varphi_1 + f(\underline{U}) - f'(0)(1 - Me^{-\delta\xi})\varphi_1 e^{-\lambda_1\xi}\\ &= -e^{-(\delta+\lambda_1)\xi} ML_{(\lambda_1+\delta)}\varphi_1 + f(\underline{U}) - f'(0)\underline{U}\\ &\ge 0 \end{aligned} \tag{3.11}$$

provided that M is sufficiently large. Note that, in the last inequality, we have used the fact that $L_{\lambda_1}\varphi_1 = 0$ and $L_{(\lambda_1+\delta)}\varphi_1 < 0$ when $\lambda_1 + \delta < \lambda_2$, and $[f(\underline{U}) - f'(0)\underline{U}] \sim O(\varphi_1^2 e^{-2\lambda_1\xi})$ as $\underline{U}$ is close to 0. By (3.11), Definition A.1 and Lemma A.2, it then follows that there exist positive numbers δ and $M = M(\delta)$ such that $\underline{U}$ is a lower solution of (1.3)–(1.4) when $c = c_0 + \epsilon_1$. This completes the proof. ■

Now, with the construction of an upper and a lower solution above, it is easy to find a ξ_1 so that $\overline{U}(\xi - \xi_1)$ is still an upper solution with $0 \le \underline{U} \le \overline{U}(\xi - \xi_1)$. Therefore, we are ready to give our results for the linear speed selection.

Theorem 3.4. *When (3.7) is satisfied, the minimal wave speed c_{min} of the system (1.3)–(1.4) is linearly selected, i.e., $c_{min} = c_0$.*

Proof. When $c = c_0 + \epsilon_1$, by Lemmas 3.1 and 3.3, we have a pair of an upper and a lower solution. Thus, the existence of a monotone traveling wave solution U of (1.3)–(1.4) with speed $c = c_0 + \epsilon_1$ follows from Theorem A.4 and the traveling wave solution satisfies $U(+\infty, y) = 0$ and $U(-\infty, y) = \beta(y)$.

In the case when $c = c_0$, a limiting argument can be applied to obtain the existence of traveling waves.

To be exact, we choose a sequence $\{c_n\}$ such that $c_n \in (c_0, c_0 + 1]$ and $\lim_{n\to+\infty} c_n = c_0$. For instance, we can choose $c_n = c_0 + \frac{1}{n}$, which clearly satisfies the requirement. Corresponding to each c_n, by the above arguments and Theorem A.4, there exists a monotone decreasing traveling wave solution $U_n(\xi, y)$ of (1.3)–(1.4). Since $U_n(\xi + \bar{\xi}_0, y)$, $\bar{\xi}_0 \in \mathbb{R}$ is also such a solution, by translation, we can always assume $U_n(0, y_0) = \frac{1}{2}\beta(y_0)$ for a given $y_0 \in \Omega$.

Notice that $U_n(\xi, y)$ is uniformly bounded, that is, $|U_n(\xi, y)| \le \beta(y) \le \max \beta(y)$, $\forall(\xi, y) \in \mathbb{R} \times \bar{\Omega}$, $n \ge 1$. According to Theorem A.4, U_n is the fixed point of the solution map $T_{-ct}Q_t$, that is, $T_{-ct}Q_t[U_n](x, y) = U_n(x, y)$. Moreover, $\{T_{-ct}Q_t[U_n]\}_{n\ge1}$ is precompact. It then follows that there exists a convergent subsequence of U_n, say $\{U_{n_k}\}_{k\ge1}$, converging to a function $W \in \mathcal{C}_\beta$ as $k \to +\infty$. That is, there exists a function W satisfying $Q_t[W](x, y) = W(x - c_0t, y) = W(\xi, y)$, or equivalently the equation

$$W_{\xi\xi} + \Delta_y W + (\alpha(y) + c_0)W_\xi + f(W) = 0,\ (\xi, y) \in \mathbb{R} \times \Omega.$$

Clearly, $W(\xi, y)$ is non-increasing in $\xi \in \mathbb{R}$ and $W(0, y_0) = \frac{1}{2}\beta(y_0)$. Moreover, $W(\xi, y)$ connects β to 0 with $W(-\infty, y) = \beta(y)$ and $W(+\infty, y) = 0$ for all $y \in \Omega$. Consequently, when (3.7) is satisfied, (1.3)–(1.4) has a monotone traveling wave solution connecting $\beta(y)$ to 0 with $c = c_0$. The proof is complete. ■

Next, we want to investigate the nonlinear speed selection. To proceed, we first prove the following lemma.

Lemma 3.5. *For $c_1 > c_0$, suppose that there exists a lower solution $\underline{U}(\xi, y)$ to system (1.3)–(1.4), which is non-increasing in ξ and satisfies $0 < \underline{U} < \beta(y)$ and*

$$\underline{U} \sim \varphi_2(y)e^{-\lambda_2(c_1)\xi}$$

as $\xi \to +\infty$, where $(\lambda_2(c_1), \varphi_2)$ is defined in Theorem 2.2 and $\xi = x - c_1t$, i.e., $\underline{U}(\xi, y)$ has the faster decay rate near positive infinity. Then there is no traveling wave solution to system (1.3)–(1.4) connecting $\beta(y)$ to 0 with speed $c \in [c_0, c_1)$.

Proof. By this assumption, there exists a lower solution $\underline{U}(x - c_1t, y)$ with $c_1 > c_0$ to

$$u_t = u_{xx} + \Delta_y u + \alpha(y)u_x + f(u), \tag{3.12}$$

with the initial data

$$u(x, y, 0) = \underline{U}(x, y).$$

By way of contradiction, we assume that, for some $c \in [c_0, c_1)$, there exists a monotonic traveling wave solution $U(x - ct, y)$, which connects $\beta(y)$ to 0 and has the initial data as

$$u(x, y, 0) = U(x, y).$$

We should note that if $c = c_0$, then we have traveling wave solutions for all $c > c_0$. Thus we can always assume that $c \in (c_0, c_1)$.

Following the calculations from the previous section (see, e.g., from (2.1) to (2.4)), it is easy to find the asymptotic behavior of $U(x - ct, y)$ with

$$U(\xi, y) \sim C_1\varphi_1(y)e^{-\lambda_1(c)\xi} + C_2\varphi_2(y)e^{-\lambda_2(c)\xi},\ \ \xi \to \infty,$$

for $C_1 > 0$, or $C_1 = 0, C_2 > 0$. A rigorous proof of this can be obtained by the comparison principle and the linearization of the model. Moreover, we have $\lambda_2(c_1) > \lambda_2(c) > \lambda_0(c_0) > \lambda_1(c) > \lambda_1(c_1)$ when $c \in [c_0, c_1)$. Thus, we can always assume $\underline{U}(x, y) \le U(x, y)$ for $(x, y) \in \mathbb{R} \times \Omega$ (by shifting of U if necessary). Since $\underline{U}(\xi, y)$, $\xi = x - ct$, is assumed to be a lower solution to Eq. (3.12) and $\underline{U}(x, y) \le U(x, y)$, by comparison, we have

$$\underline{U}(x - c_1t, y) \le U(x - ct, y),\ (x, y, t) \in \mathbb{R} \times \Omega \times \mathbb{R}_+. \tag{3.13}$$

Now, if we fix $\xi_1 = x - c_1t$, then $\underline{U}(\xi_1, y) > 0$ is fixed. On the other hand, from $U(x - ct, y)$, it is clear to see

$$U(x - ct, y) = U(\xi_1 + (c_1 - c)t, y) \sim U(+\infty, y) = 0,\ \text{as } t \to +\infty.$$

By (3.13), we therefore get $\underline{U}(\xi_1, y) \le 0$. This is a contradiction. Thus, there is no traveling wave solution when $c \in [c_0, c_1)$. This completes the proof. ■

Remark 3.6. This lemma implies that if there is a lower solution $\underline{U}$ satisfying $0 < \underline{U} < \beta(y)$ and $\underline{U} \sim \varphi_2(y)e^{-\lambda_2(c_1)\xi}$ as $\xi \to +\infty$, for $c_1 > c_0$, then the nonlinear selection is realized.

Now, let $c_1 = c_0 + \epsilon_2$ and define a continuous function as follows

$$\underline{U}_1 = \frac{\beta\varphi_2}{\left[\beta^\gamma e^{\lambda_2(c_1)\gamma\xi} + \varphi_2^\gamma\right]^{\frac{1}{\gamma}}}. \tag{3.14}$$

Similarly to the previous computations, we get

$$\mathcal{L}(\underline{U}_1) = \frac{\underline{U}_1^{(\gamma+1)}}{\beta^\gamma}\left(1-\frac{\underline{U}_1^\gamma}{\beta^\gamma}\right)\Bigg\{-(\gamma+1)\,\lambda_2^2(c_1)-(\gamma+1) \times \frac{\varphi_2^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_2}\right)\right]^2 + G_2(\xi,y)\Bigg\}, \tag{3.15}$$

where

$$G_2(\xi,y) = \frac{[f(\underline{U}_1)-f'(0)\underline{U}_1] + \left(\frac{\underline{U}_1^{\gamma+1}}{\beta^\gamma}\right)\left[f'(0)-\frac{f(\beta)}{\beta}\right]}{\frac{\underline{U}_1^{\gamma+1}}{\beta^\gamma}\left(1-\frac{\underline{U}_1^\gamma}{\beta^\gamma}\right)}. \tag{3.16}$$

To obtain a condition for the nonlinear selection, we will take $\underline{U}_1$ as the lower solution which satisfies $\underline{U}_1 \sim \varphi_2(y)e^{-\lambda_2(c_1)\xi}$ as $\xi \to +\infty$. Notice that when $\epsilon_2 \to 0$, we have $\lambda_2(c_1) \to \lambda_0(c_0)$ and $\varphi_2 \to \varphi_0$. Thus, if the following condition

$$\min_{(\xi,y)\in\mathbb{R}\times\Omega} G_2(\xi,y) > (\gamma+1)\lambda_0^2(c_0) + (\gamma+1)\frac{\varphi_0^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_0}\right)\right]^2 \tag{3.17}$$

is true, then the nonlinear selection is realized.

In the case of Neumann boundary conditions, we have $\beta(y) \equiv 1$, and thus (3.14) can be simplified as

$$\underline{U}_1 = \frac{\varphi_2}{\left(e^{\lambda_2(c)\gamma\xi}+\varphi_2^\gamma\right)^{\frac{1}{\gamma}}}. \tag{3.18}$$

We thus have

$$\mathcal{L}(\underline{U}_1) = \underline{U}_1^{(\gamma+1)}\left(1-\underline{U}_1^\gamma\right)\Bigg\{-(\gamma+1)\,\lambda_2^2(c)-(\gamma+1) \times \left[\varphi_2\nabla\left(\frac{1}{\varphi_2}\right)\right]^2 + G_2(\xi,y)\Bigg\}, \tag{3.19}$$

and

$$G_2(\xi,y) = \frac{f(\underline{U}_1)-f'(0)\underline{U}_1+\underline{U}_1^{\gamma+1}f'(0)}{\underline{U}_1^{\gamma+1}\left(1-\underline{U}_1^\gamma\right)}.$$

Moreover, when $\epsilon_2 \to 0$, under the condition

$$\min_{(\xi,y)\in\mathbb{R}\times\Omega} G_2(\xi,y) > (\gamma+1)\lambda_0^2(c_0)+(\gamma+1)\left[\varphi_0\nabla\left(\frac{1}{\varphi_0}\right)\right]^2, \tag{3.20}$$

we are ready to have the nonlinear selection as follows.

Theorem 3.7. *If the inequality (3.20) is satisfied, then the minimal speed of system (1.3)–(1.4) prescribed by the Neumann boundary condition is nonlinearly selected.*

In the case of Dirichlet boundary condition, through similar analysis to that in Remark 3.2, we obtain

$$\lim_{\xi\to-\infty} G_2(\xi,y) = -\frac{f'(\beta)}{\gamma} + \frac{\gamma+1}{\gamma\beta}f(\beta) - f'(0)$$

or

$$\lim_{\xi\to-\infty} G_2(\xi,y) = -\frac{\beta}{\gamma}g'(\beta) - g(0) + g(\beta)$$

where $g(u) = f(u)/u$. This gives $\lim_{y\to\partial\Omega}\lim_{\xi\to-\infty} G_2(\xi,y) = 0$. Thus (3.17) cannot be true, i.e., this choice of the lower solution (i.e., $\underline{U}_1$ in (3.14)) is not valid when (1.3)–(1.4) is prescribed by the Dirichlet boundary condition. We suspect that other challenging types of lower solutions need to be constructed. This will be a subject of our future study.

4. Applications

In this section, we apply the results of Section 3 to the reaction–diffusion model with a cubic reaction term and a subcritical quintic Ginzburg–Landau equation respectively. By applying numerical simulations to each case, we will find the linear wave speed, i.e., c_0 defined in Theorem 2.2 as well as the numerical minimal wave speed. Comparison of them is carried out to illustrate our theoretical results.

4.1. A cubic reaction term

The first application is a cubic reaction term given as $f(u) = u(1-u)(1+2\epsilon u)$ with $\epsilon \geq 0$ and $\Omega = (-L_y, L_y)$, that is, we consider traveling wave solutions of the following equation

$$u_t = u_{xx} + u_{yy} + \alpha(y)u_x + u(1-u)(1+2\epsilon u), \quad (x,y)\in\mathbb{R}\times(-L_y,L_y),\ t>0. \tag{4.1}$$

The corresponding wave profile (i.e., letting $u(t,x,y) = U(\xi,y)$ and $\xi = x - ct$) equation becomes

$$U_{\xi\xi} + U_{yy} + (\alpha(y)+c)\,U_\xi + U(1-U)(1+2\epsilon U) = 0, \tag{4.2}$$

satisfying

$$\lim_{\xi\to+\infty} U(\xi,y) = 0, \text{ and } \lim_{\xi\to-\infty} U(\xi,y) = \beta(y). \tag{4.3}$$

The speed selection of such an equation in one dimensional case was first considered by Hadeler and Rothe [27] in 1975. They studied the equation

$$u_t = u_{xx} + u(1-u)(1+2\epsilon u),\ \epsilon > -\frac{1}{2},\ x\in\mathbb{R},\ t>0, \tag{4.4}$$

and obtained that the minimal speed of the traveling waves is linearly selected when $\epsilon \leq 1$ and nonlinearly selected when $\epsilon > 1$. For more details of this result, please refer to [27].

In the sequel, for the model (4.1) we always assume that $\epsilon > 0$ and also show that there exists a critical number of ϵ to classify the linear and nonlinear selection mechanism.

The reaction term f is smooth on [0, 1] and

$$f(0) = f(1) = 0,\ f'(0) = 1 > 0 > f'(1) = -1-2\epsilon, \quad \text{and } f(u) > 0 \text{ for } u \in (0,1).$$

Thus f satisfies (A1), (A4) and (A5) for all ϵ. Moreover, there are equilibria 0 and a nonzero function $\beta(y)$ with $0 \leq \beta(y) \leq 1$ for all $y \in \overline{\Omega}$. Since $-2-8\epsilon \leq f''(u) = 4\epsilon - 2 - 12\epsilon u \leq 4\epsilon - 2$, we can choose $\gamma = 1$ in (3.5). Then, by substituting the formula of f into Eq. (3.5) and simplifying it, we obtain

$$\mathcal{L}(\overline{U}) = \frac{\overline{U}^2}{\beta}\left(1-\frac{\overline{U}}{\beta}\right)\left\{-2\lambda_1^2(c) - 2\frac{(\beta'\varphi_1-\beta\varphi_1')^2}{\varphi_1^2\beta^2} + 2\epsilon\beta^2\right\}. \tag{4.5}$$

Here, $G_1(\xi,y) = 2\epsilon\beta^2$ is clearly monotonic in ϵ. Thus, the condition (3.7) for the linear selection becomes

$$\epsilon < \min_{y\in\Omega}\left[\frac{\lambda_0^2(c_0)}{\beta^2} + \frac{(\beta'\varphi_0-\beta\varphi_0')^2}{\varphi_0^2\beta^4}\right]. \tag{4.6}$$

Similarly, the condition (3.20) for the nonlinear selection becomes

$$\epsilon > \max_{y\in\Omega}\left[\frac{\lambda_0^2(c_0)}{\beta^2} + \frac{(\beta'\varphi_0-\beta\varphi_0')^2}{\varphi_0^2\beta^4}\right]. \tag{4.7}$$

Next, we will show the existence of a threshold value of ϵ so that, when ϵ increases to cross through this critical value, the speed selection changes from linear to nonlinear. To this end, we want to prove the following lemma first.

Lemma 4.1. *Let (4.2)–(4.3) be prescribed by Neumann boundary conditions (or Dirichlet boundary conditions). If the wave speed is linearly selected when $\epsilon = \epsilon_l$ for some $\epsilon_l > 0$, then it is linearly selected for all $\epsilon < \epsilon_l$.*

Proof. By this assumption, when $\epsilon = \epsilon_l$, we have U_l as a solution, which is decreasing in $\xi \in \mathbb{R}$, with $c = c_0 + \epsilon_1$ to (4.2) for any small $\epsilon_1 > 0$. Thus, it satisfies

$$(U_l)_{\xi\xi} + (U_l)_{yy} + (\alpha(y) + c)(U_l)_\xi + U_l(1 - U_l)(1 + 2\epsilon_l U_l) = 0. \quad (4.8)$$

Then, by substituting $U_l(\xi, y)$ into (4.2) with $\epsilon < \epsilon_l$, we obtain

$$\begin{aligned} &(U_l)_{\xi\xi} + (U_l)_{yy} + (\alpha(y) + c)(U_l)_\xi + U_l(1 - U_l)(1 + 2\epsilon U_l) \\ =\ &(U_l)_{\xi\xi} + (U_l)_{yy} + (\alpha(y) + c)(U_l)_\xi + U_l(1 - U_l) \\ &\times(1 + 2\epsilon_l U_l - 2\epsilon_l U_l + 2\epsilon U_l) \\ =\ &-2U_l^2(1 - U_l)(\epsilon_l - \epsilon) \\ \le\ &0. \end{aligned}$$

This implies that U_l can be viewed as an upper solution to (4.2) for $\epsilon < \epsilon_l$. Then taking the lower solution defined in Lemma 3.3, we conclude that the wave speed is linearly selected for $\epsilon < \epsilon_l$. This completes the proof. ■

From the above lemma, we can define the threshold value of ϵ as

$$\epsilon_c := \sup\{\epsilon |\ \text{the linear speed selection of (4.2)–(4.3) is realized}\}. \quad (4.9)$$

Remark 4.2. By the above definition, we have $0 \le \epsilon_c \le \infty$. Furthermore, if $\epsilon_c = 0$, then the interval $0 < \epsilon \le \epsilon_c$ is empty, thus the nonlinear speed selection is realized for all $\epsilon > 0$; if $\epsilon_c = \infty$, then the linear speed selection is realized for all $\epsilon \ge 0$.

Depending on the choice of boundary conditions, the critical value ϵ_c may differ. We start with the case where (4.2)–(4.3) is prescribed by Neumann boundary conditions, i.e., $U_y(\xi, -L_y) = U_y(\xi, L_y) = 0$. In this case, $\beta(y) \equiv 1$ and we have the following theorem about the value of ϵ_c.

Theorem 4.3. *If the system (4.2)–(4.3) is prescribed by the Neumann boundary condition, then*

$$\lambda_0^2(c_0) \le \epsilon_c \le \lambda_0^2(c_0) + \max_{y\in[-L,L]} \left(\frac{\varphi_0'}{\varphi_0}\right)^2,$$

where λ_0 and φ_0 are defined in Theorem 2.2.

Proof. For the Neumann boundary case, we have $\beta \equiv 1$; thus, (4.6) reduces to $\epsilon < \lambda^2(c_0)$ due to the fact that $\min(\varphi_0')^2 = 0$ (at the boundary). It leaves us to prove the linear selection in the case when $\epsilon = \lambda_0^2(c_0)$. To this end, we choose a sequence $\epsilon_n \to \lambda_0^2(c_0)$. By Theorem 3.4, it follows that (4.2)–(4.3) has a monotone traveling wave solution when $c = c_0$ for any $\epsilon = \epsilon_n$. Due to the compactness of the solution map, a limiting argument gives the existence of traveling waves when $\epsilon = \lambda_0^2(c_0)$ for all $c \ge c_0$. In other words, when $\epsilon = \lambda_0^2(c_0)$, the minimal speed of (4.2)–(4.3) is linearly selected.

To obtain an upper bound of the critical value ϵ_c, we will concentrate on the nonlinear selection. From (4.7) and Theorem 3.7, it follows that the nonlinear selection is realized when $\epsilon > \lambda_0^2 + \max_{y\in\Omega}\left(\frac{\varphi_0'}{\varphi_0}\right)^2$. Consequently, combining those results, this theorem holds. ■

Remark 4.4. From Theorem 4.3, for the Neumann boundary case with $\alpha(y) = 0$, we obtain that $\epsilon_c = 1$ since $\varphi_0 = 1$ and $\lambda_0 = 1$ under such a condition. This recovers the result of [27],

For the case where (4.2)–(4.3) is prescribed by Dirichlet boundary conditions, i.e.,

$$U(\xi, -L_y) = U(\xi, L_y) = 0,$$

we have $0 \le \beta(y) \le 1$ for $y \in [-L_y, L_y]$. From (4.6), it immediately follows that (4.2) is linearly selected when $\epsilon = \lambda_0^2(c_0) < \min_{y\in\Omega} \frac{\lambda_0^2(c_0)}{\beta^2}$. Furthermore, it is easy to see that $\min_{y\in\Omega} \frac{\lambda_0^2(c_0)}{\beta^2} = \frac{\lambda_0^2(c_0)}{\max_{y\in\Omega}(\beta^2)}$. Similarly to Theorem 4.3, we arrive at the following result for the linear selection.

Theorem 4.5. *Let the system (4.2)–(4.3) be prescribed by Dirichlet boundary conditions. Then the linear selection is realized for all $\epsilon \le \bar{\epsilon}$, where $\bar{\epsilon} = \frac{\lambda_0^2}{\max_{y\in\Omega}(\beta^2)}$.*

Let us now perform some numerical simulations on (4.2)–(4.3) using the Matlab software. To make our numeric method look more convincing, we first compare the numerical results with the accurate solution obtained in [27]. The authors have found that the formula of the minimal wave speed is

$$c_{\min} = \begin{cases} 2, & \epsilon \le 1, \\ \sqrt{\epsilon} + \sqrt{\frac{1}{\epsilon}}, & \epsilon \ge 1, \end{cases} \quad (4.10)$$

and the traveling wave solution is a so-called Huxley's solution

$$u(x, t) = \frac{1}{1 + e^{\sqrt{\epsilon}(x - ct)}}, \ \text{ with } c = \sqrt{\epsilon} + \sqrt{\frac{1}{\epsilon}}.$$

The comparison results are summarized in Fig. 1. The figures show results related to the minimal wave speed. The left figure tells us that our numerically computed speeds match the speeds predicted by the accurate formula (4.10); the right one shows the absolute difference between them, which are as small as $O(10^{-3})$. Thus, our numeric methods are reliable and will be explained in detail in the following context.

Throughout simulations in the rest of this section, we fix $\alpha(y) = \sin(y)$ if not specified otherwise, and $L_y = 5\pi$. The simulations are also taken into two cases: one is the Neumann boundary condition case and the other one is the Dirichlet boundary condition case.

(1) When (4.2)–(4.3) is prescribed by the Neumann boundary condition, we do the following numerical computations. Through applying the central difference method to the eigenvalue problem (2.4), we determine that $c_0 = 2.58$ and $\lambda_0 = 0.93$. As we can see in Fig. 2, the large one manifests the relation between μ and λ, which verifies the convexity of $\mu(\lambda)$ with respect to λ; the small one is an enlarged figure when $\lambda \in [0.6, 1.2]$, which implies $c_0 = 2.58$.

Furthermore, to obtain a traveling wave solution, we do numerical simulations on (4.1). By applying the central difference method on space variables, the 4th-order Runge–Kutta method on the time variable, and choosing an initial condition as

$$u_0(x, y) = \frac{1}{1 + e^{10^5(x + x_0)}}, \ (x, y) \in \mathbb{R} \times \overline{\Omega}, \ x_0 = 900, \quad (4.11)$$

we will obtain a solution that stabilizes to a traveling wave solution. We conjecture without proof that the wave takes the minimal speed due to the fast decaying initial function. To have a stable wave profile, we start to store all the data after $t = 200$. As shown in Fig. 3, the left panel is a 3-D figure that displays the shape of the solution; the right panel is obtained through fixing $y = 0$ and letting $t = 210, 211, \ldots, 220$. Actually, in Fig. 3(b), by letting $u(t, x, 0) \equiv 0.5$, we can find the level set $x(t)$ for every t through linear interpolation, and use it to compute the spreading wave speed. Through this method, we calculate the minimal wave speed c whose result is shown in Fig. 4. As we

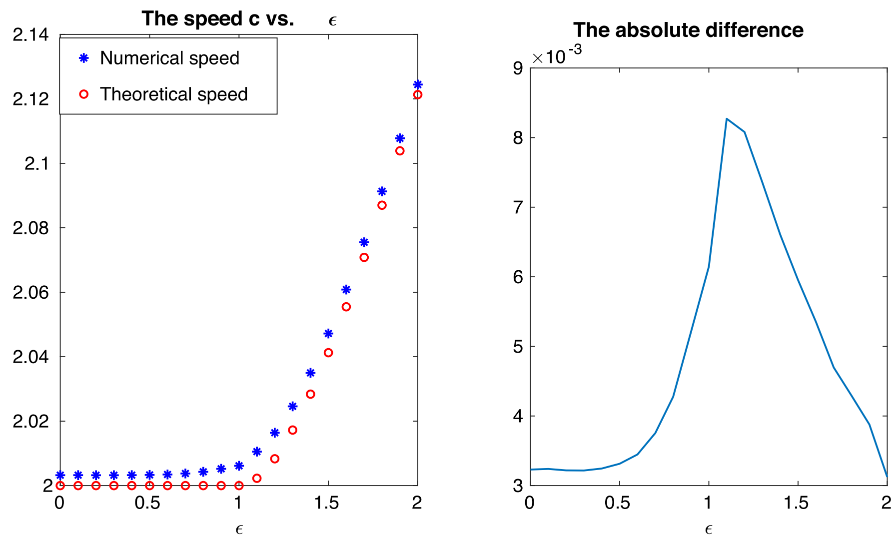


Fig. 1. (Color online) The minimal-speed comparison of numerical results and theoretical results. The figures show the speed for $\epsilon \in [0, 2]$.

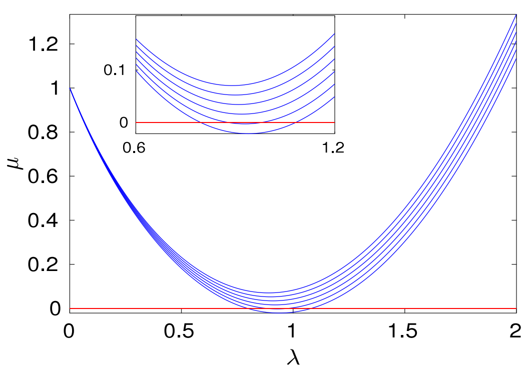


Fig. 2. (Color online) The relation between the principal eigenvalue $\mu(\lambda)$ and λ. From top to bottom, $c = 2.5,\ 2.52,\ 2.54,\ 2.56,\ 2.58$ and 2.6 respectively.

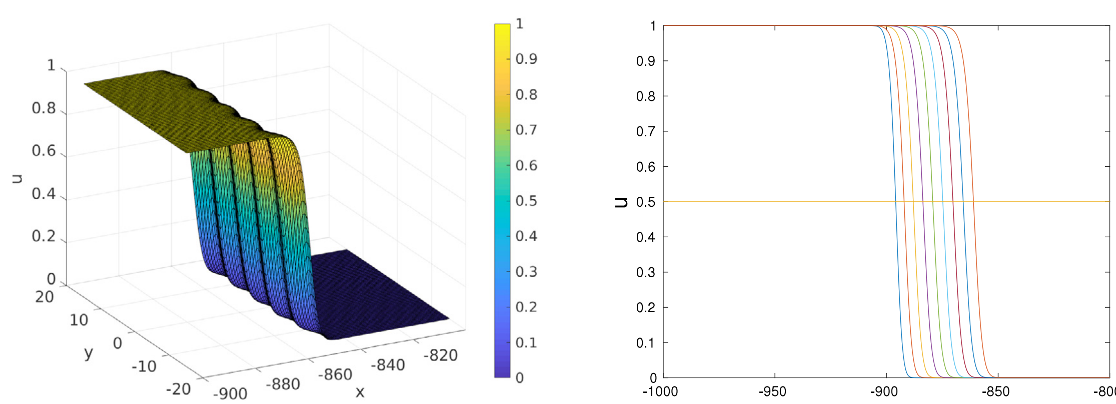
can see in this figure, the numerically computed speed $c_{\text{num}} \simeq c_0$ when $\epsilon \leq \lambda_0^2 = 0.865$. By substituting the value of c_0 and λ_0 into the eigenvalue problem (2.4), we can numerically solve φ_0 and by which we find $\max_{y\in[-L,L]}\left(\frac{\varphi_0'}{\varphi_0}\right)^2 = 0.2205$. Therefore, by Theorem 4.3, the system is nonlinearly selected if $\epsilon > 1.091$, which has been verified by the figure. Actually, from the numerical simulation, we find that $\epsilon_c \simeq 1$.

(2) When (4.2)–(4.3) is prescribed by the Dirichlet boundary condition, we do similar simulations. The same method applied to the eigenvalue problem (2.4) with Dirichlet boundary conditions, we obtain $c_0 = 2.36$ and $\lambda_0 = 0.885$. Next, to obtain a traveling wave solution here, we choose the initial data as

$$u_0(x, y) = \frac{\cos(\pi y/2L_y)}{1 + e^{10^5(x+x_0)}},\ (x, y) \in \mathbb{R} \times \overline{\Omega},\ x_0 = 900. \tag{4.12}$$

Due to the zero boundary condition, the shape of a traveling wave solution in this case looks like an arch, which is quite different from the former one and is shown in Fig. 5. Finally, using the same method as the one used in the previous case, we calculate the wave speed corresponding to different values of the parameter ϵ. The results are shown in Fig. 6. As shown in the figure, there is a critical number ϵ_c such that the speed is linearly selected when $\epsilon \leq \epsilon_c$, and nonlinearly selected when $\epsilon > \epsilon_c$. Here, by the numerical simulation, we can see $\epsilon_c \simeq 0.8 > \lambda_0^2 = 0.783$.

To complete the numerical simulations for the model with cubic nonlinearity, we provide some discussions of the effect of $\alpha(y)$ on the critical number ϵ_c when the Neumann boundary condition occurs. When $\alpha(y) \equiv \alpha$ with α being a constant, through a direct computation, we find that the eigenfunction of



(a) (b)

Fig. 3. (Color online) Figure (a) depicts the solution of (4.1) with the Neumann boundary condition when $t = 220$. Figure (b) depicts the solution when $y = 0$, $t = 210, 211, \ldots, 220$. The parameter set corresponds to: $(x, y) \in [-1000, 1000] \times [-5\pi, 5\pi]$ and $x_0 = 900$.

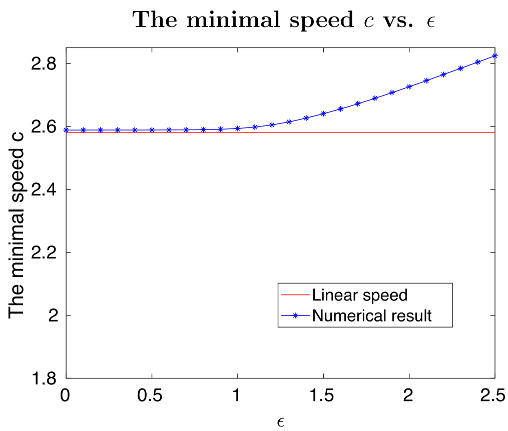


Fig. 4. (Color online) The relation between the asymptotic spreading speed c and ϵ. The blue line with stars denotes the numerically computed speed obtained by direct simulation, and the red line is $c_0 = 2.58$.

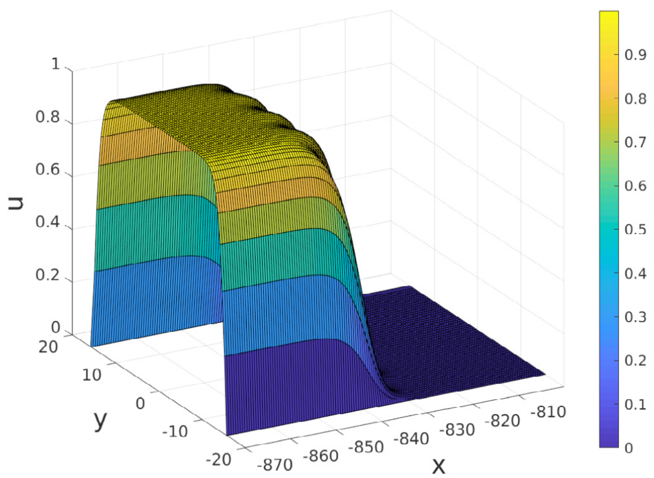


Fig. 5. (Color online) The solution of (4.1) with the Dirichlet boundary condition when $t = 220$.

(2.2) can be always normalized to be "$\varphi_0 = 1$" and the eigenvalue

$$\lambda_0 = \frac{\alpha + c_0}{2} \equiv 1 \text{ where } c_0 = 2 - \alpha.$$

By Theorem 4.3, $\epsilon_c \equiv 1$ for all $\alpha \in \mathbb{R}$. In other words, α only affects the value of the linear speed c_0 but it does not affect the critical value ϵ_c.

When $\alpha(y)$ is not a constant, with the help of numerical simulations, we also find that ϵ_c always equals 1. We first give a table to manifest the influence of α on c_0, λ_0, and the range of ϵ_c by Theorem 4.3. As Table 1 shows, when $\alpha_2(y) \le \alpha_1(y)$ for all $y \in [-L, L]$, $\lambda_{0,2} \ge \lambda_{0,1}$ while $\max_{y\in[-L,L]}\left(\frac{\varphi_{0,2}'}{\varphi_{0,2}}\right)^2 \le \max_{y\in[-L,L]}\left(\frac{\varphi_{0,1}'}{\varphi_{0,1}}\right)^2$, where $\lambda_{0,i}$ $(i = 1, 2)$ denotes λ_0 corresponding to $\alpha_i(y)$ $(i = 1, 2)$ and the same notations are used for $\varphi_{0,i}$. The last column of Table 1 shows the range of ϵ_c. It is clear that all of them contain the value 1. Furthermore, we apply the same numerical method used for $\alpha(y) = \sin(y)$ to other two cases: (a) $\alpha(y) = 1.5\sin(y)$ and (b) $\alpha(y) = 0.5\sin(y)$. The details are shown in Fig. 7. From those figures, we can see that $\epsilon_c = 1$ for both cases. It can be interesting to prove this result rigorously.

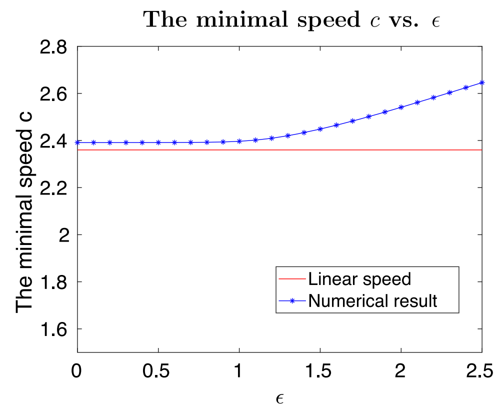


Fig. 6. (Color online) The relation between the minimal speed c and ϵ. The blue line with stars denotes the numerically computed speed obtained by direct numerical simulations and the red line is $c_0 = 2.36$.

Table 1
The influence of $\alpha(y)$ on the range of ϵ_c.

$\alpha(y)$	c_0	λ_0	$\max_{y\in[-L,L]}\left(\frac{\varphi_0'}{\varphi_0}\right)^2$	The range of ϵ_c
$1.5\sin(y)$	2.95	0.914	0.3362	[0.8354,1.1713]
$1.25\sin(y)$	2.7	0.92	0.2747	[0.8464,1.1211]
$\sin(y)$	2.58	0.93	0.2195	[0.8649,1.0844]
$0.75\sin(y)$	2.4	0.938	0.1683	[0.8798,1.0481]
$0.5\sin(y)$	2.23	0.951	0.1191	[0.9044,1.0235]
$0.2\sin(y)$	2.08	0.974	0.0645	[0.9478,1.0132]
0	2	1	0	1

4.2. Subcritical quintic Ginzburg–Landau equation

In our second application, we consider a subcritical quintic Ginzburg–Landau equation in a cylindrical domain. The equation is given by

$$u_t = u_{xx}+u_{yy}+\alpha(y)u_x+\mu u+u^3-u^5,\ (x, y) \in \mathbb{R}\times\Omega,\ \mu > 0. \quad (4.13)$$

Here $f(u) = \mu u + u^3 - u^5$ and $\Omega = (-L_y, L_y)$. Thus, for traveling wave solutions, we mean $u(t, x, y) = U(\xi, y)$ where $\xi = x - ct$. Then, the equation for the wave profile is

$$U_{\xi\xi} + U_{yy} + (\alpha(y) + c)\, U_\xi + \mu U + U^3 - U^5 = 0, \quad (4.14)$$

satisfying

$$\lim_{\xi\to+\infty} U(\xi, y) = 0,\quad \lim_{\xi\to-\infty} U(\xi, y) = \beta(y) \le \mu_+,\ y \in \Omega, \quad (4.15)$$

where

$$\mu_+ = \sqrt{\frac{1 + \sqrt{1 + 4\mu}}{2}} > 1.$$

It is easy to have

$$f(0) = f(\mu_+) = 0,\ f'(0) = \mu > 0 > f'(\mu_+) = -2\mu_+^2\sqrt{1 + 4\mu_+}.$$

Clearly, f satisfies (A1) and (A4). Notice that $f'(0)$ depends on the parameter μ. Thus, we may require some extra conditions on μ for f to satisfy (A5) when (4.14) is prescribed by the Dirichlet boundary condition.

Since $f''(u) = 6u - 20u^3$ and $f'''(u) = 6 - 60u^2$, $u = 0$ is a solution of $f''(u) = 0$ with multiplicity $k = 1$. Following Remark 3.2, we will choose $\gamma = 2$ in (3.2). By substituting the formula of f into Eq. (3.5) and simplifying it, we then obtain

$$\mathcal{L}(U_1) = \frac{U_1^3}{\beta^2}\left(1 - \frac{U_1^2}{\beta^2}\right)\left\{-3\lambda_1^2(c) - 3\frac{(\beta'\varphi_1 - \beta\varphi_1')^2}{\varphi_1^2\beta^2} + \beta^4\right\}, \quad (4.16)$$

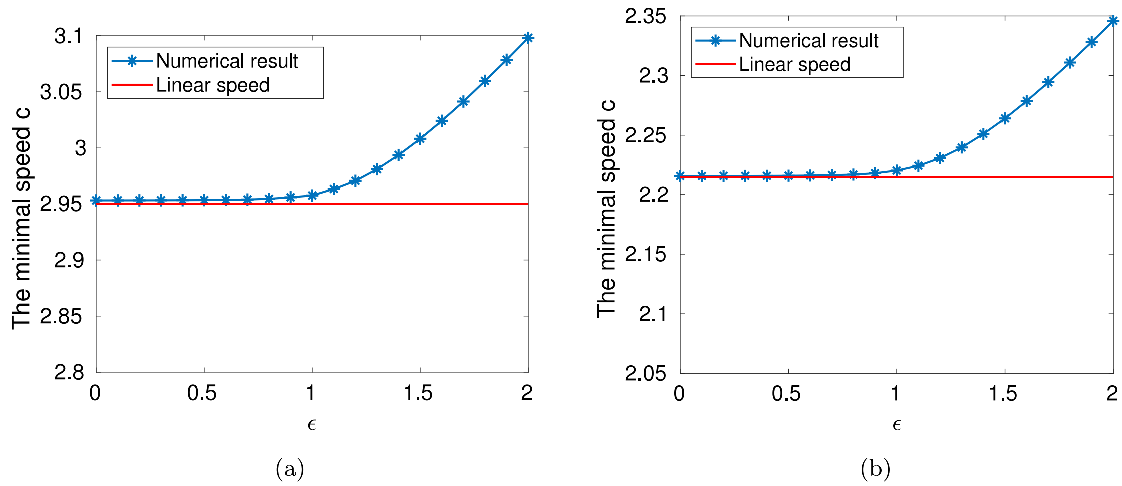


Fig. 7. (Color online) The numerical speed c corresponding to different ϵ. Figure (a) is depicted when $\alpha(y) = 1.5\sin(y)$ while (b) is depicted when $\alpha(y) = 0.5\sin(y)$.

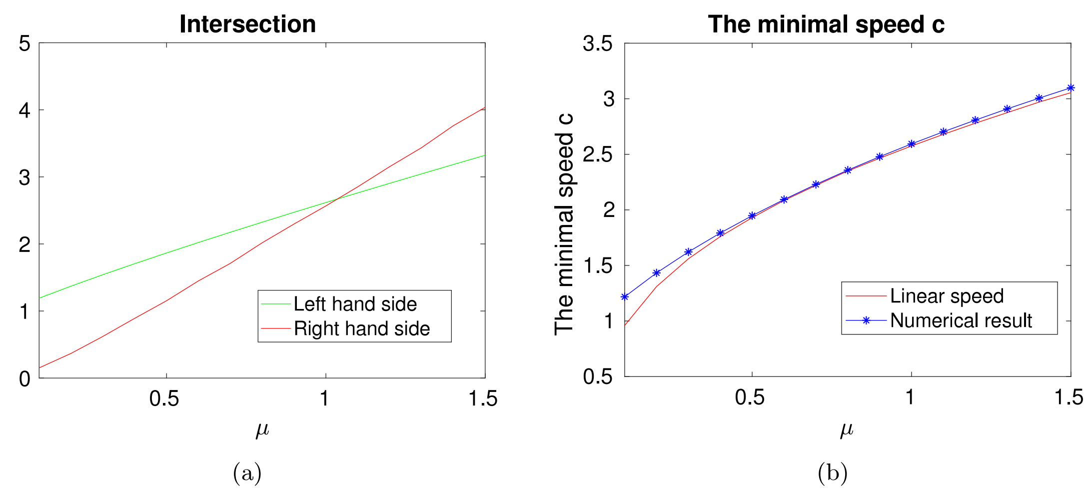


Fig. 8. (Color online) The left panel shows the intersection predicted by the inequality (4.18) with the green line representing the function $\frac{1}{2} + \mu + \frac{1}{2}\sqrt{1+4\mu}$ and the red line representing $3\lambda_0^2$; the right panel depicts the relation between the parameter μ and c_0 (red line) or c_{num} (blue line with stars).

and now $G_1 = \beta^4$. With the condition $0 \le \beta(y) \le \mu_+$ for $y \in \Omega$, we further have

$$\max_{(\xi,y)\in\mathbb{R}\times\Omega} G_1(\xi, y) \le \mu_+^4 = \frac{1}{2} + \mu + \frac{1}{2}\sqrt{1+4\mu}. \tag{4.17}$$

Thus, the condition (3.7) for the linear selection becomes

$$\frac{1}{2} + \mu + \frac{1}{2}\sqrt{1+4\mu} < 3\lambda_0^2(c_0). \tag{4.18}$$

We then have the following theorem.

Theorem 4.6. *When (4.14)–(4.15) is prescribed by Neumann (or Dirichlet) boundary conditions, the minimal wave speed is linearly selected if the inequality (4.18) holds.*

As for the nonlinear selection, we give a condition for the Neumann boundary condition case as follows. Substituting the formula of f into (3.20) gives $G_2 = \beta^4 = \mu_+^4 = \frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\mu}$. Then, we arrive at the following theorem.

Theorem 4.7. *When (4.14)–(4.15) is prescribed by Neumann boundary conditions, the minimal wave speed is nonlinearly selected if*

$$\frac{1}{2} + \mu + \frac{1}{2}\sqrt{1+4\mu} > 3\lambda_0^2(c_0) + 3\left(\frac{\varphi_0'}{\varphi_0}\right)^2, \tag{4.19}$$

where $\lambda_0(c_0)$ and $\varphi_0(c_0)$ are defined in Theorem 2.2.

Remark 4.8. Actually, if the Neumann boundary condition case occurs with $\alpha(y) = 0$, (4.18) and (4.19) imply that there is a critical value $\mu_c = 0.75$ such that the minimal wave speed of (4.14) is linearly selected if $\mu \ge \mu_c$ and nonlinearly selected if $\mu < \mu_c$. This means, our results include the one in [26]. When $\alpha(y) \ne 0$, there is a gap between conditions (4.18) and (4.19), we conjecture that there exists a critical number μ_c and its exact value can be found by numerical simulations.

Next, we perform numerical simulations on (4.14)–(4.15). Here, we also fix $\alpha(y) = \sin(y)$ and $L_y = 5\pi$. Similarly, we apply the same method as that in the previous application and carry out simulations in two cases.

(1) We first do simulations for the Neumann boundary condition case. By direct calculations on (4.18), we obtain the left panel of Fig. 8. In the figure, we use the green line to represent the left hand side of (4.18), i.e., $\frac{1}{2} + \mu + \frac{1}{2}\sqrt{1+4\mu}$, and the red

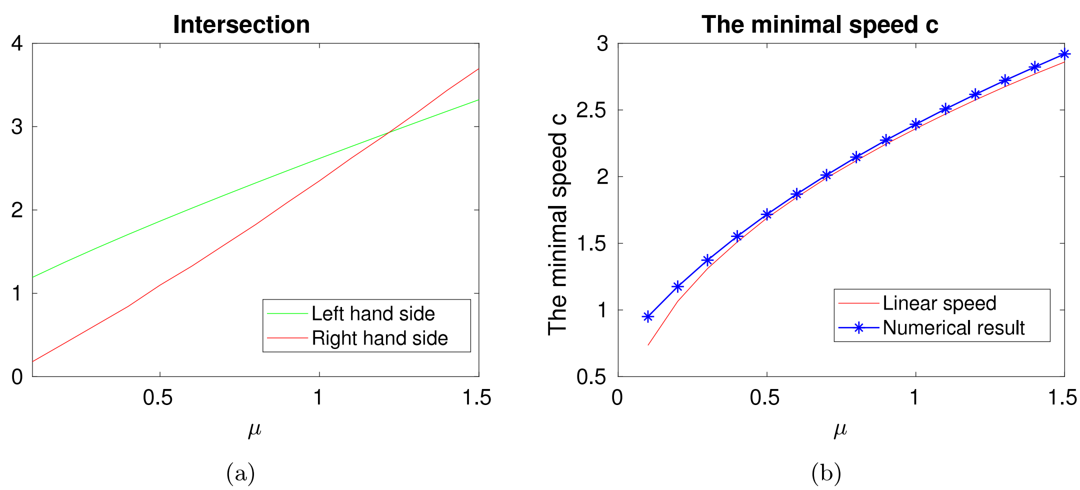


Fig. 9. (Color online) In the left panel, the green line denotes the left hand side of (4.18), i.e., $\frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\mu}$, and the red line denotes the right hand side, i.e., $3\lambda_0^2$. The right panel depicts the relation between the parameter μ and c_0 (red line) or c_{num} (blue line with stars).

line to denote the right hand side, that is, $3\lambda_0^2$. Clearly, there is an intersection $\mu_c \simeq 1$ shown in Fig. 9(b). Following Theorems 4.6 and 4.7, we expect that the system (4.13) is linearly selected when $\mu > \mu_c$ and nonlinearly selected when $\mu \le \mu_c$. In the right panel of Fig. 8, we illustrate the relation between c_{num} and c_0. By choosing the same initial condition given in Eq. (4.11), we obtain the traveling wave solution for (4.13). The shape of this solution is similar to the one shown in Fig. 3, so we will not repeat showing it here. To obtain a stable traveling wave solution, we record all the speed data after 200 s. On the other hand, c_0 is from (2.4) and its value differs as μ varies. Then, we use the blue line with stars to denote c_{num} and the red line to denote c_0. As we can see, the system is nonlinearly selected when $\mu \le 1$ and linearly selected when $\mu > 1$. Thus, with the help of numerical simulations, we indeed have verified the theoretical results.

(2) In the Dirichlet boundary condition case, we carry out similar procedures. In the left panel of Fig. 9, we find an intersection $\mu_d \simeq 1.25$ from (4.18). Following Theorem 4.6, we expect that the system (4.13) is nonlinearly selected when $\mu \le \mu_d$. To verify this, we choose the same initial condition defined in (4.12) to obtain the traveling wave solution for (4.13). Again, we store all the data after 200 s and use the red line to denote c_0 while the blue line with stars to denote c_{num}. As we can see in the right panel of Fig. 9, in our depicted region $\mu \in [0.1, 1.5]$, the blue line is always above the red one, which means the system is nonlinearly selected for all $\mu \in [0.1, 1.5]$. Thus, we have verified that the system is indeed nonlinearly selected when $\mu \le \mu_d$.

5. Conclusion and discussion

In summary, by the method of upper and lower solutions, we have obtained speed selection mechanism (including linear and nonlinear) for traveling wave solutions of a reaction–diffusion–advection equation in a cylindrical domain. Precisely, we found conditions on the linear selection when the model is prescribed by Neumann (or Dirichlet) boundary conditions, see the inequality (3.7) and Theorem 3.4. We also give results on the nonlinear selection when the model is prescribed by Neumann boundary conditions, see the inequality (3.20) and Theorem 3.7. To see the speed selection mechanism more specifically, we gave two applications in Section 4. In each application, we obtained the corresponding simplified conditions for the speed selection mechanism and then demonstrated them by direct numerical simulations.

We should emphasize that, because of our newly constructed upper and lower solutions, our results make significant progress in the study of the speed selection in higher dimension models such as (1.1). These constructed solutions are more accurate for approaching the true traveling wave solutions. With this method, we extend the previous results in the Neumann boundary condition case, and even give a sufficient condition on the linear selection in the Dirichlet boundary condition case, which was thought to be very difficult to study.

There are many interesting but open problems related to the topic of speed selections. One open problem arising in this paper is how to find a suitable lower solution to analyze the nonlinear selection in the Dirichlet boundary condition case. Furthermore, concerning the problem of wave speeds, it is interesting and challenging to find an estimation of $c_{\min}$ or even give an exact formula when the nonlinear selection is realized.

Appendix

The upper and lower solution method has proved to be a very powerful tool to investigate the existence of monotone traveling wave solutions (see e.g. [28]). This method originates in Diekmann [29], and has been extended by many academics, such as in [28,30]. The main idea is as follows. By transforming the wave profile equation (1.3) or its original partial differential equation (1.1) into an integral one, we can define a monotone solution map. Then with the definition of the solution map, we can construct a pair of upper and lower solutions of (1.1) to set up an iteration scheme. Through the scheme, we then obtain the existence of traveling wave solutions of (1.1).

To proceed, we present the phase space used in our model. Let $\mathcal{C}$ ($\tilde{\mathcal{C}}$) be the set of all bounded continuous functions from $\mathbb{R}\times\Omega$ to $\mathbb{R}$ (or $\tilde{\mathcal{C}} = C(\mathbb{R}, X)$, where $X = C_0(\Omega)$), and $\mathcal{C}_\beta := \{\varphi \in \mathcal{C} : 0 \le \varphi \le \beta\}$ ($\tilde{\mathcal{C}}_\beta := \{\varphi \in \tilde{\mathcal{C}} : 0 \le \varphi \le \beta\}$). Here, $\mathcal{C}$ is used for the Neumann boundary condition case, while $\tilde{\mathcal{C}}$ is used for the Dirichlet boundary condition case. Since the process in each case is similar, we then only take the Neumann boundary condition case to show the scheme.

To obtain a monotone solution map, we let M_1 be a sufficiently large positive number such that $F_1(u) := f(u) + M_1 u$ is monotone in u. Thus, Eq. (1.1) is equivalent to the following one:

$$u_t = u_{xx} + \Delta_y u + \alpha(y)u_x - M_1 u + F_1(u). \tag{A.1}$$

Next, we want to transform it into an integral form. To this end, we first investigate the corresponding homogeneous equation, that is,

$$u_t = u_{xx} + \Delta_y u + \alpha(y)u_x - M_1 u. \tag{A.2}$$

Let $\Gamma(t,x,y)$ (or $\tilde{\Gamma}(t,x,y)$) be the Green's function of (A.2) prescribed by the Neumann (or Dirichlet) boundary conditions (see, e.g., [31]). Then the solution of (A.2) with the initial value $u(0,\cdot) = \varphi(\cdot)$ can be expressed as

$$u(t,x,y) = \Gamma(t, x-x_0, y-y_0) * \varphi(x_0, y_0).$$

By the comparison principle (see, e.g., [32]), the above Green's function is monotone in u, that is, $\Gamma * u_1 \geq \Gamma * u_2$ when $u_1 \geq u_2$ for $(x,y) \in \mathbb{R} \times \overline{\Omega}$. Now, by variation of parameters, Eq. (A.1) can be written in an integral form as

$$u(t,x,y) = \Gamma(t, x-x_0, y-y_0) * \varphi(x_0, y_0) + \int_0^t \Gamma(t-t_0, x-x_0, y-y_0) * F_1(u(t_0, x_0, y_0))dt_0, \tag{A.3}$$

where the initial data $\varphi \in \mathcal{C}_\beta$ and $*$ denotes the convolution as

$$\Gamma(t, x-x_0, y-y_0) * \varphi(x_0, y_0) = \int_{\mathbb{R}} \int_{\Omega} \Gamma(t, x-x_0, y-y_0) \cdot u_0(x_0, y_0) dy_0 dx_0.$$

We define

$$Q_t[\varphi] = u(t, \cdot, \varphi).$$

It then follows that $\{Q_t\}_{t=0}^{\infty}$ is a semiflow on $\mathcal{C}_\beta$ with $Q_t(0) = 0$ and $Q_t(\beta) = \beta$. Then, by a traveling wave solution of the map Q_t for each $t \geq 0$, we mean a special solution $U(x,y)$ satisfying

$$Q_t[U](x,y) = U(x-ct, y)$$

for some constant c, and $U(x,y)$ connecting β to 0 if $U(-\infty, y) = \beta(y)$ and $U(+\infty, y) = 0$. Notice that, in the literature of Q_t, the minimal wave speed defined in the Introduction means that Q_t has a non-increasing traveling wave connecting β to 0 if and only if $c \geq c_{\min}$. Furthermore, for any $t \geq 0$, the solution map Q_t has the following properties:
(1) Q_t is monotone in the sense that $Q_t[U_1] \geq Q_t[U_2]$ whenever $U_1 \geq U_2$ for $(x,y) \in \mathbb{R} \times \overline{\Omega}$;
(2) If $U \in \mathcal{C}_\beta$ is decreasing with respect to $\xi \in \mathbb{R}$, so is $Q_t[U]$;
(3) $Q_t[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$ (see, e.g., [13] for the Neumann boundary conditions and [14] for the Dirichlet boundary conditions).

Then, corresponding to the solution map Q_t, we introduce the definition of an upper (or a lower) solution. Given $x_0 \in \mathbb{R}$, we define the translation operator T_{x_0} by $T_{x_0}[U](x,y) = U(x-x_0, y)$.

Definition A.1. For any given c, a continuous function $u(x,y)$ is called an upper solution to the integral equation (A.3) if

$$T_{-ct}\left[Q_t[u]\right](x,y) \leq u(x,y), \ \forall (x,y) \in \mathbb{R} \times \Omega.$$

A lower solution of (A.3) is defined by reversing the inequality.

In the following lemma, we give the inequality in Definition A.1 in terms of the differential equation for the wave profile, since these differential form inequalities are straightforward in our analysis.

Lemma A.2. *A continuous function $U(\xi, y) = T_{ct}[U](x,y)$, where $\xi = x - ct$, is twice continuously differentiable on $\mathbb{R} \times \Omega$ except finite many points ξ_i with*

$$U_\xi(\xi_i^+, y) \leq U_\xi(\xi_i^-, y), \ i = 1, 2, \ldots, m, \tag{A.4}$$

and

$$U_{\xi\xi} + \Delta_y U + [\alpha(y) + c]U_\xi + f(U) \leq 0, \ \forall (\xi, y) \in \mathbb{R} \setminus \{\xi_i\} \times \overline{\Omega}, \quad i = 1, 2, \ldots, m. \tag{A.5}$$

Then, it is an upper solution of (A.3). A lower solution is obtained by reversing the afore-mentioned inequalities.

Proof. Suppose there is a solution $\overline{U}(\xi, y)$ satisfying (A.5). We denote

$$u(t,x,y) = \overline{U}(x-ct, y).$$

Substituting it into (A.1) gives $u_t = -c\overline{U}_\xi$, $u_{xx} = \overline{U}_{\xi\xi}$ and $\Delta_y u = \Delta_y \overline{U}$. Then, (A.5) implies

$$\begin{cases} u_t \geq u_{xx} + \Delta_y u + \alpha(y)u_x + f(u), \\ u(0,x,y) = \overline{U}(x,y). \end{cases} \tag{A.6}$$

Since $Q_t[\overline{U}](x,y)$ is the solution of (A.1) with the initial data as $\overline{U}(x,y)$. By the comparison principle (see, e.g., [32]), we then obtain $u(t,x,y) \geq Q_t[\overline{U}](x,y)$ for all $t \geq 0$. That is, $\overline{U}(x-ct, y) = T_{ct}[\overline{U}](x,y) \geq Q_t[\overline{U}](x,y)$ for all $t \geq 0$. Thus, $\overline{U}(x,y) \geq T_{-ct}[Q_t[\overline{U}]](x,y)$, which exactly meets the requirement for an upper solution in Definition A.1. A similar proof can be applied to the lower solution of (A.3) if we reverse (A.4) and (A.5). This completes the proof. ■

The existence of an upper and a lower solution to the system (A.3) will give the existence of an actual traveling wave solution. Indeed, for our problem, we assume the following hypothesis.

Hypothesis A.3. For $c > c_0$, there exists a monotone non-increasing upper solution $\overline{U}(x,y)$ with respect to x and a non-zero lower solution $\underline{U}(x,y)$ to the system (A.1) with the following properties:
(1) $\underline{U}(x,y) \leq \overline{U}(x,y)$, for all $(x,y) \in \mathbb{R} \times \Omega$;
(2) $\overline{U}(-\infty, y) = \beta(y)$, $\overline{U}(+\infty, y) = 0$, for all $y \in \Omega$;
(3) $\underline{U}(-\infty, y) = \beta^*(y)$, $\underline{U}(+\infty, y) = 0$, where $0 \leq \beta^* \leq \beta$ for all $y \in \Omega$.

When the above hypothesis holds true, we can define an iteration scheme as

$$U_0(x,y) = \overline{U}(x,y), \ U_{n+1}(x,y) = T_{-ct}[Q_t[U_n]](x,y), \quad n = 0, 1, 2, \ldots. \tag{A.7}$$

With the construction of upper and lower solutions and the iteration scheme, we then arrive at the following existence theorem for a traveling wave solution (see, e.g., [13,29,30] for the Neumann boundary condition case, and [14] for the Dirichlet boundary condition case).

Theorem A.4. *If Hypothesis A.3 holds and Q_t is defined in (A.3), then the iteration (A.7) converges to a function $U(x,y)$. This function is a solution to (1.3)–(1.4) with $U(x-ct, y) = Q_t[U](x,y)$. Furthermore, $U(x-ct, y) = U(\xi, y)$ with $\xi = x - ct$ is non-increasing in $\xi \in \mathbb{R}$ with $U(-\infty, y) = \beta(y)$ and $U(+\infty, y) = 0$ uniformly for $y \in \Omega$.*

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