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Spreading dynamics of a Lotka-Volterra competition model in periodic habitats

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Abstract

Spreading speed of spatio-temporal nonlinear dynamical system can sometimes be determined either by its corresponding linear system with an explicit speed formula, or by the complicated nonlinear system itself with the existence of a pushed wavefront. In this paper, the spreading speed (the minimal speed of wavefronts) for a Lotka-Volterra competition model in spatially periodic habitats is investigated. We establish new results on the linear and nonlinear selections in terms of the spatio-periodic coefficient functions. In the case of nonlinear selection, lower and upper bound estimates of the minimal speed are provided.

MSC: 35K57; 35B20; 92D25

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1. Introduction

This paper is devoted to the spreading speed (the minimal wave speed of traveling waves) determinacy for the following Lotka-Volterra competition model in periodic habitats:

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$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1(x)\dfrac{\partial^2 u_1}{\partial x^2} + u_1(b_1(x) - a_{11}(x)u_1 - a_{12}(x)u_2), \\ \dfrac{\partial u_2}{\partial t} = d_2(x)\dfrac{\partial^2 u_2}{\partial x^2} + u_2(b_2(x) - a_{21}(x)u_1 - a_{22}(x)u_2),\ t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with all of the coefficients in system (1.1) to be L-periodic functions for a positive number L. Here, $u_1(t,x)$ and $u_2(t,x)$ are the population densities of two species respectively. The functions $d_1(x)$ and $d_2(x)$ represent the diffusive coefficients that are bounded below by positive constants so that the operators $d_1(x)\frac{\partial^2 u_1}{\partial x^2}$ and $d_2(x)\frac{\partial^2 u_2}{\partial x^2}$ are uniformly elliptic. Moreover, $a_{ij}(x)$, $1 \le i, j \le 2$ are positive, while $b_1(x)$ and $b_2(x)$ can change signs. We will also assume that all the coefficients are in $C^\nu(R)(\nu \in (0,1))$. For more details on the biological interpretation of the coefficient functions, readers are referred to [31].

Dynamics for reaction-diffusion models, time-delayed reaction-diffusion models and scalar evolutions in heterogeneous media (including periodic habitat), have been widely investigated in literature (see for examples [2–6,8,11,25–28,30,32]). In the spatial bounded domain with some particular boundary conditions, Dockery et al. [7] proved that the phenotype of a species with lowest diffusing speed dominates the other phenotypes, under conditions that the rate of mutation is small and the species is of haploid genetics. The uniqueness, stability of the coexistence steady states of a classic diffusive Lotka-Volterra competition model with spatially heterogeneous intrinsic growth rate have been studied in [19]. By incorporating advection-effect, Lutscher et al. in [21] studied pattern formations of a Lotka-Volterra competition model.

Spreading dynamics plays an important role in understanding the invasion of foreign species in competition for a common resource. In the case of homogeneous environment (the coefficients in system (1.1) are constant), patterns of traveling waves and their spreading speed (the minimal speed) determinacy have been studied extensively in [1,13,15–17,20,22] and references therein. However, when the coefficients are spatial-periodic functions, wave-propagation study becomes challenging, since not only the dynamics of co-existence steady state(s) is unclear, but also the method of phase plane analysis, which is useful in the study of classical KPP-Fisher model [18], cannot be applied to find the existence of wavefronts.

In the present paper, we adapt the upper-lower solution method to investigate the minimal speed determinacy of traveling waves, connecting two semi-trivial equilibria, of system (1.1) with monostable nonlinearity that is illustrated in section 2. With a deep understanding of the dependence of wavefront V on the invasion front U, we first investigate the essential feature of nonlinear selection. As an application, several criteria in terms of the coefficients are established for the nonlinear speed and some estimates are provided. For the linear selection, we obtain a set of sufficient conditions that are completely different from the condition in [31] that implies the nonlinear system is bounded by the corresponding linear system along the direction of a particular eigenvector. The new upper and lower solutions constructed in this paper play an important role, which establish nonlinear selection of the minimal wave speed that has not been investigated before in [31].

The paper is arranged as follows. In section 2, we establish the monostable nonlinearity of the system. Detailed derivation for the linear speed c_0 is provided. In section 3, we proceed to give a priori estimate of the wave profile and indicate the dependence of V on U. In sections 4 and 5, we establish the nonlinear and linear speed selection respectively. Section 6 is a short conclusion of our paper.

2. Monostable nonlinearity and the linear spreading speed

Let us first study the dynamics of (1.1) when the initial data are spatially-periodic. Obviously $(0, 0)$ is a steady state and it is unstable if
(A1) $\lambda(d_i(x), b_i(x)) > 0,\ i = 1, 2.$
Here, the notation $\lambda(d(x), b(x))$ represents the principal eigenvalue of the system

$$\begin{cases} \lambda\varphi(x) = d(x)\varphi''(x) + b(x)\varphi(x), \\ \varphi(x) = \varphi(x + L),\ x \in \mathbb{R}. \end{cases}$$

Under this condition **(A1)**, for (1.1) there exist two semi-trivial equilibria

$$(p(x), 0) \text{ and } (0, q(x)) \tag{2.1}$$

with unknown explicit formulas. Both $p(x)$ and $q(x)$ are twice-differentiable with respect to x (see, e.g., [31, Proposition 2.1]). Linearizing the system around $(0, q(x))$, we know that it is unstable under a further condition
(A2) $\lambda(d_1(x), b_1(x) - a_{12}(x)q(x)) > 0,\ \forall x \in \mathbb{R}.$
However, due to such a complicated coupled system, we still don't know whether there exists a co-existence positive equilibrium, even though we assume that $(p(x), 0)$ is stable under the condition

$$\lambda(d_2(x), b_2(x) - a_{21}(x)p(x)) < 0. \tag{2.2}$$

As such, we follow [31] to assume
(A3) there is no steady solution in the interior of $\mathbb{P}_+$ for the system (1.1),
where $\mathbb{P}_+ = \{\varphi \in \mathbb{P} : \varphi(x) \geq 0, \forall x \in \mathbb{R}\}$ and $\mathbb{P}$ is the set of all continuous and L-periodic functions from $\mathbb{R}$ to $\mathbb{R}^2$.

Remark 2.1. The condition **(A3)** combined with **(A2)** implies that $(p(x), 0)$ is stable for all initial data in $\mathbb{P}_+$. For details, readers are referred to [31, Theorem 2.1]. Hence, it follows that $\lambda(d_2(x), b_2(x) - a_{21}(x)p(x)) \leq 0$. In addition, we remark that **(A1)** holds whenever $b_i(x) > 0, i = 1, 2$ for all $x \in \mathbb{R}$ and **(A2)** holds whenever $b_1(x) - a_{12}(x)q(x) > 0$ for all $x \in \mathbb{R}$. As far as **(A3)** is concerned, we can consider a special case: $d_1(x) = d_1, d_2(x) = d_2$ with $0 < d_1 < d_2$, $b_1(x) = b_2(x)$, $a_{11}(x) = a_{21}(x) = a_{22}(x) = 1$ and $a_{12}(x) = c$ with $c \in [0, 1]$. Such a model was also reported in [31], where the authors showed that **(A1)**-**(A3)** hold true provided that $\bar{b}_1 = \frac{1}{L}\int_0^L b_1(x)dx \geq 0$ (see page 58 of [31]).

From a biological point of view, conditions **(A1)**-**(A3)** are referred as the case of competition exclusion or the so-called monostable nonlinearity.

By making use of a transformation

$$v_1(t, x) = \frac{u_1(t, x)}{p(x)},\ v_2(t, x) = 1 - \frac{u_2(t, x)}{q(x)}, \tag{2.3}$$

we can rewrite the competitive system (1.1) into the following cooperative system

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = d_1(x)\dfrac{\partial^2 v_1}{\partial x^2} + 2d_1(x)\dfrac{p'(x)}{p(x)}\dfrac{\partial v_1}{\partial x} + v_1[a_{11}(x)p(x)(1-v_1) - a_{12}(x)q(x)(1-v_2)], \\ \dfrac{\partial v_2}{\partial t} = d_2(x)\dfrac{\partial^2 v_2}{\partial x^2} + 2d_2(x)\dfrac{q'(x)}{q(x)}\dfrac{\partial v_2}{\partial x} + (1-v_2)[a_{21}(x)p(x)v_1 - a_{22}(x)q(x)v_2]. \end{cases} \tag{2.4}$$

The two equilibria $(p(x), 0)$ and $(0, q(x))$ are changed into $(1, 1)$ and $(0, 0)$ respectively. As such, (2.4) has three equilibria

$$e_0 = (0, 1),\ e_1 = (1, 1),\ e_2 = (0, 0).$$

In the current paper, we are interested in the existence of positive traveling wave solution, connecting e_1 and e_2, to the system (2.4). It is a special solution in the form of

$$(v_1(t, x), v_2(t, x)) = (U(x, z), V(x, z)),\ \ z = x - ct. \tag{2.5}$$

Here, c is the wave speed, and the pair of functions (U, V) is called the wavefront. Substituting (2.5) into (2.4), we have

$$\begin{cases} d_1(x)(U_{xx} + 2U_{xz} + U_{zz}) + 2d_1(x)\dfrac{p'(x)}{p(x)}(U_x + U_z) + cU_z \\ \qquad\qquad + U[a_{11}(x)p(x)(1-U) - a_{12}(x)q(x)(1-V)] = 0, \\ d_2(x)(V_{xx} + 2V_{xz} + V_{zz}) + 2d_2(x)\dfrac{q'(x)}{q(x)}(V_x + V_z) + cV_z \\ \qquad\qquad + (1-V)[a_{21}(x)p(x)U - a_{22}(x)q(x)V] = 0, \end{cases} \tag{2.6}$$

prescribed by boundary conditions

$$(U, V)(x, -\infty) = (1, 1),\ (U, V)(x, +\infty) = (0, 0). \tag{2.7}$$

In general, (2.6)-(2.7) does not always have a non-negative monotone solution for every c. However, it was proved in [31, Theorem 3.1] that there exists a number $c_{\min}$ such that the system possesses a non-negative monotone solution if and only if $c \ge c_{\min}$. Usually, it is difficult to obtain the explicit expression for the minimal speed $c_{\min}$. To estimate it, we can first linearize the system (2.4) near e_2 to get a linear system

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = d_1(x)\dfrac{\partial^2 v_1}{\partial x^2} + 2d_1(x)\dfrac{p'(x)}{p(x)}\dfrac{\partial v_1}{\partial x} + (a_{11}(x)p(x) - a_{12}(x)q(x))v_1, \\ \dfrac{\partial v_2}{\partial t} = d_2(x)\dfrac{\partial^2 v_2}{\partial x^2} + 2d_2(x)\dfrac{q'(x)}{q(x)}\dfrac{\partial v_2}{\partial x} + a_{21}(x)p(x)v_1 - a_{22}(x)q(x)v_2. \end{cases} \tag{2.8}$$

The first equation can define a linear speed

$$c_0 = \inf_{\mu>0}\{\widetilde{\lambda}(\mu)/\mu\}, \tag{2.9}$$

where $\widetilde{\lambda}(\mu)$ is the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} \lambda\phi = d_1(x)\phi'' - \left(2d_1(x)\mu - 2d_1(x)\dfrac{p'(x)}{p(x)}\right)\phi' \\ \qquad\qquad + \left(d_1(x)\mu^2 - 2d_1(x)\dfrac{p'(x)}{p(x)}\mu + a_{11}(x)p(x) - a_{12}(x)q(x)\right)\phi, \\ \phi(x+L) = \phi(x),\ x \in \mathbb{R}. \end{cases} \tag{2.10}$$

Remark 2.2. By the well known Krein-Rutman theorem, it follows that for each μ, (2.10) has a principal eigenvalue $\tilde{\lambda}(\mu)$ and a corresponding positive eigen function $\phi_\mu(x)$.

Besides the existence of traveling waves for $c \geq c_{\min}$, it follows also from [31, Proposition 4.1] that $c_{\min} \geq c_0$ under the conditions (**A1**)-(**A3**). Therefore, c_0 becomes a lower bound of $c_{\min}$, but we don't know whether or when they are equal or not. We say this minimal speed is *linearly* selected if $c_{\min} = c_0$, and *nonlinearly* selected if $c_{\min} > c_0$. In [31], the authors studied the linear determinacy with two further technique conditions

(D1) $\lambda_0(\mu_0) > \overline{\lambda}(\mu_0)$;
(D2) $\frac{\psi_1^*(x)}{\psi_2^*(x)} \geq \max\left\{\frac{a_{12}(x)}{a_{11}(x)}, \frac{a_{22}(x)}{a_{21}(x)}\right\},\ \forall x \in \mathbb{R}$,

where $\lambda_0(\mu)$ is the principal eigenvalue of the equation

$$\begin{cases} \lambda\psi = d_1(x)\psi'' - 2d_1(x)\mu\psi' + [d_1(x)\mu^2 + b_1(x) - a_{12}(x)q(x)]\psi, \\ \psi(x) = \psi(x+L),\ \ x \in \mathbb{R}, \end{cases} \tag{2.11}$$

and $\overline{\lambda}(\mu)$ is the principal eigenvalue of the equation

$$\begin{cases} \lambda\psi = d_2(x)\psi'' - 2d_2(x)\mu\psi' + [d_2(x)\mu^2 + b_2(x) - 2a_{22}(x)q(x)]\psi, \\ \psi(x) = \psi(x+L),\ \ x \in \mathbb{R}. \end{cases}$$

The constant μ_0 is the point such that $\frac{\lambda_0(\mu)}{\mu}$ attains its minimum. Moreover, $\psi_1^*(x)$ and $\psi_2^*(x)$ are two positive L-periodic eigenfunctions corresponding to $\lambda_0(\mu_0)$ to the following periodic eigenvalue problem

$$\begin{cases} \lambda\psi_1 = d_1(x)\psi_1'' - 2d_1(x)\mu\psi_1' + [d_1(x)\mu^2 + b_1(x) - a_{12}(x)q(x)]\psi_1, \\ \lambda\psi_2 = d_2(x)\psi_2'' - 2d_2(x)\mu\psi_2' + a_{21}(x)q(x)\psi_1 + [d_2(x)\mu^2 + b_2(x) - 2a_{22}(x)q(x)]\psi_2, \\ \psi_i(x) = \psi_i(x+L),\ i = 1, 2,\ x \in \mathbb{R}. \end{cases} \tag{2.12}$$

Remark 2.3. Note that [31] requires assumption (D1) so that the eigenfunctions of (2.12) are positive. In addition, (D2) implies that the nonlinear system is bounded by the corresponding linear system along the direction of the initial data $(e^{-\mu_0 x}\psi_1^*(x), e^{-\mu_0 x}\psi_2^*(x))$. In our study, we can remove both restrictions to have new results.

Remark 2.4. We should mention $\widetilde{\lambda}(\mu) = \lambda_0(\mu)$, with the eigenfunction ψ in (2.11) satisfying $\psi = p(x)\phi$ for ϕ in (2.10).

In summary, we will study the minimal wave speed selection for the traveling waves of (2.6) under conditions (**A1**)-(**A3**) with the linear speed c_0 defined in (2.9).

3. A priori estimate of the wave profile and the dependence of V on U

As we know, for $c \geq c_{\min}$, traveling wavefronts (U, V) of (2.6) exist and we want to give a priori estimate of their behaviors as $z \to \infty$. This will also give a derivation of (2.10). Indeed as $z \to \infty$, we will see that the nonlinear system is well-approximated by its linear system. Linearization of system (2.6) around e_2 gives

$$\begin{cases} d_1(x)(U_{xx} + 2U_{xz} + U_{zz}) + 2d_1(x)\dfrac{p'(x)}{p(x)}(U_x + U_z) + cU_z + U[a_{11}(x)p(x) - a_{12}(x)q(x)] \\ \quad = 0, \\ d_2(x)(V_{xx} + 2V_{xz} + V_{zz}) + 2d_2(x)\dfrac{q'(x)}{q(x)}(V_x + V_z) + cV_z + a_{21}(x)p(x)U - a_{22}(x)q(x)V \\ \quad = 0, \end{cases} \tag{3.1}$$

where $p(x)$ and $q(x)$ satisfy

$$\begin{cases} 0 = d_1(x)p''(x) + p(x)[b_1(x) - a_{11}(x)p(x)], \\ 0 = d_2(x)q''(x) + q(x)[b_2(x) - a_{22}(x)q(x)]. \end{cases}$$

The first equation in (3.1) is decoupled. Let $U = e^{-\mu z}\phi_1(x)$, for some positive functions $\phi_1(x)$ and constant μ. Substituting it into the first equation of (3.1), we get the following problem

$$\begin{cases} 0 = d_1(x)\phi_1'' - \left(2d_1(x)\mu - 2d_1(x)\dfrac{p'(x)}{p(x)}\right)\phi_1' \\ \qquad\qquad + \left(d_1(x)\mu^2 - 2d_1(x)\dfrac{p'(x)}{p(x)}\mu - c\mu + a_{11}(x)p(x) - a_{12}(x)q(x)\right)\phi_1, \\ \phi_1(x+L) = \phi_1(x), \ x \in \mathbb{R}. \end{cases} \tag{3.2}$$

For later use, we need to investigate the concavity of $\widetilde{\lambda}(\mu)$ that is defined in (2.10). The following lemma can be found in [31].

Lemma 3.1 *(Lemma 5.1 in [31]). Assume that L-periodic functions $d(x), g(x), m(x)$ are in $C^\nu(R)(\nu \in (0,1))$. Let $\lambda_m(\mu)$ be the principal eigenvalue of the following elliptic eigenvalue problem*

$$\begin{cases} \lambda\psi = d(x)\psi'' - [2d(x)\mu + g(x)]\psi' + [d(x)\mu^2 + g(x)\mu + m(x)]\psi, \\ \psi(x+L) = \psi(x), \ x \in \mathbb{R}. \end{cases}$$

Then $\lambda_m(\mu)$ *is a convex function of* μ *on* $\mathbb{R}$.

Therefore, it follows from Lemma 3.1 that $\widetilde{\lambda}(\mu)$ in (2.10) is a convex function of μ on $\mathbb{R}$. This together with the condition (**A2**) implies that, for each $c > c_0$, $\widetilde{\lambda}(\mu) - c\mu = 0$ has two positive roots $\mu_1(c)$ and $\mu_2(c)$, with $\mu_1(c) < \mu_2(c)$ for each $c > c_0$. The corresponding positive eigenfuctions are denoted as $\phi_{1,\mu_1}(x)$ and $\phi_{1,\mu_2}(x)$. When $c = c_0$, we have $\overline{\mu} := \mu_1(c_0) = \mu_2(c_0)$. Moreover, it follows that μ_1 is a decreasing function and μ_2 is an increasing function with respect to c. A straightforward way to understand this property is as follows. Divide both sides of the first equation of (3.2) by $d_1(x)\phi_1(x)$ and integrate it from 0 to L to obtain the following characteristic equation

$$\mu^2 - c\bar{A}\mu + \bar{B} = 0,$$

where

$$\bar{A} = \frac{1}{L}\int_0^L \frac{1}{d_1(x)}dx,\ \ \bar{B} = \frac{1}{L}\int_0^L (\frac{a_{11}(x)p(x) - a_{12}(x)q(x)}{d_1(x)} + \frac{\phi_1''}{\phi_1} + 2\frac{p'(x)\phi_1'(x)}{p(x)\phi_1(x)})dx.$$

The condition (**A2**) implies $\bar{B} > 0$. Hence, the two solutions are

$$\mu_1(c) = \frac{c\bar{A} - \sqrt{(c\bar{A})^2 - 4\bar{B}}}{2} > 0,\ \ \mu_2(c) = \frac{c\bar{A} + \sqrt{(c\bar{A})^2 - 4\bar{B}}}{2} > 0, \tag{3.3}$$

as $c \geq c_0$. Here c_0 satisfies

$$c_0^2\bar{A}^2 = 4\bar{B}.$$

Remark 3.2. For each $c > c_0$, the asymptotic behavior of positive $U(x,z)$ is given by

$$U(x,z) \sim C_1\phi_{1,\mu_1}(x)e^{-\mu_1(c)z} + C_2\phi_{1,\mu_2}(x)e^{-\mu_2(c)z},\ \ as\ z \to \infty,$$

with $C_1 > 0$, or $C_1 = 0, C_2 > 0$. What should be pointed out is that the asymptotic behavior of $U(x,z)$, as $z \to \infty$, can not contain generalized eigen-models like $\phi_{i,\mu_i}(x)(1 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n)e^{-\mu_i(c)z}$, $n \geq 1$ as long as $c > c_0$. For this result, one is referred to the work [14, Theorem 1.3]. We can give a short explanation. Take a single term $\alpha_1\phi_{1,\mu_1}(x)ze^{-\mu_1(c)z}$ as an example. Putting it into the first equation of (3.1) leads to

$$2\frac{\phi'(x)}{\phi(x)} - 2\mu_1 + 2\frac{p'(x)}{p(x)} + \frac{c}{d_1(x)} = 0.$$

Integrating the above equation from 0 to L with respect to x gives $\mu_1 = \frac{c\bar{A}}{2}$, which is impossible for $c > c_0$. Hence, the term like $\alpha_1\phi_{1,\mu_1}(x)ze^{-\mu_1(c)z}$ can not be appeared in the asymptotic behavior of $U(x,z)$ as $z \to \infty$. Similar discussions can rule out the other terms with high order power of z.

For the above given behavior of U, we can use the second equation of (3.1) to get the behavior of V. Denote the principal eigenvalue of the operator

$$\mathfrak{L}_\mu(\phi_2) = d_2(x)\phi_2'' - \left(2d_2(x)\mu - 2d_2(x)\frac{q'(x)}{q(x)}\right)\phi_2' + \left(d_2(x)\mu^2 - 2d_2(x)\frac{q'(x)}{q(x)}\mu - a_{22}(x)q(x)\right)\phi_2$$

by $\kappa(\mu)$, which is also a convex function. Then condition (**A1**) together with Remark 2.1 implies $q(x) > 0$, $\kappa(0) \le 0$, and $\kappa(\mu) - c\mu = 0$ has a unique positive root for each $c > c_0$. Also denote this positive root by $\mu_3(c)$. Then the behavior of V is given by

$$-\lim_{z\to\infty}\frac{1}{z}\log V = \begin{cases} \min\{\mu_1(c), \mu_3(c)\}, & \text{if } C_1 > 0, \\ \min\{\mu_2(c), \mu_3(c)\}, & \text{if } C_1 = 0, \end{cases} \tag{3.4}$$

where C_1 is defined in Remark 3.2.

The next lemma plays an important role in obtaining our main result. It enables us to avoid constructing simultaneously the upper solution U and V (or lower solution) to the equations in system (2.6) and shows the dependence of V on U. Instead, we only need to focus on the construction of the solution U.

By an upper or lower solution, we have the following definitions.

Definition 3.3. (Upper/Lower solution) A pair of continuous functions $(U, V)(x, z)$, which is twice continuously differentiable in x and z, is called a regular upper solution of (2.6) if it satisfies

$$\begin{cases} d_1(x)(U_{xx} + 2U_{xz} + U_{zz}) + 2d_1(x)\dfrac{p'(x)}{p(x)}(U_x + U_z) + cU_z + \\ \qquad U[a_{11}(x)p(x)(1-U) - a_{12}(x)q(x)(1-V)] \le 0, \\ d_2(x)(V_{xx} + 2V_{xz} + V_{zz}) + 2d_2(x)\dfrac{q'(x)}{q(x)}(V_x + V_z) + cV_z + \\ \qquad (1-V)[a_{21}(x)p(x)U - a_{22}(x)q(x)V] \le 0, \end{cases} \tag{3.5}$$

for $(x, z) \in [0, L) \times R$. The definition of a regular lower solution follows by reversing all the inequalities in (3.5).

Usually it is difficult to find upper or lower solution pairs twice-differentiable in subdomains. We can relax to find upper or lower solutions domain-wisely. This results in the definition of irregular upper or lower solutions. The notion of "irregular solution" is now more commonly known as "weak solutions" in the H^1 sense.

Definition 3.4. (see, [12]) A pair of continuous functions $(\bar{U}, \bar{V})$ is said to be an irregular upper solution of (2.6), if there exist regular upper solutions $(\bar{U}^1, \bar{V}^1)$, ..., $(\bar{U}^k, \bar{V}^k)$ of (2.6) such that $(\bar{U}, \bar{V}) = \min_{1\le i\le k} (\bar{U}^i, \bar{V}^i)$ componentwise. Similarly $(\underline{U}, \underline{V})$ is called an irregular lower solution of (2.6) if there exist regular lower solutions $(\underline{U}^1, \underline{V}^1)$, ..., $(\underline{U}^k, \underline{V}^k)$ of (2.6) such that $(\underline{U}, \underline{V}) = \max_{1\le i\le k} (\underline{U}^i, \underline{V}^i)$ componentwise.

From now on, by an upper solution (lower solution), we always mean an irregular upper solution (lower solution).

Lemma 3.5. *For $c > 0$ and any given continuous function $U(x, z)$ which is non-increasing in z and is L-periodic in x, and satisfies $U(x, +\infty) = 0$, with $\frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\infty) > 1$ or $\frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\infty) < 1$, the equation*

$$\begin{cases} d_2(x)(V_{xx} + 2V_{xz} + V_{zz}) + 2d_2(x)\dfrac{q'(x)}{q(x)}(V_x + V_z) + cV_z \\ \qquad\qquad + (1 - V)[a_{21}(x)p(x)U - a_{22}(x)q(x)V] = 0, \\ V(x, -\infty) = \min\{1, \dfrac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\infty)\},\ V(x, +\infty) = 0,\ x \in \mathbb{R}, \\ V(x, z) = V(x + L, z), \end{cases} \tag{3.6}$$

has a continuous solution V which is non-increasing in z and is L-periodic in x. The solution V is also monotone in U.

Proof. By putting $\xi = -z$ and $W(x, \xi) = 1 - V(x, z)$, the equation (3.6) can be rewritten as

$$\begin{cases} d_2(x)(W_{xx} - 2W_{x\xi} + W_{\xi\xi}) + 2d_2(x)\dfrac{q'(x)}{q(x)}(W_x - W_\xi) - cW_\xi + a_{22}(x)q(x)W[a(x, \xi) - W] \\ \quad = 0, \\ W(x, -\infty) = 1,\ W(x, +\infty) = 1 - \min\{1, \dfrac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\infty)\},\ x \in \mathbb{R}, \\ W(x, \xi) = W(x + L, \xi), \end{cases} \tag{3.7}$$

where

$$a(x, \xi) = 1 - \frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\xi),$$

with

$$a(x, -\infty) = 1,\ a(x, \infty) = 1 - \frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\infty),\ x \in \mathbb{R}.$$

Next, we use the upper and lower solution method to prove the existence of W which then gives rise to the existence of V. To proceed, we will consider two cases: (i) $\frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\infty) > 1$ and (ii) $\frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x, -\infty) < 1$. Firstly, it is easy to check that $W \equiv 1$ is an upper solution of (3.7), no matter for case (i) or case (ii). For the construction of lower solution, it is completely different for the two cases.

Case 1. We intend to apply the result of bistable wave in [10]. Indeed, for a sufficient small number ϵ, we define

$$f_\epsilon(x,\hat{W}) = \begin{cases} a_{22}(x)q(x)\hat{W}(1-\epsilon-\hat{W}), & \hat{W}\geq 0,\\ a_{22}(x)q(x)\hat{W}(\epsilon+\hat{W}), & \hat{W}<0,\end{cases}$$

and consider the following equation

$$d_2(x)(\hat{W}_{xx}-2\hat{W}_{x\xi}+\hat{W}_{\xi\xi})+2d_2(x)\frac{q'(x)}{q(x)}(\hat{W}_x-\hat{W}_\xi)+\hat{c}_\epsilon\hat{W}_\xi+f_\epsilon(x,\hat{W})=0, \tag{3.8}$$

which has three equilibria $-\epsilon, 0, 1-\epsilon$. This is a bistable system and by following the idea in [10], one can prove that equation (3.8) has a non-increasing solution $\hat{W}=\hat{W}_\epsilon$, subject to

$$\hat{W}(x,-\infty)=1-\epsilon,\ \ \hat{W}(x,+\infty)=-\epsilon. \tag{3.9}$$

As $\epsilon\to 0$, we claim that $\hat{c}_\epsilon\to\hat{c}_0$ and the limit of $\hat{W}_\epsilon$, say $\hat{W}$, satisfies

$$d_2(x)(\tilde{W}_{xx}-2\tilde{W}_{x\xi}+\tilde{W}_{\xi\xi})+2d_2(x)\frac{q'(x)}{q(x)}(\tilde{W}_x-\tilde{W}_\xi)+\hat{c}_0\tilde{W}_\xi+a_{22}(x)q(x)\tilde{W}(1-\tilde{W})=0, \tag{3.10}$$

which admits a non-negative solution, connecting 1 and 0, with

$$\hat{c}_0=\max_{\mu>0}\left\{\frac{\hat{\lambda}(\mu)}{\mu}\right\}>0.$$

Here $\hat{\lambda}(\mu)$ is the principal eigenvalue of the following eigenvalue problem

$$\begin{cases}\hat{\lambda}(\mu)\phi(x)=d_2(x)\phi''(x)+2d_2(x)\left(\mu+\dfrac{q'(x)}{q(x)}\right)\phi'(x)\\ \qquad\qquad +\left(d_2(x)\mu^2+2\mu d_2(x)\dfrac{q'(x)}{q(x)}+a_{22}(x)q(x)\right)\phi(x),\\ \phi(x+L)=\phi(x),\ \ x\in R.\end{cases}$$

Assume that this claim is true and we proceed to prove our main result. There exists a number ξ_0 such that $a(x,\xi)\geq 1-\epsilon$ if $\xi\leq\xi_0$ since $a(x,-\infty)=1$. Due to any translation of $\hat{W}(x,\xi)$ in ξ is still a solution to (3.8), we can assume that $\hat{W}(x,\xi)\geq 0$ if $\xi\leq\xi_0$, and $\hat{W}(x,\xi)<0$ if $\xi>\xi_0$. Let $\hat{W}(x,\xi)$ be the solution to (3.8)-(3.9). Then we can define the following function

$$\underline{W}(x,\xi)=\max\{0,\hat{W}(x,\xi)\}=\begin{cases}0, & \text{for } \xi>\xi_0,\\ \hat{W}(x,\xi), & \text{for } \xi\leq\xi_0.\end{cases}$$

We readily see that $\underline{W}$ is a lower solution to (3.7). Actually, the proof of the case $\xi>\xi_0$ is obvious. As for the case $\xi\leq\xi_0$, we have

$$d_2(x)(\underline{W}_{xx} - 2\underline{W}_{x\xi} + \underline{W}_{\xi\xi}) + 2d_2(x)\frac{q'(x)}{q(x)}(\underline{W}_x - \underline{W}_\xi) - c\underline{W}_\xi + a_{22}(x)q(x)\underline{W}[a(x,\xi) - \underline{W}]$$
$$= -(c + \hat{c}_\epsilon)\hat{W}_\xi + a_{22}(x)q(x)\hat{W}[a(x,\xi) - (1-\epsilon)]$$
$$\geq 0,$$

for $c > 0$. Thus, for given U, we find a solution V. The monotonicity of V on U comes from the positivity of $p(x)$ and $a_{21}(x)$ in (3.6).

We are left to prove our claim.

By developing the ideas in [33, Lemma 3.6], we first show that $\lim_{\epsilon\to 0^+} \hat{c}_\epsilon \leq \hat{c}_0$. We start with showing that $\hat{c}_\epsilon$ is nonincreasing with respect to ϵ. Suppose $0 < \epsilon_1 < \epsilon_2$ and $v_1(x,t) = \hat{W}_{\epsilon_1}(x, x - \hat{c}_{\epsilon_1}t)$ and $v_2(x,t) = \hat{W}_{\epsilon_2}(x, x - \hat{c}_{\epsilon_2}t)$ are solutions of (3.8) with (3.9). Translate so that

$$\hat{W}_{\epsilon_2}(x,\eta) > 0 \tag{3.11}$$

for some η. It is easy to see that $\hat{W}_{\epsilon_1}(x,-\infty) = 1-\epsilon_1 > 1-\epsilon_2 = \hat{W}_{\epsilon_2}(x,-\infty)$ and $\hat{W}_{\epsilon_1}(x,\infty) = -\epsilon_1 > -\epsilon_2 = \hat{W}_{\epsilon_2}(x,\infty)$. As a result, due to the fact that any translation of $\hat{W}(x,\xi)$ in ξ is still a solution to (3.8), one can assume that $\hat{W}_{\epsilon_1}(x,x) > \hat{W}_{\epsilon_2}(x,x)$ for $x \in \mathbb{R}$. Noting $f_{\epsilon_1} \geq f_{\epsilon_2}$ and by use of the comparison principle to the parabolic equation, we obtain

$$v_1(x,t) = \hat{W}_{\epsilon_1}(x, x - \hat{c}_{\epsilon_1}t) > \hat{W}_{\epsilon_2}(x, x - \hat{c}_{\epsilon_2}t) = v_2(x,t) \text{ for } (x,t) \in (\mathbb{R}, \mathbb{R}^+).$$

To the contrary, if $\hat{c}_{\epsilon_1} < \hat{c}_{\epsilon_2}$, by letting $\eta = x - \hat{c}_{\epsilon_2}t$, it follows that

$$\hat{W}_{\epsilon_2}(x,\eta) < \hat{W}_{\epsilon_1}(x, (\hat{c}_{\epsilon_2} - \hat{c}_{\epsilon_1})t) \to -\epsilon_1, \text{ as } t \to \infty,$$

which contradicts to (3.11). Consequently, we get $\hat{c}_{\epsilon_1} \geq \hat{c}_{\epsilon_2}$ if $\epsilon_1 < \epsilon_2$, which indicates $\hat{c}_\epsilon$ is nonincreasing in ϵ. This together with $\hat{c}_\epsilon \leq \hat{c}_0$ implies the existence of limit of $\hat{c}_\epsilon$ as $\epsilon \to 0^+$. We denote it by $\bar{c}$ which satisfies $\bar{c} \leq \hat{c}_0$.

Next, we shall prove that $\bar{c} \geq \hat{c}_0$. Replacing ϵ by $\epsilon_n, n = 1, 2, \cdots, k$ in (3.8) and denoting the unique solution by $(c_{\epsilon_n}, \hat{W}_{\epsilon_n})$. Here, the sequence ϵ_n is chosen so that $\lim_{n\to\infty} \epsilon_n = 0$ and $\hat{W}_{\epsilon_n}(0,0) = \frac{1}{2}$ by a translation. Passing to a subsequence of ϵ_n if it is necessary and by virtue of some a priori estimates, one can show that $(c_{\epsilon_n}, \hat{W}_{\epsilon_n})$ converges to a solution $(\bar{c}, \bar{W})$ of the following system

$$\begin{cases} d_2(x)(\bar{W}_{xx} - 2\bar{W}_{x\xi} + \bar{W}_{\xi\xi}) + 2d_2(x)\dfrac{q'(x)}{q(x)}(\bar{W}_x - \bar{W}_\xi) + \bar{c}\bar{W}_\xi + a_{22}(x)q(x)\bar{W}(1-\bar{W}) = 0, \\ \bar{W}(0,0) = \dfrac{1}{2}, \bar{W}(x+L,z) = \bar{W}(x,z), \ (x,z) \in \mathbb{R}\times\mathbb{R}. \end{cases}$$

We deduce from the monotonicity and boundedness of $\hat{W}_{\epsilon_n}(x,z)$ that $\bar{W}_z \leq 0$ and $0 \leq \bar{W} \leq 1$. Hence, $\bar{W}(x,\pm\infty)$ exists and satisfies

$$d_2(x)\bar{W}''(x) + 2d_2(x)\frac{q'(x)}{q(x)}\bar{W}_x + a_{22}(x)q(x)\bar{W}(1-\bar{W}) = 0. \tag{3.12}$$

By the assumption (A3), the second equation of (1.1) has and only has two steady states 0 and $q(x)$. Taking the transformation (2.3) into account, it is obvious that (3.12) has and only has two steady states 1 and 0. This is equivalent to $\bar{W}(x,\pm\infty) \in \{0,1\}$. In view of the monotonicity of $\hat{W}(x,z)$ in z, we get

$$\bar{W}(0,\infty) \le \bar{W}(0,0) = \frac{1}{2} \le \bar{W}(0,-\infty).$$

Therefore $\bar{W}(x,-\infty) = 1$ and $\bar{W}(x,\infty) = 0$. It is proven that $\bar{W}(x,x-\bar{c}t)$ is a traveling wavefront of (3.10) connecting 1 to 0. This further implies $\bar{c} \ge \hat{c}_0$, since (3.10) is a KPP-Fisher type system with minimal speed $\hat{c}_0$. Thus, we have shown that $\bar{c} = \hat{c}_0$.

Case 2. For this case, we can simply choose

$$\underline{W} = \min_{x\in[0,L)} \left\{ 1 - \frac{a_{21}(x)p(x)}{a_{22}(x)q(x)} U(x,-\infty) \right\}$$

as the lower solution. Thus, the proof is complete. □

4. Nonlinear speed selection

In what follows, we will first concentrate on the nonlinear speed selection of system (2.6)-(2.7). Our result shows that the nonlinear selection is realized if we can find a pair of lower solutions of system (2.6) with U to be decayed with a faster rate μ_2 at infinity. Note that the choice of larger decay rate μ_2 in z for "pushed fronts" was also observed by Roques et al. [24] in a conjecture. To some extent, the following theorem provides a justification for that conjecture. Moveover, lower and upper bounds of the minimal speed $c_{\min}$ can be derived.

Theorem 4.1 *(Nonlinear selection). For a given $c = c_1 > c_0$, let $(\underline{U}, \underline{V})(x,z)$ be a pair of nonnegative functions which are non-increasing in z and L-periodic in x with $z = x - c_1 t$. If $(\underline{U}, \underline{V})(x,z)$ is a lower solution to the system (2.6), and satisfies*

$$\lim_{z\to-\infty} \underline{U}(x,z) < 1,\ \underline{U}(x,z) \sim \phi_{\mu_2}(x)e^{-\mu_2 z},\ z \to +\infty,$$

where $\mu_2 := \mu_2(c_1)$ is the eigenvalue that is given in (3.3), $\phi_{\mu_2}(x)$ is the corresponding eigenfunction in (2.10), then for any speed $c \in [c_0, c_1)$, no traveling wave solution exists for system (2.6)-(2.7). This means $c_{\min} \ge c_1$.

Proof. Conversely, suppose that there exists a traveling wave solution $(U,V)(x,x-ct)$ for a given number c in $[c_0,c_1)$ to the system (2.4). In other words, $(U,V)(x,x-ct+\xi_0)$ is an exact solution of (2.4) with the following initial conditions

$$u(x,0) = U(x,x+\xi_0) \text{ and } v(x,0) = V(x,x+\xi_0)$$

for any constant ξ_0. Additionally, $(\underline{U}, \underline{V})(x,z)$ is a lower solution to the system (2.4), subject to the initial conditions

$$u(x,0) = \underline{U}(x,x) \text{ and } v(x,0) = \underline{V}(x,x).$$

Because $\mu_1(c)$ is decreasing and $\mu_2(c)$ is increasing with respect to c, it follows from Remark 3.2 that $\underline{U}(x, x) \le U(x, x + \xi_0)$ for a chosen negative enough ξ_0. By Lemma 3.5 we conclude that $(\underline{U}, \underline{V})(x, x) \le (U, V)(x, x + \xi_0)$ for $x \in \mathbb{R}$. Then it follows from comparison principle that

$$\underline{U}(x, x - c_1 t) \le U(x, x - ct + \xi_0) \text{ and } \underline{V}(x, x - c_1 t) \le V(x, x - ct + \xi_0). \tag{4.1}$$

By the assumption, we can fix a value for $z = x - c_1 t$ such that $\underline{U}(x, z)$ is fixed and positive. Thus, we obtain

$$U(x, x - ct + \xi_0) = U(x, z + \xi_0 + (c_1 - c)t) \sim U(x, \infty) = 0 \text{ as } t \to \infty.$$

Therefore, the first inequality of (4.1) gives $\underline{U}(x, z) \le 0$ which is contradicted to $\underline{U}(x, z) > 0$. The proof is complete. □

The above theorem not only gives a criterion for nonlinear selection, but also provides a lower bound to the minimal speed $c_{\min}$. Our next result gives an upper bound to this speed.

Theorem 4.2 *(Upper bound for the minimal wave speed). For $c_2 > c_0$, suppose that there exists a positive and nonincreasing upper solution pair $(U_2, V_2)(x, x - c_2 t)$ of (2.4), satisfying*

$$\liminf_{z \to -\infty} (U_2(x, z), V_2(x, z)) > (0, 0), \qquad U_2(x, z) = \phi_{\mu_2(c_2)}(x) e^{-\mu_2(c_2) z} \ \textit{as} \ z \to \infty, \tag{4.2}$$

where $\mu_2(c_2)$ is defined in (3.3). Then, $c_{\min} \le c_2$.

Proof. First we recall the definition of the spreading speed in [9] (see also the paper [23]). For any initial data $(\rho_1(x), \rho_2(x))$ to (2.4), its solution $(v_1(t, x, \rho_1(x)), v_2(t, x, \rho_2(x)))$ defines a monotone semiflow map Q_t. At time $t = 1$, we have a map Q_1. Now we can define the spreading speed c^* of Q_1 as

$$c^* := \sup\{c : \lim_{i \to \infty, x \in [iL, (i+1)L]} a(c; x) = (1, 1)\}, \tag{4.3}$$

where

$$a(c; x) = \lim_{n \to \infty} a_n(c; x).$$

For a given real number c, the sequence of functions $\{a_n\}_{n=0}^{\infty}$ is defined as

$$a_0(c; x) = \phi(x), \quad a_{n+1}(c; x) = R_c[a_n(c; \cdot)](x), \tag{4.4}$$

and

$$R_c[a](x) = \max\{\phi(x), T_{-c}[Q_1[a]](x)\}, \tag{4.5}$$

where

$$T_y(u)(x) = u(x-y),$$

and $\phi(x)$ is a non-increasing function that satisfies

$$\phi(x) = (0,0) \;\text{ for }\; x > 0 \text{ and } \lim_{x\to-\infty}(\phi(x)-\omega) = 0,$$

$(0,0) < \omega < (1,1)$. Here, c^* is well-defined and independent of the choice of ϕ, see [9,23]. As it can be seen from [31], there is no traveling wave connecting $(0,1)$ and $(0,0)$. As such, a single spreading speed c^* exists and it is equal to the minimal wave speed $c_{\min}$. Therefore, we can let $\phi(-\infty)$ be small so that the upper solution (U_2, V_2) (or a shift of (U_2, V_2) if needed) satisfies

$$a_0(c_2; x) \le (U_2, V_2)(x, x) \tag{4.6}$$

for all $x \in (-\infty, \infty)$. From (4.4) and (4.5), by induction, it follows that

$$a_{n+1}(c_2; x) \le (U_2, V_2)(x, x), \quad n \ge 0.$$

Thus $a(c_2; -\infty) = 0$. By (4.3), we have $c^* \le c_2$, that is $c_{\min} \le c_2$. The proof is complete. □

Based on Theorem 4.1, we will derive results on nonlinear selection by construction of lower solutions. To proceed, for $0 < \underline{k} < 1$ and sufficient small number ϵ, we choose

$$\underline{U}(x,z) = \frac{\underline{k}}{1 + \frac{e^{\mu_2 z}}{\phi_{\mu_2}(x)}}, \; \mu_2 := \mu_2(c),$$

where $c = c_1 = c_0 + \varepsilon$. Let $\underline{V}(x,z)$ be the function determined by Lemma 3.5. By substituting $(\underline{U}, \underline{V})$ into the U-equation, one can show directly that $(\underline{U}, \underline{V})$ is a lower solution to the system (1.1) provided that

$$-2d_1(x)\left(\mu_2 - \frac{\phi'_{\mu_2}}{\phi_{\mu_2}}\right)^2 + Y_1(x,z) \ge 0, \tag{4.7}$$

where

$$Y_1(x,z) = \frac{a_{12}(x)q(x)\underline{V} + \underline{U}\left[a_{11}(x)p(x)(\frac{1}{\underline{k}} - 1) - a_{12}(x)q(x)\frac{1}{\underline{k}}\right]}{\frac{\underline{U}}{\underline{k}}\left(1 - \frac{\underline{U}}{\underline{k}}\right)}.$$

Thus, for the nonlinear selection, we have the following result.

Proposition 4.3. *If (4.7) is true, then $c_{\min} > c_0$.*

Furthermore, by virtue of choosing explicit function $\underline{V}$, we can obtain the following results concerning the nonlinear selection.

Theorem 4.4. *The minimal wave speed of (2.6)-(2.7) is nonlinearly selected provided that there exist* ε, $\underline{k} > 0$ *such that*

$$\begin{aligned}&\frac{a_{22}(x)q(x) + 2d_2(x)\big(\mu_2 - \frac{\phi'_{\mu_2}}{\phi_{\mu_2}}\big)^2 - \frac{d_2(x)}{d_1(x)}(c\mu_2 - \Delta(x)) + c\mu_2 - Q_{\mu_2}(x)}{a_{21}(x)p(x)} \\ &\qquad \le \underline{k} \le \min_{x\in[0,L)} \left\{ 1 - \frac{2d_1(x)\big(\mu_2 - \frac{\phi'_{\mu_2}}{\phi_{\mu_2}}\big)^2}{a_{11}(x)p(x)} \right\},\end{aligned} \tag{4.8}$$

holds for $c = c_0 + \varepsilon$, *where*

$$\Delta(x) := a_{11}(x)p(x) - a_{12}(x)q(x), \tag{4.9}$$

and

$$Q_{\mu_2}(x) := 2d_2(x)\left(\frac{\phi'_{\mu_2}(x)}{\phi_{\mu_2}(x)} - \mu_2\right)\left(\frac{q'(x)}{q(x)} - \frac{p'(x)}{p(x)}\right).$$

Proof. By a substitution of

$$\underline{V}(x,z) = \frac{1}{\underline{k}}\underline{U}(x,z) \text{ with } \underline{U}(x,z) = \frac{\underline{k}}{1 + \frac{e^{\mu_2 z}}{\phi_{\mu_2}(x)}},$$

formula (4.7) can be estimated by

$$-2d_1(x)\left(\mu_2 - \frac{\phi'_{\mu_2}}{\phi_{\mu_2}}\right)^2 + a_{11}(x)p(x)(1 - \underline{k}) \ge 0. \tag{4.10}$$

It is true due to the right part of (4.8). In view of (4.8), a substitution of $(\underline{U}, \underline{V})$ into the V-equation enables us to get

$$\begin{aligned}&\underline{V}(1-\underline{V})\left\{ \frac{d_2(x)}{d_1(x)}(c\mu_2 - \Delta(x)) - c\mu_2 + Q_{\mu_2}(x) \right. \\ &\qquad \left. + (a_{21}(x)p(x)\underline{k} - a_{22}(x)q(x)) - 2d_2(x)\left(\mu_2 - \frac{\phi'_{\mu_2}}{\phi_{\mu_2}}\right)^2 \right\} \ge 0.\end{aligned} \tag{4.11}$$

Thus, (4.10) and (4.11) guarantee that $(\underline{U}, \underline{V})$ is a lower solution. By Theorem 4.1, the proof is complete. □

Remark 4.5. In Theorem 4.4, we have chosen $c = c_0 + \epsilon$. As $\epsilon \to 0^+$, we have $c \to c_0$, $\mu_2 \to \bar{\mu}$ and $\phi_{\mu_2}(x) \to \phi(x)$. As a result, (4.8) can reduce to

$$\frac{a_{22}(x)q(x) + 2d_2(x)\big(\bar{\mu} - \frac{\phi'}{\phi}\big)^2 - \frac{d_2(x)}{d_1(x)}(c_0\bar{\mu} - \Delta(x)) + c_0\bar{\mu} - Q_{\bar{\mu}}(x)}{a_{21}(x)p(x)}$$
$$\leq \underline{k} \leq \min_{x\in[0,L)} \left\{ 1 - \frac{2d_1(x)\big(\bar{\mu} - \frac{\phi'}{\phi}\big)^2}{a_{11}(x)p(x)} \right\},$$

where

$$Q_{\bar{\mu}}(x) := 2d_2(x)\left(\frac{\phi'(x)}{\phi(x)} - \bar{\mu}\right)\left(\frac{q'(x)}{q(x)} - \frac{p'(x)}{p(x)}\right). \tag{4.12}$$

Inspired by Remark 4.5, hereafter, we will always denote the principal eigenfunction corresponding to $\tilde{\lambda}(\bar{\mu})$ in (2.10) by $\phi(x)$ for brevity. Alternatively, we can choose

$$\underline{U}(x,z) = \frac{\underline{k}}{1 + \frac{e^{\mu_2(c)z}}{\phi_{\mu_2(c)}(x)}}, \quad \underline{V}(x,z) = \frac{\underline{U}(x,z)}{\underline{k}}\left(2 - \frac{\underline{U}(x,z)}{\underline{k}}\right), \tag{4.13}$$

where $c = c_0 + \varepsilon$ and $\phi_{\mu_2(c)}(x)$ is the principal eigenfunction corresponding to $\tilde{\lambda}(\mu_2(c))$ (see (2.10)), to have a different result on the nonlinear selection. By performing an analogous limiting analysis as in Remark 4.5, we have the following conclusion.

Theorem 4.6. *The minimal wave speed of (2.6)-(2.7) is nonlinearly selected provided that*

$$\begin{cases} 6d_2(x)\left(\overline{\mu} - \dfrac{\phi'}{\phi}\right)^2 - a_{22}(x)q(x) < 0, \\ \dfrac{2a_{22}(x)q(x) - 2\Theta(x)}{a_{21}(x)p(x)} < \min\limits_{x\in[0,L)} \left\{ 1 - \dfrac{2d_1(x)\big(\bar{\mu} - \frac{\phi'}{\phi}\big)^2 - a_{12}(x)q(x)}{a_{11}(x)p(x)} \right\}, \end{cases} \tag{4.14}$$

where

$$\Theta(x) := \frac{d_2(x)}{d_1(x)}[c_0\bar{\mu} - \Delta(x)] - c_0\bar{\mu} + Q_{\bar{\mu}}(x).$$

Proof. From the second condition in (4.14), we can select the constant $\underline{k}$ in (4.13) satisfying

$$\frac{2a_{22}(x)q(x) - 2\Theta(x)}{a_{21}(x)p(x)} < \underline{k} < \min_{x\in[0,L)} \left\{ 1 - \frac{2d_1(x)\big(\bar{\mu} - \frac{\phi'_{\bar{\mu}}}{\phi_{\bar{\mu}}}\big)^2 - a_{12}(x)q(x)}{a_{11}(x)p(x)} \right\}. \tag{4.15}$$

Substituting (4.13) into $Y_1(x,z)$ gives

$$Y_1(x,z) \geq a_{12}(x)q(x) + a_{11}(x)p(x)(1 - \underline{k}),$$

hence, for small ε, (4.7) holds true if

$$-2d_1(x)\left(\bar{\mu}-\frac{\phi'_{\bar{\mu}}(x)}{\phi_{\bar{\mu}}(x)}\right)^2+a_{12}(x)q(x)+a_{11}(x)p(x)(1-\underline{k})>0, \tag{4.16}$$

which can be deduced from the second inequality of (4.15). Next, by a substitution of (4.13) into V-equation, we can rewrite the left-side of V-equation as $\frac{U}{\underline{k}}(1-\frac{U}{\underline{k}})F_0(\frac{U}{\underline{k}})$, where

$$\begin{aligned}F_0(\frac{U}{\underline{k}}) =&2\Theta(x)+a_{21}(x)p(x)\underline{k}-2a_{22}(x)q(x)-\left[6d_2(x)\left(\mu_2-\frac{\phi'_{\mu_2}}{\phi_{\mu_2}}\right)^2+2\Theta(x)\right.\\&\left.+a_{21}(x)p(x)\underline{k}-3a_{22}(x)q(x)\right]\frac{U}{\underline{k}}+\left[6d_2(x)\left(\mu_2-\frac{\phi'_{\mu_2}}{\phi_{\mu_2}}\right)^2-a_{22}(x)q(x)\right]\left(\frac{U}{\underline{k}}\right)^2.\end{aligned} \tag{4.17}$$

It is easy to check that $F_0(1)=0$. As $\varepsilon\to 0$, the first condition of (4.14) indicates that $F_0(\frac{U}{\underline{k}})$ is a concave downward function, and the first inequality of (4.15) implies

$$F_0(0)=2\Theta(x)+a_{21}(x)p(x)\underline{k}-2a_{22}(x)q(x)\geq 0.$$

Consequently, we have $F_0(\frac{U}{\underline{k}})\geq 0$ for $\underline{U}\in(0,\underline{k})$. This combined with (4.16) shows that the pair of functions $(\underline{U},\underline{V})$ is a lower solution. Thus, by use of Theorem 4.1, the proof is complete. □

Remark 4.7. We illustrate an application of Theorem 4.6 to the following constant-coefficient system

$$\begin{cases}\dfrac{\partial u_1}{\partial t}=\dfrac{\partial^2 u_1}{\partial x^2}+u_1(1-u_1-a_1u_2),\\[2mm] \dfrac{\partial u_2}{\partial t}=d\dfrac{\partial^2 u_2}{\partial x^2}+ru_2(1-a_2u_1-u_2),\ t>0,\ x\in\mathbb{R}.\end{cases} \tag{4.18}$$

As a result, we obtain that the minimal wave speed of (4.18) is nonlinearly selected provided that

$$\begin{cases}6d(1-a_1)-r<0,\\[1mm] \dfrac{2r-2(d-2)(1-a_1)}{ra_2}<3a_1-1,\end{cases} \tag{4.19}$$

which is new comparing with results in [1].

5. Linear speed selection

In this section, we are concerned with the linear speed selection. We will construct a pair of upper solutions to establish the existence of traveling waves for $c=c_0$. For (2.6), recall that there are three equilibria $(0,0)$, $(0,1)$ and $(1,1)$. It is easy to see that there is no traveling wavefront of (2.6), connecting $(0,1)$ and $(0,0)$ for $c\geq c_0$. As such, by Theorem 5.3 in [9], a single spreading speed exists, which is equal to the minimal speed of traveling waves to (2.6)-(2.7).

Theorem 5.1 *(Linear selection). For $c = c_0$, assume that there exists a continuous and positive function $(\overline{U}(x,z), \overline{V}(x,z))$ as an upper solution of (2.6) such that*

$$\liminf_{z\to-\infty}(\overline{U}(x,z), \overline{V}(x,z)) > (0,0), \quad \lim_{z\to\infty}(\overline{U}(x,z), \overline{V}(x,z)) = (0,0). \tag{5.1}$$

Then the minimal wave speed is linearly selected. This means $c_{\min} = c_0$.

Proof. The proof here is similar to that of Theorem 4.2 and omitted. □

Next, we prepare to construct an upper solution to (2.6). Let $\overline{\mu} = \mu_1(c_0)$, and $\phi(x)$ be the corresponding eigenfunction, see (2.10). Then, by keeping Lemma 3.5 in mind, we can define

$$\overline{U}(x,z) = \frac{1}{1+\frac{e^{\overline{\mu}z}}{\phi(x)}}, \ \overline{V}(x,z) \text{ is the corresponding solution, see Lemma 3.5,} \tag{5.2}$$

which satisfy

$$\overline{U}_x = \frac{\phi'}{\phi}\overline{U}(1-\overline{U}), \ \ \overline{U}_z = -\overline{\mu}\,\overline{U}(1-\overline{U}), \ \ \overline{U}_{zz} = \overline{\mu}^2\overline{U}(1-\overline{U})(1-2\overline{U}), \tag{5.3}$$

and

$$\overline{U}_{xz} = -\overline{\mu}\frac{\phi'}{\phi}\overline{U}(1-\overline{U})(1-2\overline{U}), \ \overline{U}_{xx} = \frac{\phi''\phi-\phi'^2}{\phi^2}\overline{U}(1-\overline{U}) + \left(\frac{\phi'}{\phi}\right)^2\overline{U}(1-\overline{U})(1-2\overline{U}). \tag{5.4}$$

By virtue of (5.3) and (5.4), the U-equation (the left of the first equation of (2.6)) becomes

$$\begin{aligned}
&\overline{U}(1-\overline{U})\Bigg\{\bigg[d_1(x)\frac{\phi''}{\phi} - \left(2d_1(x)\overline{\mu} - 2d_1(x)\frac{p'(x)}{p(x)}\right)\frac{\phi'}{\phi} + \left(d_1(x)\overline{\mu}^2 - 2d_1(x)\frac{p'(x)}{p(x)}\overline{\mu} - c\overline{\mu}\right)\bigg] \\
&\qquad - 2d_1(x)\overline{U}\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 + \overline{U}\big[a_{11}(x)p(x)(1-\overline{U}) - a_{12}(x)q(x)(1-\overline{V})\big]\Bigg\} \\
&= \overline{U}(1-\overline{U})\Bigg\{ - (a_{11}(x)p(x) - a_{12}(x)q(x)) - 2d_1(x)\overline{U}\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2\Bigg\} \\
&\qquad + \overline{U}\big[a_{11}(x)p(x)(1-\overline{U}) - a_{12}(x)q(x)(1-\overline{V})\big] \\
&= \overline{U}^2(1-\overline{U})\Bigg\{ - 2d_1(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)\frac{\overline{V}-\overline{U}}{\overline{U}(1-\overline{U})}\Bigg\}.
\end{aligned}$$

Thus, the pair of functions $(\overline{U}, \overline{V})$ turns to be an upper solution if

$$-2d_1(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)Y_2(x,z) \le 0, \ \text{where } Y_2(x,z) = \frac{\overline{V}-\overline{U}}{\overline{U}(1-\overline{U})}. \tag{5.5}$$

By Theorem 5.1, we arrive at the following result.

Proposition 5.2. *The minimal wave speed of (2.6)-(2.7) is linearly selected if there exists an upper solution $(\bar{U}, \bar{V})$ in the form of (5.2) such that (5.5) is satisfied.*

Now we provide explicit formulas for the upper solution pair to establish new results on the linear selection. Bearing the notations $\Delta(x)$ and $Q_{\bar{\mu}}(x)$ (see (4.9) and (4.12) respectively) in mind, we have the following result.

Theorem 5.3. *Assume that there exists a number $\eta > 1$ such that*

$$\frac{a_{21}(x)p(x)}{a_{22}(x)q(x) - Q_{\bar{\mu}}(x) - \left[(\frac{d_2(x)}{d_1(x)} - 1)c_0\overline{\mu} - \frac{d_2(x)}{d_1(x)}\Delta(x)\right]} \le \eta \le \min_{x\in[0,L)} \left\{ \frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}, \frac{2d_1(x)(\overline{\mu} - \frac{\phi'}{\phi})^2}{a_{12}(x)q(x)} \right\}. \tag{5.6}$$

Then the minimal wave speed of (2.6) is linearly selected if

$$\frac{d_2(x)}{d_1(x)} \le \frac{a_{22}(x)q(x) + c_0\overline{\mu} - Q_{\bar{\mu}}(x)}{c_0\overline{\mu} - \Delta(x)}, \tag{5.7}$$

where c_0 is the wave speed of the linear system and is defined in (2.9).

Proof. Let $\overline{U}(x,z)$ be the function defined in (5.2), and redefine an irregular upper solution

$$\overline{V}(x,z) = \min\{1, \eta\overline{U}(x,z)\}. \tag{5.8}$$

In the case $\overline{V}(x,z) = 1$, it is obvious that

$$\begin{aligned} d_2(x)(\overline{V}_{xx} + 2\overline{V}_{xz} + \overline{V}_{zz}) + 2d_2(x)\frac{q'(x)}{q(x)}(\overline{V}_x + \overline{V}_z) + c_0\overline{V}_z \\ + (1-\overline{V})(a_{21}(x)p(x)\overline{U} - a_{22}(x)q(x)\overline{V}) = 0. \end{aligned} \tag{5.9}$$

As for the case $\overline{V}(x,z) = \eta\overline{U}(x,z)$, a straightforward calculation combined with (5.6) and (5.7) results in

$$\begin{aligned} & d_2(x)(\overline{V}_{xx} + 2\overline{V}_{xz} + \overline{V}_{zz}) + 2d_2(x)\frac{q'(x)}{q(x)}(\overline{V}_x + \overline{V}_z) + c_0\overline{V}_z \\ & \qquad + (1-\overline{V})(a_{21}(x)p(x)\overline{U} - a_{22}(x)q(x)\overline{V}) \\ & \le \eta\overline{U}(1-\overline{U})\left\{\frac{d_2(x)}{d_1(x)}(c_0\overline{\mu} - \Delta(x)) - c_0\overline{\mu} + Q_{\bar{\mu}}(x) + \left(\frac{a_{12}(x)p(x)}{\eta} - a_{22}(x)q(x)\right)\right\} \\ & \le 0. \end{aligned} \tag{5.10}$$

In addition, from (5.5), it is easy to check that

$$Y_2(x,z) = \begin{cases} \dfrac{1}{\overline{U}} \le \eta, & \text{if } \ \eta\overline{U} \ge 1, \\ \dfrac{\eta-1}{1-\overline{U}} \le \eta, & \text{if } \ \eta\overline{U} < 1. \end{cases} \tag{5.11}$$

Hence, in view of the condition (5.6), we obtain

$$-2d_1(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)\eta \le 0. \tag{5.12}$$

Based on (5.9), (5.10) and (5.12), we can conclude that $(\overline{U}, \overline{V})$ is an upper solution to the system (2.6). By Theorem 5.1, the proof is complete. □

Next, we want to construct a new pair of upper solution to establish some sharper sufficient conditions for linear selection. In fact, we can set up

$$\overline{U} = \frac{1}{1 + \frac{e^{\overline{\mu}z}}{\phi(x)}}, \ \ \overline{V} = \overline{U}(a + b\overline{U} + (1-a-b)\overline{U}^2), \tag{5.13}$$

where a, b are two constants that will be determined later. By substituting (5.13) into the U-equation, we have a new condition for linear selection. Before stating it, we need some notations:

$$\begin{aligned}
&\Delta_1(x) = a_{21}(x)p(x) - a_{22}(x)q(x)a, \\
&A = a\Theta(x) + \Delta_1(x), \\
&B = -2ad_2(x)\left(\frac{\phi'}{\phi} - \bar{\mu}\right)^2 + 2b\left[\Theta(x) + d_2(x)\left(\frac{\phi'}{\phi} - \bar{\mu}\right)^2\right] \\
&\qquad + [(1-a)\Delta_1(x) - ba_{22}(x)q(x)], \\
&C = -6bd_2(x)\left(\frac{\phi'}{\phi} - \bar{\mu}\right)^2 + 3(1-a-b)\left[\Theta(x) + 2d_2(x)\left(\frac{\phi'}{\phi} - \bar{\mu}\right)^2\right] \\
&\qquad + (1-a-b)\Delta_1(x) - (1-a)ba_{22}(x)q(x), \\
&D = -12(1-a-b)d_2(x)\left(\frac{\phi'}{\phi} - \bar{\mu}\right)^2.
\end{aligned} \tag{5.14}$$

Theorem 5.4. *The minimal wave speed of (2.6)-(2.7) is linearly selected provided that there exist two numbers a and b satisfying*

$$a \ge \frac{a_{21}(x)p(x)}{a_{22}(x)q(x) - \Theta(x)}, \ \ a_{12}(x)q(x)(b + 2(a-1)) \le 2d_1(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 \tag{5.15}$$

and

$$1 - a - b \le 0, \ \ B \le 0, \ \ B + C + D \le 0. \tag{5.16}$$

Proof. Substituting (5.13) into the function $Y_2(x, z)$ gives

$$Y_2(x,z) = a - 1 + (a+b-1)\overline{U}. \tag{5.17}$$

Therefore, the conditions (5.15) and $1 - a - b \le 0$ in (5.16) imply that (5.5) can be evaluated by

$$\begin{aligned} &-2d_1(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)(a - 1 + (a+b-1)\overline{U}) \\ &\le -2d_1(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)(b + 2(a-1)) \\ &\le 0. \end{aligned} \tag{5.18}$$

The following formulas can be obtained by a direct computation (n is an integer),

$$\begin{aligned} &(\overline{U}^n)_x = n\frac{\phi'}{\phi}\overline{U}^n(1-\overline{U}),\ (\overline{U}^n)_z = -n\bar{\mu}\overline{U}^n(1-\overline{U}), \\ &(\overline{U}^n)_{xx} = n\overline{U}^n(1-\overline{U})\frac{\phi''\phi - \phi'^2}{\phi^2} + n\left(\frac{\phi'}{\phi}\right)^2\overline{U}^n(1-\overline{U})(n-(n+1)\overline{U}), \\ &(\overline{U}^n)_{zx} = -n\bar{\mu}\frac{\phi'}{\phi}\overline{U}^n(1-\overline{U})(n-(n+1)\overline{U}),\ (\overline{U}^n)_{zz} = -n\bar{\mu}^2\overline{U}^n(1-\overline{U})(n-(n+1)\overline{U}). \end{aligned}$$

By taking $n = 1, 2, 3$ respectively and plugging the above relationships into the V-equation, we can rewrite the left-side of the V-equation as $\overline{U}(1-\overline{U})F(\overline{U})$, with $F(\overline{U}) = A + B\overline{U} + C\overline{U}^2 + D\overline{U}^3$, where A, B, C and D are defined in (5.14). Under the assumption (5.15), we can further estimate the left-side of the V-equation as

$$\overline{U}^2(1-\overline{U})G(\overline{U}),$$

with $G(\overline{U}) = B + C\overline{U} + D\overline{U}^2$. Formulas in (5.16) imply that $G(\overline{U}) \le 0$ for $\overline{U} \in [0, 1]$. This, combined with (5.18), indicates $(\overline{U}, \overline{V})$ is an upper solution. In view of Theorem 5.1, the proof is complete. □

Remark 5.5. By letting $a = 2, b = -1, c = 0$, the choice in (5.13) becomes

$$\overline{U}(x,z) = \frac{1}{1 + \frac{e^{\overline{\mu}z}}{\phi(x)}},\ \overline{V} = 2\overline{U} - \overline{U}^2. \tag{5.19}$$

It follows that the minimal wave speed of (2.6) is linearly selected provided that

$$\begin{cases} -2d_1(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x) \le 0, \\ 6d_2(x)\left(\overline{\mu} - \frac{\phi'}{\phi}\right)^2 - a_{22}(x)q(x) \ge 0, \\ 2\Theta(x) + a_{21}(x)p(x) - 2a_{22}(x)q(x) \le 0. \end{cases} \tag{5.20}$$

A further application of (5.20) on the constant-coefficient system (4.18) indicates that the minimal wave speed of (2.6) is linearly selected provided that

$$a_1 \le \frac{2}{3},\ 6d(1-a_1)-r \ge 0,\ (2d-4)(1-a_1) \le r(2-a_2). \tag{5.21}$$

This is a new result, and is not covered in [16,29]. In fact, to some extent, our result is much sharper than the previous ones. For example, if we take $a_1 = 0.5, a_2 = 1.5, r = 2, d = 1$, then (5.21) holds true. Hence, the linear selection of the minimal wave speed of (2.6) is realized, but this case cannot be handled in [16,29].

Remark 5.6. It is possible to establish a unified type of upper solution with the following form

$$\overline{U}(x,z) = \frac{1}{1+\frac{e^{\overline{\mu}z}}{\phi(x)}},\ \overline{V}(x,z) = \max\{1, \overline{U}^{\gamma}(\alpha_1 + \alpha_2\overline{U} + \cdots)\},$$

where

$$\gamma = \frac{\mu_3(c_0)}{\mu_1(c_0)}.$$

Taking the system (4.18) as an example, it gives

$$\gamma = \frac{1+\sqrt{1+\frac{dr}{1-a_1}}}{d}.$$

According to the previous analysis, one can find that the sufficient conditions obtained in Theorem 5.3 and Theorem 5.4 rely heavily on the choice of upper solutions. In particular, smooth functions can be constructed so that we can have regular upper or lower solutions. For instance, we let

$$U = \frac{1}{(1+\frac{e^{\frac{1}{2}\overline{\mu}z}}{\phi(x)})^2},\quad V = \sqrt{U}(a+(1-a)\sqrt{U}),\quad a \ge 1. \tag{5.22}$$

Then the following relations are readily to verify.

$$\begin{aligned}
&U_z = -\overline{\mu}U(1-U^{\frac{1}{2}}),\quad U_x = 2\frac{\phi'}{\phi}U(1-U^{\frac{1}{2}}),\\
&U_{zz} = \overline{\mu}^2 U(1-U^{\frac{1}{2}})(1-\frac{3}{2}U^{\frac{1}{2}}),\quad U_{xz} = -2\overline{\mu}\frac{\phi'}{\phi}U(1-U^{\frac{1}{2}})(1-\frac{3}{2}U^{\frac{1}{2}}),\\
&U_{xx} = 2\frac{\phi''\phi-\phi'^2}{\phi^2}U(1-U^{\frac{1}{2}}) + 4\left(\frac{\phi'}{\phi}\right)^2 U(1-U^{\frac{1}{2}})(1-\frac{3}{2}U^{\frac{1}{2}}).
\end{aligned} \tag{5.23}$$

For simplicity, we need the following notations:

$$E_1 = d_2(x)\frac{\phi''}{\phi} - d_2(x)\left(\bar{\mu} - 2\frac{q'(x)}{q(x)}\right)\frac{\phi'}{\phi} + \frac{1}{4}d_2(x)\bar{\mu}^2 - d_2(x)\bar{\mu}\frac{q'(x)}{q(x)} - \frac{1}{2}c_0\bar{\mu},$$
$$E_3 = 2d_2(x)\frac{\phi''}{\phi} + 2d_2(x)\frac{\phi'^2}{\phi^2} - d_2(x)\left(4\bar{\mu} - 4\frac{q'(x)}{q(x)}\right)\frac{\phi'}{\phi} + d_2(x)\bar{\mu}^2 - 2d_2(x)\bar{\mu}\frac{q'(x)}{q(x)} - c_0\bar{\mu},$$
$$E_2 = \frac{1}{2}d_2(x)\left(\bar{\mu} - 2\frac{\phi'}{\phi}\right)^2, \quad E_4 = \frac{3}{2}d_2(x)\left(\bar{\mu} - 2\frac{\phi'}{\phi}\right)^2. \tag{5.24}$$

Theorem 5.7. *Assume that*

$$\begin{cases} \dfrac{\phi''}{\phi} + 2\left(\dfrac{p'(x)}{p(x)} - \overline{\mu}\right)\dfrac{\phi'}{\phi} + 2\dfrac{\phi'^2}{\phi^2} \le 0, \\ -E_4 + a_{11}(x)p(x) - (1-a)a_{12}(x)q(x) \le 0, \end{cases} \tag{5.25}$$

or

$$\begin{cases} \dfrac{\phi''}{\phi} + 2\left(\dfrac{p'(x)}{p(x)} - \overline{\mu}\right)\dfrac{\phi'}{\phi} + 2\dfrac{\phi'^2}{\phi^2} + \dfrac{1}{d_1(x)}[-E_4 + a_{11}(x)p(x) - (1-a)a_{12}(x)q(x)] \le 0, \\ -E_4 + a_{11}(x)p(x) - (1-a)a_{12}(x)q(x) > 0. \end{cases} \tag{5.26}$$

The minimal wave speed of (2.6)-(2.7) is linearly selected provided that

$$\begin{cases} a_{21}(x)p(x) - a_{22}(x)q(x)(1-a) - E_4 \le 0, \\ E_1 - a_{22}(x)q(x) \le 0, \\ a(E_1 - E_2) + (1-a)(E_3 - E_4) + (2-a)(a_{21}(x)p(x) - a_{22}(x)q(x)) \le 0. \end{cases} \tag{5.27}$$

Proof. Let U and V be the functions defined in (5.22). Using the relations in (5.23) to the U-equation, we arrive at

$$\begin{aligned}
& d_1(x)(U_{xx} + 2U_{xz} + U_{zz}) + 2d_1(x)\frac{p'(x)}{p(x)}(U_x + U_z) + cU_z \\
& \qquad + U[a_{11}(x)p(x)(1-U) - a_{12}(x)q(x)(1-V)] \\
&= U(1-U^{\frac{1}{2}})\Big\{2d_1(x)\frac{\phi''}{\phi} + 2d_1(x)\frac{\phi'^2}{\phi^2} - d_1(x)\left(4\bar{\mu} - 4\frac{p'(x)}{p(x)}\right)\frac{\phi'}{\phi} + d_1(x)\bar{\mu}^2 \\
& \qquad - 2d_1(x)\bar{\mu}\frac{p'(x)}{p(x)} - c_0\bar{\mu} + a_{11}(x)p(x) - a_{12}(x)q(x) \\
& \qquad + U^{\frac{1}{2}}\Big[-\frac{3}{2}d_1(x)\left(\overline{\mu} - \frac{2\phi'}{\phi}\right)^2 + a_{11}(x)p(x) - (1-a)a_{12}(x)q(x)\Big]\Big\} \\
&= U(1-U^{\frac{1}{2}})\Big\{d_1(x)\frac{\phi''}{\phi} + 2d_1(x)\left(\frac{p'(x)}{p(x)} - \overline{\mu}\right)\frac{\phi'}{\phi} + 2d_1(x)\frac{\phi'^2}{\phi^2} \\
& \qquad + U^{\frac{1}{2}}\Big[-\frac{3}{2}d_1(x)\left(\overline{\mu} - \frac{2\phi'}{\phi}\right)^2 + a_{11}(x)p(x) - (1-a)a_{12}(x)q(x)\Big]\Big\}
\end{aligned}$$

$$\le 0.$$

The negativity comes from (5.25) or (5.26). Inserting (5.22) into the V-equation leads to

$$\begin{aligned}
& d_2(x)(V_{xx} + 2V_{xz} + V_{zz}) + 2d_2(x)\frac{q'(x)}{q(x)}(V_x + V_z) + c_0 V_z \\
& \quad + (1 - V)(a_{21}(x)p(x)U - a_{22}(x)q(x)V) \\
& = \sqrt{U}(1 - \sqrt{U})\Big\{aE_1 - aa_{22}(x)q(x) + [a_{21}(x)p(x) - (1-a)a_{22}(x)q(x) \\
& \qquad - a(1-a)a_{22}(x)q(x) + (1-a)E_3 - aE_2]\sqrt{U} + (1-a)[a_{21}(x)p(x) \\
& \qquad - a_{22}(x)q(x)(1-a) - E_4]U\Big\} \\
& \le 0.
\end{aligned}$$

The negativity follows from (5.27). Consequently, the pair of functions (U, V) defined in (5.22) is an upper solution. By Theorem 5.1, the proof is complete. □

6. Simulations

In order to verify numerically the result of Theorem 4.6, we consider

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\partial^2 u_1}{\partial x^2} + u_1(1 + 0.1\cos(0.2x) - u_1 - 0.98u_2), \\ \dfrac{\partial u_2}{\partial t} = 3\dfrac{\partial^2 u_2}{\partial x^2} + u_2(1 + 0.1\cos(0.2x) - 2u_1 - u_2), \ t > 0, \ x \in \mathbb{R}. \end{cases} \tag{6.1}$$

The graph of $u_1(t, x)$ is shown by Fig. 6.1, when the initial data are taken as step functions.

We numerically calculate out

$$\lambda(1, 1 + 0.1\cos(0.2x)) = 1.0785 > 0, \lambda(3, 1 + 0.1\cos(0.2x)) = 1.0722 > 0,$$

which indicates that **(A1)** is satisfied. Moreover,

$$\begin{aligned}
& \lambda(1, 1 + 0.1\cos(0.2x) - 0.98q(x)) = 0.0214 > 0, \\
& \lambda(3, 1 + 0.1\cos(0.2x) - 2p(x)) = -0.9711 < 0,
\end{aligned}$$

which implies that **(A2)** is valid and $(p(x), 0)$ is stable. By performing a similar argument as [31, Lemma 5.3], one can prove that under the condition $\overline{b_1(x)} = \overline{b_2(x)} = \frac{1}{10\pi}\int_0^{10\pi}(1 + 0.1\cos(0.2x))dx > 0$ (which is obvious true), **(A3)** is ensured. A further computation shows that (4.14) is also valid for system (6.1). Thus, the minimal wave speed of (2.6) must be nonlinearly selected. In fact, we have

$$\hat{\lambda} = 0.0425, \bar{\mu} = 0.15, c_0 = 0.02834, c_{num} = 0.6614.$$

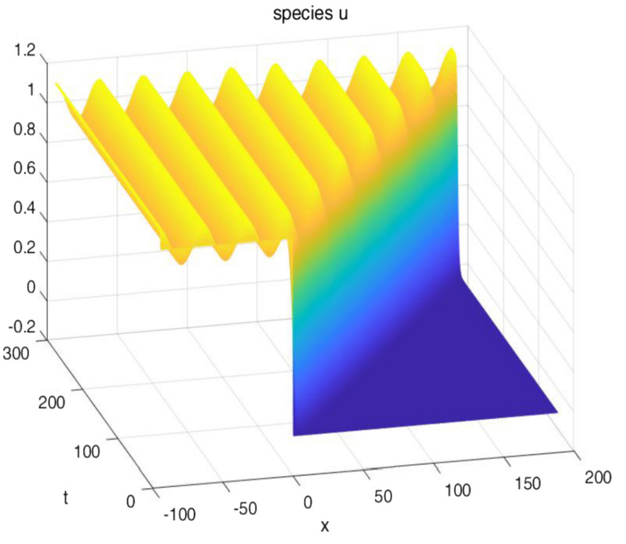

Fig. 6.1. The wave profile of $u_1(t,x)$ of (6.1).

As we can see, the numerically computed speed c_{num} is greater than the linear speed c_0. The solution evolves into a pushed wave.

To verify the condition (5.20) so that the minimal wave speed of the system (2.6) is linearly selected, we set

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\partial^2 u_1}{\partial x^2} + u_1(1 + 0.1\cos(0.2x) - u_1 - 0.5u_2), \\ \dfrac{\partial u_2}{\partial t} = 2\dfrac{\partial^2 u_2}{\partial x^2} + u_2(1 + 0.1\cos(0.2x) - u_1 - u_2), \ t > 0, \ x \in \mathbb{R}. \end{cases} \tag{6.2}$$

The graph of $u_1(t,x)$ is given by Fig. 6.2.

Noting that

$$\frac{1}{10\pi}\int_0^{10\pi} (1 + 0.1\cos(0.2x))dx = 1 > 0,$$

it follows from Remark 2.1 that conditions **(A1)-(A3)** are valid. Additionally, by the software MATLAB, one can check that the condition (5.20) is also true. Hence, the minimal wave speed is expected to be linearly selected. In fact, by the simulation, we have $\hat{\lambda} = 1.0046$, $\bar{\mu} = 0.71$, $c_0 = 1.4150$ and the numeric speed is $c_{num} = 1.4122$. As we can see, the relative error is as small as $O(10^{-3})$ which demonstrates the linear selection of the minimal wave speed.

The above graph demonstrates that our numerical simulations agree with the theoretical results from Theorems 4.14 and 5.4. One can confirm the theoretical results of Theorems 4.8, 5.3 as well as Theorem 5.7 by setting more general coefficients in (1.1). We omit them here.

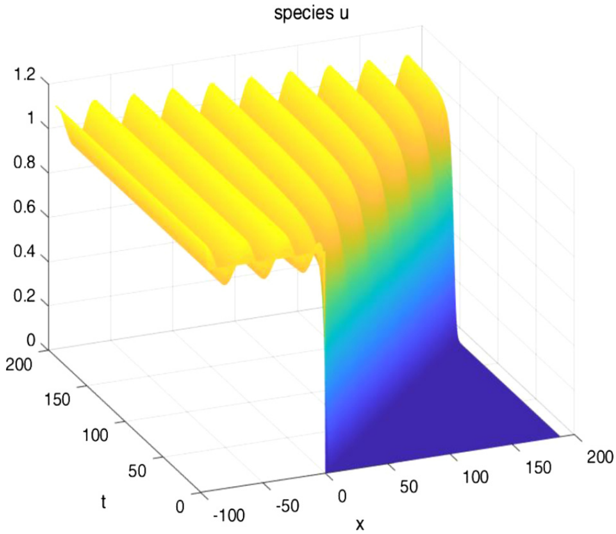

Fig. 6.2. The wave profile of $u_1(t,x)$ of (6.2).

7. Conclusions

In this paper, by means of the upper or lower solution method as well as comparison principle, we studied speed selection mechanism to the spreading speed of a two-species competitive reaction diffusion model in periodic habitats in the monostable case. Nonlinear selection can be obtained, as long as we can find a pair of lower solutions to the wave profile system with the first species decaying with a faster rate. Upper and lower bounds for the nonlinearly-selected spreading speed were provided. Explicit results for the determinacy of the nonlinear selection were established, based on various constructions of the lower solutions. On the other side, for the linear selection, our results showed that it is sufficient to find the existence of an upper solution at the linear speed c_0. This helped us to contribute a series of new results in terms of the coefficient functions.

Finally, to simplify the conditions of (4.14), we look at the special case when $a_{11}(x) = a_{22}(x) = 1$, $a_{12}(x) = a_1 < 1$, $a_{21}(x) = a_2 > 1$, $d_i(x) = 1$ and $b_i(x) = \bar{b} + \delta\cos(2\pi x/L + \xi)$, $i = 1, 2$. In this case, we have

$$\bar{b} - \delta < p(x) < \bar{b} + \delta,\ \bar{b} - \delta < q(x) < \bar{b} + \delta. \tag{7.1}$$

We can use asymptotic analysis to obtain an explicit condition of (4.14). To give an estimate for $p'(x)/p(x)$, we consider the equation

$$p''(x) + p(x)(\bar{b} + \delta\cos(2\pi x/L + \xi) - p(x)) = 0. \tag{7.2}$$

For small δ, we set $p(x) \sim \bar{b} + p_1(x)\delta$, where $p_1(x)$ is an L-periodic function. Substituting it into the above equation and equating the coefficient of δ yield

$$p_1''(x) - \bar{b}p_1(x) = -\bar{b}\cos(2\pi x/L + \xi). \tag{7.3}$$

Owing to the fact that $p_1(x)$ is an L-periodic function, we find

$$p_1(x) = \frac{\bar{b}}{(2\pi/L)^2 + \bar{b}} \cos(2\pi x/L + \xi).$$

As a result, an integral of (7.2) can lead to

$$\frac{p'(x)}{p(x)} \sim -\frac{2\pi L}{4\pi^2 + \bar{b}L^2} \sin(2\pi x/L + \xi)\delta \tag{7.4}$$

for small δ. Since $q(x)$ satisfies the same equation as (7.2), we get $q(x) = p(x)$. Next, we turn to the estimate of ϕ'/ϕ. To this end, let $\phi \sim 1 + \phi_1(x)\delta$, $\lambda \sim \lambda_0 + \lambda_1\delta$ and $\mu \sim \mu_0 + \mu_1\delta$, where $\phi_1(x)$ is an L-periodic function, $\lambda_0 = 2(1 - a_1)\bar{b}$ and $\mu_0 = \sqrt{(1 - a_1)\bar{b}}$. Plugging them into (2.10) leads to

$$\begin{aligned}(\lambda_0 + \lambda_1\delta)(1 + \phi_1\delta) = \phi_1''\delta - 2(\mu_0 + \mu_1\delta - \frac{p'(x)}{p(x)})\phi_1'\delta \\ + [(\mu_0 + \mu_1\delta)^2 - 2\frac{p'(x)}{p(x)}(\mu_0 + \mu_1\delta) + p(x) - a_1 q(x)](1 + \phi_1\delta).\end{aligned}$$

Equating the constant term, we get $\lambda_0 = \mu_0^2 + \bar{b}(1 - a_1)$. Equating the coefficient of δ, we have

$$\begin{aligned}\lambda_0\phi_1 + \lambda_1 = &\phi_1'' - 2\mu_0\phi_1' + [2\mu_0\mu_1 + 2M\mu_0 \sin(2\pi x/L + \xi) \\ &+ (1 - a_1)M_1 \cos(2\pi x/L + \xi)] + [\mu_0^2 + (1 - a_1)\bar{b}]\phi_1,\end{aligned}$$

where $M = \frac{2\pi L}{4\pi^2 + \bar{b}L^2}$ and $M_1 = \frac{\bar{b}}{(2\pi/L)^2 + \bar{b}}$. Bearing $\lambda_0 = \mu_0^2 + \bar{b}(1 - a_1)$ in mind, one can see that the above equation is equivalent to

$$\lambda_1 = \phi_1'' - 2\mu_0\phi_1' + [2\mu_0\mu_1 + 2M\mu_0 \sin(2\pi x/L + \xi) + (1 - a_1)M_1 \cos(2\pi x/L + \xi)].$$

Integrating it from 0 to L yields $\lambda_1 = 2\mu_0\mu_1$. Hence,

$$0 = \phi_1'' - 2\mu_0\phi_1' + [2M\mu_0 \sin(2\pi x/L + \xi) + (1 - a_1)M_1 \cos(2\pi x/L + \xi)]. \tag{7.5}$$

To find the expression of ϕ_1, we set

$$\phi_1 = C_1 \cos(2\pi x/L + \xi) + C_2 \sin(2\pi x/L + \xi),$$

and put it into (7.5) to get

$$\begin{aligned}(\frac{2\pi}{L})^2 C_1 + 2\mu_0(\frac{2\pi}{L})C_2 = (1 - a_1)M_1, \\ -2\mu_0(\frac{2\pi}{L})C_1 + (\frac{2\pi}{L})^2 C_2 = 2M\mu_0.\end{aligned}$$

Then we have

$$C_1 = \frac{(1-a_1)M_1\frac{2\pi}{L} - (2\mu_0)^2 M}{(\frac{2\pi}{L})^3 + (2\mu_0)^2\frac{2\pi}{L}}, C_2 = \frac{2\mu_0(1-a_1)M_1 + 2\mu_0 M\frac{2\pi}{L}}{(\frac{2\pi}{L})^3 + (2\mu_0)^2\frac{2\pi}{L}}.$$

As a result, for small δ, we get

$$\frac{\phi'}{\phi} \sim (-C_1\frac{2\pi}{L}\sin(2\pi x/L+\xi) + C_2\frac{2\pi}{L}\cos(2\pi x/L+\xi))\delta.$$

In view of (3.3) in our manuscript, we know $\bar{\mu} = \sqrt{\bar{B}}$, with

$$\begin{aligned}\bar{B} &= \frac{1}{L}\int_0^L (p(x) - a_1(x)q(x) + \frac{\phi_1''}{\phi_1} + 2\frac{p'(x)\phi_1'(x)}{p(x)\phi_1(x)})dx \\ &= \frac{1}{L}\int_0^L (p(x) - a_1(x)q(x) + (\frac{\phi_1'}{\phi_1})^2 + 2\frac{p'(x)\phi_1'(x)}{p(x)\phi_1(x)})dx.\end{aligned}$$

Consequently, we have

$$(\mu_0 + \mu_1\delta + o(\delta))^2 = \bar{b}(1-a_1) + o(\delta).$$

From which, it follows that

$$\mu_0^2 = \bar{b}(1-a_1), \mu_1 = 0.$$

In view of $\lambda_1 = 2\mu_0\mu_1$, we further obtain $\lambda_1 = 0$. Thus, $\lambda = \lambda_0 + o(\delta)$ and $\bar{\mu} = \mu_0 + o(\delta)$.

Due to the fact that $p(x) = q(x)$ in our special case, we get

$$Q_{\bar{\mu}}(x) := 2d_2(x)\left(\frac{\phi'(x)}{\phi(x)} - \bar{\mu}\right)\left(\frac{q'(x)}{q(x)} - \frac{p'(x)}{p(x)}\right) = 0.$$

Based on the above analysis, there exists a small δ_1 so that

$$p(x) > \bar{b} - \delta, \ \left(\bar{\mu} - \frac{\phi'(x)}{\phi(x)}\right)^2 < (\mu_0 + \frac{2\pi+1}{L}\sqrt{C_1^2 + C_2^2}\delta)^2$$

for $\delta < \delta_1$. The first condition of (4.14) can be ensured provided that

$$6(\mu_0 + \frac{2\pi+1}{L}\sqrt{C_1^2+C_2^2}\delta)^2 - (\bar{b} - \delta) < 0,$$

which can be further ensured if

$$\delta < \min\{\delta_1, \frac{\bar{b} - 6\bar{b}(1-a_1)}{1 + 24\mu_0(\pi+\frac{1}{2})\sqrt{C_1^2+C_2^2}/L}\}. \tag{7.6}$$

Similarly, the second equation of (4.14) can be guaranteed by

$$\frac{2+2(1-a_1)}{a_2} < 1+a_1 - \frac{2(\mu_0 + \frac{2\pi+1}{L}\sqrt{C_1^2+C_2^2}\delta}{\bar{b}-\delta},$$

which can be further ensured if

$$\delta < \min\{\delta_1, \frac{C_3\bar{b}-2(1-a_1)\bar{b}}{C_3+4\mu_0(\pi+\frac{1}{2})\sqrt{C_1^2+C_2^2}/L}\}, \tag{7.7}$$

where $C_3 = ((1+a_1)a_2 - 4 + 2a_1)/a_2$. Then one can conclude from (7.6) and (7.7) that condition (4.14) holds true provided that

$$\delta < \min\{\delta_1, \frac{\bar{b}-6\bar{b}(1-a_1)}{1+24\mu_0(\pi+\frac{1}{2})\sqrt{C_1^2+C_2^2}/L}, \frac{C_3\bar{b}-2(1-a_1)\bar{b}}{C_3+4\mu_0(\pi+\frac{1}{2})\sqrt{C_1^2+C_2^2}/L}\}.$$

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