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Journal of Differential Equations

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Speed selection for the wavefronts of the lattice Lotka-Volterra competition system ☆

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Received 26 March 2019; revised 17 September 2019; accepted 8 October 2019

Abstract

In this paper we study speed selection for traveling wavefronts of the lattice Lotka-Volterra competition model. For the linear speed selection, by constructing new types of upper solutions to the system, we widely extend the results in the literature. We prove that, for the nonlinear speed selection, the wavefront of the first species decays with a faster rate at the far end. This enables us to construct novel lower solutions to establish the existence of pushed wavefronts, a topic that has been understudied. We raise a new conjecture related to the classical Hosono's version of the diffusive system and our numerical simulations help to confirm it, while our rigorous results only provide a partial answer.
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MSC: primary 35K57, 35B20, 92D25

Keywords: Lattice Lotka-Volterra system; Traveling waves; Speed selection

☆ This work is supported by the Canada NSERC discovery grant (# RGPIN2016-04509).
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https://doi.org/10.1016/j.jde.2019.10.009
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Please cite this article in press as: H. Wang et al., Speed selection for the wavefronts of the lattice Lotka-Volterra competition system, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.10.009

1. Introduction

In this paper, we study the speed determinacy of the lattice Lotka-Volterra competition system

$$\begin{cases} u_j'(t) = \mathcal{D}_2[u_j](t) + u_j(t)[1 - u_j(t) - kv_j(t)], \\ v_j'(t) = d\mathcal{D}_2[v_j](t) + rv_j(t)[1 - v_j(t) - hu_j(t)], \ t \in \mathbb{R}^+, \ j \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $\mathcal{D}_2[u_j](t) = u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)$ and $\mathcal{D}_2[v_j](t) = v_{j+1}(t) + v_{j-1}(t) - 2v_j(t)$. In the system (1.1), $u_j(t)$ and $v_j(t)$ are the population densities at niches j and time t; d is the diffusion coefficient; r is the intrinsic growth rate; k, h are the competition coefficients. Indeed, system (1.1) can be regarded as a discretization of the classical diffusive Lotka-Volterra competition system

$$\begin{cases} u_t = u_{xx} + u(1 - u - kv), \\ v_t = dv_{xx} + rv(1 - v - hu), \ x \in \mathbb{R}, \ t \in \mathbb{R}^+. \end{cases} \tag{1.2}$$

The dynamics for (1.2) are very diverse and have been studied extensively. Three non-negative equilibria $(0, 0)$, $(1, 0)$, and $(0, 1)$ always exist. For the case when $k < 1, h < 1$, or the case when $k > 1, h > 1$, there is another unique positive co-existence equilibrium

$$(u^*, v^*) = \left(\frac{1-k}{1-kh}, \frac{1-h}{1-kh} \right).$$

In view of the phase plane portrait to the ordinary differential system of (1.2) without diffusion terms, the classification of the model (1.2) is clear. The nonlinearity of the case when $k < 1$ and $h < 1$ is called the persistence case (or co-existence). Furthermore, the nonlinearity is called the monostable case when $k < 1$ and $h > 1$ (or $k > 1$ and $h < 1$), or the bistable case when $k > 1$ and $h > 1$. Traveling waves of (1.2) have extensively attracted the interest of scientists. In the bistable case, the existence of traveling waves, connecting $(1, 0)$ and $(0, 1)$, was studied in Conley and Gardner [5], Gardner [9], and the uniqueness and parameter dependence of wave speeds can be found in [16]. For the monostable case, we refer to [12–14,17] for the existence of traveling waves, and [1,2,15,25] for the selection of the minimal speed. For the persistence (co-existence) case, the existence of traveling waves connecting $(0, 0)$ and (u^*, v^*) has also been investigated in [23,24] in great detail.

However, researchers believed that a lattice dynamic system may be more suitable than the continuous rival to model natural phenomena in some cases, such as the applications in material science, image processing, pattern formation and the phenomenon of biological invasion (see [3,4,20,22]). This is the main motivation that we focus on the dynamics of (1.1). Similarly one can see that system (1.1) at least has three equilibria: $e_0 = (0, 0)$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$ in the region $\{(u, v)|0 \le u \le 1, 0 \le v \le 1\}$. Throughout this paper, we impose an assumption on k and h with

$$0 < k < 1 < h, \tag{1.3}$$

which implies that the system exhibits the so-called monostable nonlinearity. For more biological explanations of this condition, we refer readers to [12–14,18,19,21]. For the existence of

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wavefront as well as the sign of its speed (i.e., the moving direction) in the bistable case, we will investigate them in future research.

As previously mentioned, we are interested in the special solution (i.e., wavefront) to the system (1.1) in the form

$$(u_j(t), v_j(t)) = (U(z), V(z)), \quad z = j - ct, \tag{1.4}$$

connecting e_1 and e_2. Here, c is the wave speed, and the pair (U, V) is usually called the wavefront. By substituting (1.4) into (1.1), we have

$$\begin{cases} -cU' = \mathcal{D}_2[U] + U(1 - U - kV), \\ -cV' = d\mathcal{D}_2[V] + rV(1 - V - hU), \\ (U, V)(-\infty) = (1, 0), \ (U, V)(+\infty) = (0, 1), \ z \in \mathbb{R}, \ 0 \le U, V \le 1, \end{cases} \tag{1.5}$$

where $\mathcal{D}_2[U(z)] := U(z+1) + U(z-1) - 2U(z)$, and so is $\mathcal{D}_2[V(z)]$. For further analysis, if we make a transformation $W = 1 - V$, then the system (1.5) can be rewritten as a cooperative system

$$\begin{cases} -cU' = \mathcal{D}_2[U] + U(1 - k - U + kW), \\ -cW' = d\mathcal{D}_2[W] + r(1 - W)(hU - W), \\ (U, W)(-\infty) = (1, 1), \ (U, W)(+\infty) = (0, 0), z \in \mathbb{R}, \ 0 \le U, W \le 1. \end{cases} \tag{1.6}$$

Accordingly, our aim is to find the traveling wave solution (U, W) with some unknown speed(s) c, connecting $(1, 1)$ and $(0, 0)$ for (1.6).

It has been proved in [10] that there exists a positive constant $c_{\min}$ (the minimal wave speed) such that (1.6) has a non-negative monotone traveling wavefront (U, W) if and only if $c \ge c_{\min}$. The same result can also be obtained by applying the idea in [7], as long as we can show that a single spreading speed exists. Indeed, following the idea of Theorem 5.3 in [7], we can immediately deduce that (1.6) has a single asymptotic spreading speed which implies significant biological interpretations. However, its explicit value is usually unknown. To estimate it, by linearizing the U-equation near the equilibrium solution $(0, 0)$ in (1.6), one can find a linear speed c_0 defined by

$$c_0 = \min_{\mu > 0} \frac{(e^{\mu} + e^{-\mu} - 2) + (1 - k)}{\mu}. \tag{1.7}$$

In view of the idea in [7] again, it can always be shown that $c_{\min} \ge c_0$. Only when some special restrictions are imposed on the parameters, $c_{\min}$ is equal to c_0 (pulled traveling wave exists); otherwise, $c_{\min} > c_0$ (pushed traveling wave exists). Whether they are equal or not has become a challenging problem for researchers in the study of biological invasions. Therefore, as in the continuous case in [1,2], we can set up the definition of linear or nonlinear determinacy of the minimal wave speed as follows.

Definition 1.1. The minimal wave speed is said to be linearly selected if $c_{\min} = c_0$, and nonlinearly selected if $c_{\min} > c_0$.

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Motivated by the Hosono's conjecture in [14] for the speed selection to the continuous model (1.2), which has not been completely solved, our conjecture is

Conjecture 1.1. *If $kh \le 1$, then $c_{\min} = c_0$ for all $r > 0$. If $kh > 1$, then there exists a critical positive number r_c, $0 \le r_c \le \infty$ such that $c_{\min} = c_0$ if $0 < r \le r_c$, and $c_{\min} > c_0$ if $r > r_c$.*

Remark 1.2. When $kh > 1$ and $r_c = 0$, the conjecture means that the minimal wave speed is nonlinearly selected for all $r > 0$.

As far as we know, there have been a few previous results with a partial answer to this conjecture. Guo and Wu in [10] proved that the minimal wave speed of (1.1) is linearly selected when $0 < d \le 1$, and

$$\{hk \le 1, r > 0\} \cup \left\{hk > 1, 0 < r < \frac{1-k}{hk-1}\right\}. \tag{1.8}$$

In another paper, Guo and Liang [11] extended the results to more general cases. More precisely, they proved that there exists a constant $d_* > 2$ such that the minimal wave speed is linearly selected if $0 < d \le d_*$ and $(h, k, r, d) \in A_1 \cup A_2 \cup A_3$, where

$$\begin{cases} A_1 := \{d \in (0, d_*], hk \le 1, r > 0\}, \\ A_2 := \left\{d \in (0, 1], hk > 1, 0 < r \le \dfrac{1-k}{hk-1}\right\}, \\ A_3 := \left\{d \in (1, d_*), hk > 1, 0 < r \le \dfrac{d_* - d}{d_* - 1}\dfrac{1-k}{hk-1}\right\}. \end{cases} \tag{1.9}$$

In this paper, we aim to further investigate the challenging problem of the speed selection for the lattice system (1.1) by employing the upper-lower solution method and the comparison principle. Developing the idea in [2], we first reduce the coupled system (1.6) to a scalar nonlocal equation by means of abstractly solving the W-equation (see Lemma 2.6). Due to the existence of second order center-difference operator $\mathcal{D}_2$, the proof of this lemma shows some new challenges and ideas that are quite different from the one in [2]. For the existence of traveling wavefronts as well as the speed selection, instead of using classical construction of the upper/lower solutions as in [6], technically, we provide some new upper and lower solutions for the lattice wave profile. This enables us to obtain novel results, not only on the linear selection, but also on the nonlinear selection mechanism, which has not been touched in any previous references for (1.1), including [10,11]. Furthermore, the existence of a threshold value of h, denoted as h_c, is also provided. Precisely, when h crosses over this value h_c, the speed selection mechanism changes from linear to nonlinear. The estimations of h_c are also established.

The structure of the rest of our paper is organized as follows. By virtue of the upper and lower solution method, Section 2 provides sufficient conditions which can help us to derive some explicit conditions for the speed selection. Section 3 is devoted to obtaining explicit conditions for linear and nonlinear speed selections. In Section 4, we focus on the critical value of h and the corresponding estimations. We will carry out some numerical simulations in Section 5 to demonstrate our main results and numerically verify the conjecture raised in this paper. Finally, we provide some concluding remarks in Section 6.

2. The speed selection mechanism

We will use the technique of upper/lower solution pair coupled with the comparison principle to find the existence of traveling waves as well as the speed selection mechanism. We first give the definition of upper and lower solutions.

Definition 2.1 *(Upper solution, see [10]).* Given $c \geq c_0$, a pair of continuous functions $(\overline{U}, \overline{W})$ from $\mathbb{R}$ to $(0, 1]$ is called an upper solution to the system (1.6), if $\overline{U}$ is a non-constant function, $\overline{U}(-\infty) = 1$, $\overline{W}(-\infty) = 1$, and $(\overline{U}, \overline{W})$ is differentiable a.e. in $\mathbb{R}$ such that

$$\begin{cases} -c\overline{U}' \geq \mathcal{D}_2[\overline{U}] + \overline{U}(1 - k - \overline{U} + k\overline{W}), \text{ a.e. in } \mathbb{R}, \\ -c\overline{W}' \geq d\mathcal{D}_2[\overline{W}] + r(1 - \overline{W})(h\overline{U} - \overline{W}), \text{a.e. in } \mathbb{R}. \end{cases} \tag{2.1}$$

The definition of the lower solution can be similarly given by reversing all the inequality signs, as well as some minor changes. This definition was not provided in [10], but we will need it for the nonlinear selection of the minimal wave speed.

Definition 2.2 *(Lower solution).* Given $c \geq c_0$, a pair of continuous functions $(\underline{U}, \underline{W})$ from $\mathbb{R}$ to $[0, 1)$ is called a lower solution to the system (1.6), if $\underline{U}$ is a non-constant function, $\underline{U}(\infty) = 0$, $\underline{W}(\infty) = 0$, and $(\underline{U}, \underline{W})$ is differentiable a.e. in $\mathbb{R}$ such that

$$\begin{cases} -c\underline{U}' \leq \mathcal{D}_2[\underline{U}] + \underline{U}(1 - k - \underline{U} + k\underline{W}), \text{ a.e. in } \mathbb{R}, \\ -c\underline{W}' \leq d\mathcal{D}_2[\underline{W}] + r(1 - \underline{W})(h\underline{U} - \underline{W}), \text{a.e. in } \mathbb{R}. \end{cases} \tag{2.2}$$

For the existence of traveling wavefronts, by applying the Helly's lemma for monotone functions, Guo and Wu [10] proved the following lemmas.

Lemma 2.3 *(see Lemma 2.4 in [10]). For given $c \geq c_0$, if there exists a nonincreasing upper solution $(\overline{U}, \overline{W})$ satisfying $\overline{U}(z) = 1$, $\overline{W}(z) = 1$ for $z \in (-\infty, 0]$, then (1.6) admits a traveling wavefront (c, U, W) with $U' < 0$ and $W' < 0$.*

If the upper solution is differentiable, there is another result.

Lemma 2.4 *(see Lemma 2.5 in [10]). If there exists a differentiable upper solution $(\overline{U}, \overline{W})$ satisfying $\overline{U}' < 0$ and $\overline{W}' < 0$ for a given $c \geq c_0$, then (1.6) admits a traveling wavefront (c, U, W) with $U' < 0$ and $W' < 0$.*

Remark 2.5. The method of upper/lower solution for finding traveling wavefronts originates from Diekmann [6] with two classical constructions of upper and lower solutions that have been extensively applied in the research of traveling wave solutions. Usually the upper solution is established by the truncation of an exponential function with the positive equilibrium, while the lower solution can be set up by the idea in [6] so that it always exists for $c > c_0$ for almost any monostable nonlinear systems. This means that the existence of traveling waves is completely dependent on the construction of suitable upper solutions (see the above two lemmas from [10]), a fact that has also been observed in [1,2].

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Based on the above results, for the linear speed selection, we will focus on the establishment of new upper solutions. Since (1.6) is a system, we may follow the idea in [1,2] to express W in terms of U from the second equation and substitute it into the first equation so that we will work on a nonlocal scalar equation. To this end, we will prove the following lemma.

Lemma 2.6. *Assume $c \geq c_0$. For any given continuous and nonincreasing function $U(z)$, with $U(-\infty) = a > 0$ and $U(\infty) = 0$, there exists a nonincreasing function $W(z) = W(U)(z)$ satisfying*

$$\begin{cases} d\mathcal{D}_2[W(z)] + cW' + r(1-W)(hU - W) = 0, \\ W(-\infty) = \min\{1, ha\}, \ \ W(\infty) = 0. \end{cases} \tag{2.3}$$

Moreover, the solution $W(U)$ is monotone in U in the sense that $W(U_1) \geq W(U_2)$ if $U_1 \geq U_2$.

Proof. Let $z = -t$ and $w(t) = 1 - W(z)$. Then the equation of $w(t)$ becomes

$$\begin{cases} d\mathcal{D}_2[w(t)] - cw'(t) + rw(t)(\bar{a}(t) - w(t)) = 0, \\ w(-\infty) = 1, \ \ w(\infty) = 1 - \min\{1, ha\}, \end{cases} \tag{2.4}$$

where $\bar{a}(t) = 1 - hU(z) = 1 - hU(-t)$ with $\bar{a}(-\infty) = 1, \bar{a}(\infty) = 1 - ha$. We will use the upper-lower solution method to prove the existence of $w(t)$. It is easy to see that $\bar{w}(t) = 1$ is an upper solution to the system (2.4). To construct a lower solution, we will consider two cases (i) $ha < 1$ and (ii) $ha \geq 1$ separately.

For the case (i) $ha < 1$, we can choose the lower solution as $\underline{w}(t) = 1 - ha$. This immediately gives the existence of the solution to (2.4).

For the case (ii) $ha \geq 1$, the construction of the lower solution becomes non-trivial. To this end, we choose $t_0 \in (-\infty, \infty)$ and small $\epsilon > 0$ such that $\bar{a}(t) \geq 1 - \epsilon$ for all $t \leq t_0$. Assume

$$f(\hat{w}) = \begin{cases} r\hat{w}(1 - \epsilon - \hat{w}), & \hat{w} \geq 0, \\ r\hat{w}(-\epsilon - \hat{w}), & \hat{w} < 0, \end{cases}$$

and consider the following bistable wave profile equation

$$d\mathcal{D}_2[\hat{w}(t)] + \hat{c}\hat{w}'(t) + f(\hat{w}(t)) = 0. \tag{2.5}$$

This system has three constant solutions $\hat{w} = -\epsilon, 0, 1 - \epsilon$. By Theorem 3.5 of [8], there exists a decreasing solution (bistable wavefront) $\hat{w}(t)$ for (2.5) with speed $\hat{c} = c_\epsilon$, satisfying

$$\hat{w}(-\infty) = 1 - \epsilon, \ \ \hat{w}(\infty) = -\epsilon. \tag{2.6}$$

A translation of $\hat{w}(t)$ is still a solution. Therefore, we can assume that $\hat{w}(t) \geq 0$ for $t \leq t_0$ and $\hat{w}(t) < 0$ for $t > t_0$. Here the speed c_ϵ is a continuous function of ϵ. When $\epsilon \to 0$, (2.5) reduces to

$$d\mathcal{D}_2[\tilde{w}(t)] + \hat{c}\tilde{w}'(t) + r\tilde{w}(1 - \tilde{w}) = 0, \tag{2.7}$$

which has non-negative traveling waves for $\hat{c} \geq \bar{c}_0$, connecting 1 and 0, with the minimal wave speed $\bar{c}_0$ given by

$$\bar{c}_0 = \min_{\mu>0} \frac{d(e^{\mu}+e^{-\mu}-2)+r}{\mu} > 0. \tag{2.8}$$

By the continuity of c_ϵ, it is easy to know $c_\epsilon \to \bar{c}_0$, with $c_\epsilon > 0$ as long as ϵ is sufficiently small. Now we can construct a lower solution to (2.4) with

$$\underline{w}(t) = \max\{\hat{w}(t), 0\}, \tag{2.9}$$

where $\hat{w}$ is the solution of (2.5) satisfying (2.6). It can be easy to verify that $\underline{w}(t)$ is a lower solution of (2.4).

The construction of upper and lower solutions of w implies the existence of the solution W. To see this, we turn to the original differential-difference equation (2.3) and transform it into an integral form

$$W(z) = \frac{1}{c}\int_z^{\infty} e^{\frac{L}{c}(z-s)} F(W,U)ds =: T[W,U]. \tag{2.10}$$

Here, $L>0$ is large enough so that $F(W,U)(z) = d[W(z+1)+W(z-1)-2W(z)] + r(1-W(z))(hU-W(z)) + LW(z)$ is monotone in W. We only prove the result in the case (ii) where $ha \geq 1$, since case (i) can be dealt with similarly. For the case (ii), the chosen upper and lower solutions are $\overline{W} = \min\{1, 1-\hat{w}(-z)\}$ and $\underline{W} = 0$, respectively. Clearly, $\overline{W}(+\infty) = \epsilon$ while $\overline{W}(-\infty) = 1$. We then follow the idea of the upper-lower solution method (see, e.g., [1,2]) to define an iteration scheme as

$$\begin{cases} W_{n+1} = T[W_n, U], \\ W_0 = \underline{W} = 0. \end{cases} \tag{2.11}$$

We obtain a sequence $\{W_n\}_{n=0}^{\infty}$ with $W_n(z)$ being nonincreasing in $\mathbb{R}$ and nondecreasing in n, that is, $0 \leq W_0 \leq W_1 \leq W_2 \leq \cdots \leq \overline{W} \leq 1$. By the Helly's lemma for monotone functions (see, e.g., [10]), this sequence converges to a nonincreasing function $W(z)$ pointwise, i.e., $\lim\limits_{n\to\infty} W_n = W$. It is obvious that this limit $W(z)$ is the solution of (2.3) and its continuity is ensured by (2.10). Since $W(z)$ is nonincreasing in $\mathbb{R}$ and $0 \leq W(z) \leq \overline{W}(z) \leq 1$ for all $z \in \mathbb{R}$, $W(\pm\infty)$ exist. By using equation (2.3) or (2.10), it follows that

$$\begin{cases} (1-W(-\infty))(ha - W(-\infty)) = 0, \\ (1-W(+\infty))(-W(+\infty)) = 0. \end{cases}$$

Since case (ii) means that $ha \geq 1$, it immediately follows that $W(-\infty) = 1$. Due to the fact $\overline{W}(+\infty) = \epsilon$, we obtain that $W(+\infty) = 0$.

It remains to show the monotonicity of $W(U)$. Notice that $F(W,U)$ is monotone in U since $0 \leq W \leq 1$. Let $U_1 \geq U_2$ in $\mathbb{R}$. Through the scheme (2.11) again, we obtain two sequences

$\{W_n(U_1)\}_{n=0}^{\infty}$ and $\{W_n(U_2)\}_{n=0}^{\infty}$ with the same initial data $W_0(U_1) = W_0(U_2) = 0$. By the monotonicity of $F(W, U)$ in W and U, we have $W_i(U_1) \geq W_i(U_2)$ for all $i \geq 1$. Therefore, it follows $W(U_1) = \lim_{n\to\infty} W_n(U_1) \geq \lim_{n\to\infty} W_n(U_2) = W(U_2)$. This means the monotonicity of $W(U)$ is proved. The proof is complete. □

By applying Lemma 2.6, (1.6) can reduce to the following nonlocal scalar equation

$$\begin{cases} H_1(U, W) := \mathcal{D}_2[U] + cU' + U(1 - k - U + kW(U)) = 0, \\ U(-\infty) = 1, \ U(+\infty) = 0. \end{cases} \tag{2.12}$$

We proceed to construct upper/lower solutions to (2.12). To incorporate a suitable decay rate for this type of solutions, we first linearize (1.6) near (0, 0), since the nonlinear system is dominated by its linear system at $z = \infty$. The linearized system is given by

$$\begin{cases} \mathcal{D}_2[U] + cU' + U(1 - k) = 0, \\ d\mathcal{D}_2[W] + cW' + r(hU - W) = 0. \end{cases} \tag{2.13}$$

The first equation is decoupled from the system. Let $U(z) = \xi_1 e^{-\mu z}$ for some positive constants ξ_1 and μ, as $z \to \infty$. By substituting it into (2.13), one can get a characteristic equation

$$\Gamma_1(\mu) := (e^{\mu} + e^{-\mu} - 2) - c\mu + (1 - k) = 0. \tag{2.14}$$

Let c_0 be defined in (1.7). It is easy to know that for $c > c_0$, $\Gamma_1(\mu) = 0$ has two solutions $\mu_1 = \mu_1(c)$ and $\mu_2 = \mu_2(c)$, with $\mu_1 < \mu_2$. Here $\mu_1(c)$ is a decreasing function and $\mu_2(c)$ is an increasing function of c, satisfying

$$\bar{\mu} := \mu_1(c_0) = \mu_2(c_0). \tag{2.15}$$

Remark 2.7. As $z \to \infty$ and $c > c_0$, the behavior of solution U of (1.6) (or (2.13)) can be given by

$$U(z) \sim C_1 e^{-\mu_1(c)z} + C_2 e^{-\mu_2(c)z}, \ \ z \to \infty, \tag{2.16}$$

for non-negative C_1, C_2 with $C_1 + C_2 > 0$. Substituting (2.16) into the second equation of (1.6) (or (2.13)), we can obtain the behavior of W as $z \to \infty$. Indeed, denote the characteristic equation of the second equation as

$$\Gamma_2(\mu) := d(e^{\mu} + e^{-\mu} - 2) - c\mu - r = 0.$$

It can be derived that the behavior of W is given by

$$W(z) \sim C_1 \frac{rh}{-\Gamma_2(\mu_1)} e^{-\mu_1 z} + C_2 \frac{rh}{-\Gamma_2(\mu_2)} e^{-\mu_2 z} + C_3 e^{-\mu_3 z}, \ z \to \infty, \tag{2.17}$$

if μ_3 is not equal to μ_1 or μ_2, where $C_3 > 0$, μ_3 is the unique positive solution of $\Gamma_2(\mu) = 0$. For a detailed and rigorous analysis of the asymptotic behavior of the wave profile, we refer to Section 3 in [10].

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We are ready to construct an upper solution pair. Let $c = c_0$. Define $\overline{U}(z)$ as

$$\overline{U} = \frac{1}{1+e^{\bar{\mu}z}}, \quad \bar{\mu} = \mu_1(c_0), \tag{2.18}$$

and $\overline{W} = W(\overline{U})$ is defined by Lemma 2.6. Thus, we get

$$\begin{aligned}\mathcal{D}_2(\overline{U}) &= \overline{U}(z+1) + \overline{U}(z-1) - 2\overline{U}(z)\\ &= \frac{1}{1+e^{\bar{\mu}(z+1)}} + \frac{1}{1+e^{\bar{\mu}(z-1)}} - 2\frac{1}{1+e^{\bar{\mu}z}}\\ &= \overline{U}\left[\frac{e^{\bar{\mu}z}(2-e^{\bar{\mu}}-e^{-\bar{\mu}}) + e^{2\bar{\mu}z}(e^{\bar{\mu}}+e^{-\bar{\mu}}-2)}{(1+e^{\bar{\mu}(z+1)})(1+e^{\bar{\mu}(z-1)})}\right]\\ &= \overline{U}\left(1-\overline{U}\right)\left[\frac{(e^{\bar{\mu}z}+1)(e^{\bar{\mu}z}-1)(e^{\bar{\mu}}+e^{-\bar{\mu}}-2)}{(1+e^{\bar{\mu}(z+1)})(1+e^{\bar{\mu}(z-1)})}\right].\end{aligned}$$

Substituting it into the expression of $H_1(U, W)$ and using $\overline{U}' = -\bar{\mu}\overline{U}(1-\overline{U})$, we have

$$H_1(\overline{U}, \overline{W}) = \overline{U}^2\left(1-\overline{U}\right)\left(-2(e^{-\bar{\mu}}+e^{\bar{\mu}}-2) + (e^{\bar{\mu}}+e^{-\bar{\mu}}-2)^2 R_1(\overline{U}) + k\frac{\overline{W}-\overline{U}}{\overline{U}\left(1-\overline{U}\right)}\right) \tag{2.19}$$

where

$$R_1(\overline{U}) := \frac{e^{\bar{\mu}z}(1-e^{\bar{\mu}z})}{1+e^{\bar{\mu}z}(e^{\bar{\mu}}+e^{-\bar{\mu}}) + e^{2\bar{\mu}z}}.$$

Therefore, $(\overline{U}, \overline{W})$ is an upper solution if

$$-2(e^{-\bar{\mu}}+e^{\bar{\mu}}-2) + (e^{\bar{\mu}}+e^{-\bar{\mu}}-2)^2 R_1(\overline{U}) + kY_1(z) \le 0 \tag{2.20}$$

is satisfied, where

$$Y_1(z) = \frac{\overline{W}-\overline{U}}{\overline{U}\left(1-\overline{U}\right)}. \tag{2.21}$$

The maximum of $R_1(\overline{U})$ can be derived by setting $x = e^{\bar{\mu}z} \in (0, \infty)$ and it is given by

$$\chi = \max_{z\in(-\infty,\infty)} R_1(\overline{U})(z) = \frac{1}{\tau+4+2\sqrt{\tau+4}}, \quad \tau = e^{\bar{\mu}} + e^{-\bar{\mu}} - 2, \tag{2.22}$$

which implies that the pair of functions $(\overline{U}, \overline{W})$ is an upper solution, provided that

$$-2\tau + \tau^2\chi + kY_1(z) \le 0 \tag{2.23}$$

holds.

With this choice of $(\overline{U}, \overline{W})$ as an upper solution, one can conclude the following theorem.

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Theorem 2.8. *The minimal wave speed of the system (1.6) is linearly selected, if (2.20), or (2.23), is satisfied.*

Remark 2.9. Similar to the discussion in [1], we can derive that the function $Y_1(z)$ is bounded above for all $z \in \mathbb{R}$.

Next, we proceed to investigate the nonlinear speed selection.

Theorem 2.10. *For $c_1 > c_0$, assume that there exists a continuous and monotonic function pair $(\underline{U}, \underline{W})(z) = (\underline{U}, \underline{W})(j - c_1 t)$ as a lower solution to (1.6), with $(0, 0) \le (\underline{U}, \underline{W}) < (1, 1)$, satisfying $\underline{U}(z) \sim \zeta_1 e^{-\mu_2(c_1)z}$ for some positive ζ_1 as $z \to \infty$ and $\underline{U}(-\infty) < 1$. Then no traveling wave solution exists for (1.6) with $c \in [c_0, c_1)$.*

Proof. First for $c \in (c_0, c_1)$, assume to the contrary that there is a monotonic traveling wavefront $(U, W)(z)$ to (1.6) with $(U, W)(j - ct)$ satisfying

$$\begin{cases} u_j'(t) = \mathcal{D}_2[u_j] + u_j(1 - k - u_j + k w_j), \\ w_j'(t) = d\mathcal{D}_2[w_j] + r(1 - w_j)(h u_j - w_j), \quad j \in \mathbb{Z}, \end{cases} \tag{2.24}$$

subject to the initial conditions

$$u_j(0) = U(j) \text{ and } w_j(0) = W(j).$$

Applying the monotonicity of $\mu_1(c)$ and $\mu_2(c)$ as well as Remark 2.7, we can assume $\underline{U}(z) < U(z)$, by shifting if necessary. Using the equation in Lemma 2.6 and the monotonicity of $W(U)$, we can obtain $(\underline{U}, \underline{W})(z) \le (U, W)(z)$. Due to the fact that $(\underline{U}, \underline{W})(j - c_1 t)$ is a lower solution to (2.24), it follows by comparison that

$$\underline{U}(j - c_1 t) \le U(j - ct), \text{ and } \underline{W}(j - c_1 t) \le W(j - ct). \tag{2.25}$$

Choose a point z so that $\underline{U}(z) > 0$ is fixed. On the line $z = j - c_1 t$, it follows that

$$U(j - ct) = U(z + (c_1 - c)t) \sim U(\infty) = 0 \text{ as } t \to \infty.$$

By (2.25), this implies that $\underline{U}(z) \le 0$, which is a contradiction of the positivity of $\underline{U}(z)$.

Finally, if there exists a traveling wavefront when $c = c_0$, naturally there exists a traveling wave for $c \in (c_0, c_1)$, since c_0 becomes the minimal wave speed. The above argument still operates to deduce a contradiction. Thus, the proof is complete. □

Now, we apply Theorem 2.10 to investigate the nonlinear selection of the minimal wave speed. For $0 < \underline{k} < 1$, define

$$\underline{U} = \frac{\underline{k}}{1 + e^{\mu_2(c)z}},$$

for $c = c_0 + \varepsilon$, where ε is a sufficiently small number. Similar to (2.19), we can obtain

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$$H_1(\underline{U},\underline{W}) = \frac{\underline{U}^2}{k}\left(1-\frac{\underline{U}}{k}\right)\left(-2(e^{-\mu_2(c)}+e^{\mu_2(c)}-2)+R_2(\underline{U})+k\frac{\underline{W}-\underline{U}\left(\frac{k-1+\underline{k}}{k\underline{k}}\right)}{\frac{\underline{U}}{\underline{k}}\left(1-\frac{\underline{U}}{\underline{k}}\right)}\right). \tag{2.26}$$

It is easy to conclude that the pair of functions $(\underline{U}(z),\underline{W}(z))$ is a lower solution to system (1.6) if

$$-2(e^{\mu_2(c)}+e^{-\mu_2(c)}-2)+R_2(\underline{U})+kY_2(z)>0, \quad \text{for all } z\in(-\infty,\infty), \tag{2.27}$$

where

$$Y_2(z) = \frac{\underline{W}-\underline{U}\left(\frac{k-1+\underline{k}}{k\underline{k}}\right)}{\frac{\underline{U}}{\underline{k}}\left(1-\frac{\underline{U}}{\underline{k}}\right)}, \quad R_2(\underline{U}) := \frac{(e^{\mu_2(c)}+e^{-\mu_2(c)}-2)^2 e^{\mu_2(c)z}(1-e^{\mu_2(c)z})}{1+e^{\mu_2(c)z}(e^{\mu_2(c)}+e^{-\mu_2(c)})+e^{2\mu_2(c)z}}. \tag{2.28}$$

By replacing $R_2(\underline{U})$ by its minimum $-(e^{\mu_2(c)}+e^{-\mu_2(c)}-2)^2$, it follows from (2.27) that $(\underline{U}(z),\underline{W}(z))$ is a lower solution if

$$-2(e^{\mu_2(c)}+e^{-\mu_2(c)}-2)-(e^{\mu_2(c)}+e^{-\mu_2(c)}-2)^2+kY_2(z)>0 \tag{2.29}$$

is satisfied. Based on the above results, we have the following theorem.

Theorem 2.11. *The minimal wave speed of the system (1.6) is nonlinearly selected if (2.27), or (2.29), is satisfied for some $c=c_0+\varepsilon$.*

3. Explicit conditions for the speed selection

We give the linear/nonlinear speed selection in the previous section based on the existence of implicit formula $W(U)$. Numerically this provides a good judgment for the speed selection. In this section, we want to establish some explicit conditions for the speed selection with the estimate of $W(U)$. In other words, we will give formulas for U and W simultaneously.

Theorem 3.1. *The minimal wave speed of (1.6) is linearly selected for all $r>0$, if*

$$0\le d\le 1+\frac{1-k}{\tau}, \quad \textit{and} \quad kh\le 2\tau-\chi\tau^2, \tag{3.1}$$

where

$$\tau = e^{\bar\mu}+e^{-\bar\mu}-2, \quad \bar\mu=\mu_1(c_0), \tag{3.2}$$

and χ is defined in (2.22), i.e.,

$$\chi = \frac{1}{\tau+4+2\sqrt{\tau+4}}.$$

Proof. Let $\overline{U}$ be defined in (2.18) and $\overline{W}(z)$ be given by

$$\overline{W}(z) = \min\{1, h\overline{U}(z)\} = \begin{cases} 1, & z \leq z_1, \\ h\overline{U}, & z > z_1, \end{cases} \tag{3.3}$$

where z_1 is the unique root of $h\overline{U}(z) = 1$. Let $c = c_0$. For the W-equation, when $z \leq z_1 - 1$, we have

$$d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) = 0.$$

When $z_1 - 1 < z \leq z_1$, we have

$$d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) = d(h\overline{U}(z+1) - 1) < 0.$$

While for $z_1 < z \leq z_1 + 1$, by noticing $h\overline{U}(z-1) \geq 1$, we get

$$\begin{aligned} & d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) \\ & = d(1-2h\overline{U}(z) + h\overline{U}(z+1)) + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) \\ & = dh\overline{U}(z+1) - 2dh\overline{U}(z) + dh\overline{U}(z-1) + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) + d(1 - h\overline{U}(z-1)) \\ & \leq dh\overline{U}(z+1) - 2dh\overline{U}(z) + dh\overline{U}(z-1) + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}). \end{aligned}$$

For $z > z_1 + 1$, it is easy to see that

$$\begin{aligned} & d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) \\ & = dh\overline{U}(z+1) - 2dh\overline{U}(z) + dh\overline{U}(z-1) + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}). \end{aligned}$$

As a result, when $z > z_1$, we can deal with the W-equation in the same way. In view of the first condition of (3.1), we have

$$d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) \leq h\overline{U}\left(1 - \overline{U}\right)\{d\tau - c\bar{\mu}\} \leq 0.$$

For the U-equation, we get the following estimation

$$Y_1(z) = \begin{cases} \dfrac{1}{\overline{U}} \leq h, & \text{when } z \leq z_1, \\ \dfrac{h-1}{1-\overline{U}} \leq h, & \text{when } z > z_1. \end{cases} \tag{3.4}$$

Then we have $-2\tau + kh + \chi\tau^2 \leq 0$ for all r. This ensures that $(\overline{U}, \overline{W})$ is an upper solution. By Lemma 2.3, our proof is complete. □

Remark 3.2. We point out that the number $2\tau - \chi\tau^2$ is positive which can be verified directly.

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Remark 3.3. If we choose the function $(U, W) = (-\Gamma_2(\bar{\mu}), rh)e^{-\bar{\mu}z}$ as an upper solution, and substitute it into (1.6), then we have

$$r(kh-1) \leq (1-d)\tau + (1-k). \tag{3.5}$$

This recovers the result in [10,11].

A combination of Theorem 3.1 and Remark 3.3 leads to the following corollary.

Corollary 3.4. *If $0 \leq d < 1$ and $kh \leq \max\left\{1, 2\tau - \chi\tau^2\right\}$, then the minimal wave speed of the lattice system (1.6) is linearly selected.*

Interestingly, by choosing another different upper solution for W, we can obtain the following theorem.

Theorem 3.5. *Linear speed selection of (1.6) is realized if*

$$\begin{cases} 0 \leq d < 1 + \dfrac{1-k}{\tau}, \ k \leq 2\tau - \chi\tau^2, \\ kh > 2\tau - \chi\tau^2, \ r \leq \dfrac{L[(1-d)\tau + (1-k)]}{h-L}, \end{cases} \tag{3.6}$$

where

$$L := \frac{1}{k}(2\tau - \chi\tau^2) \geq 1.$$

Proof. For the same $\overline{U}$ defined in (2.18), choose $\overline{W}$ as

$$\overline{W}(z) = \min\left\{1, L\overline{U}(z)\right\} = \begin{cases} 1, & z \leq z_2, \\ L\overline{U}(z), & z > z_2, \end{cases}$$

where z_2 satisfies $L\overline{U}(z) = 1$. Similar to the treatment in Theorem 3.1, we know that, for $c = c_0$, $z \leq z_2$, we obtain

$$d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1-\overline{W})(h\overline{U} - \overline{W}) \leq 0.$$

While for $z > z_2$, we can show

$$\begin{aligned} d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1-\overline{W})(h\overline{U} - \overline{W}) &\leq \overline{U}\left(1-\overline{U}\right)L(d\tau - c\bar{\mu}) + r\left(1 - L\overline{U}\right)\left(h\overline{U} - L\overline{U}\right) \\ &\leq \overline{U}(1-\overline{U})\left\{L[(d-1)\tau - (1-k)] + r(h-L)\right\} \\ &\leq 0 \end{aligned}$$

if

$$L \geq 1, \quad kh > 2\tau - \chi\tau^2, \ \text{and } r \leq \frac{L[(1-d)\tau + (1-k)]}{h-L}.$$

From the choice of $(\overline{U}, \overline{W})$, it can be derived that $Y_1(z) \leq \frac{1}{k}(2\tau - \chi\tau^2)$. Therefore, (2.20) is satisfied. In view of Lemma 2.3, the proof is complete. □

If we define $M := \max\{1, 2\tau - \chi\tau^2\}$, then the combination of (3.5) and the bound for r in (3.6) gives the following result.

Corollary 3.6. *The minimal wave speed is linearly selected provided that $0 < d < 1 + \frac{1-k}{\tau}$, $hk > M$, $k \leq 2\tau - \chi\tau^2$ and*

$$r \leq \frac{M[(1-d)\tau + (1-k)]}{hk - M}.$$

Theorem 3.7. *For $1 + \frac{1-k}{\tau} < d < 1 + \frac{(1-k)+r}{\tau}$, the minimal wave speed of (1.6) is linearly selected if*

$$\frac{r + (1-k) - (d-1)\tau}{hr} > \max\left\{\frac{(d-1)\tau - (1-k)}{d(2\tau - \chi\tau^2)}, \frac{k}{2\tau - \chi\tau^2}\right\}.$$

Proof. Take the choice of

$$\overline{W}(z) = \min\{1, h\gamma\overline{U}(z)\} = \begin{cases} 1, & z \leq z_3, \\ h\gamma\overline{U}(z), & z > z_3, \end{cases}$$

where z_3 satisfies $h\gamma\overline{U}(z) = 1$ and

$$\gamma = \gamma(d, k, r) = \frac{r}{r + (1-k) - (d-1)\tau} + \eta_1,$$

with η_1 being a sufficiently small positive number such that

$$\frac{1}{h\gamma} > \max\left\{\frac{(d-1)\tau - (1-k)}{d(2\tau - \chi\tau^2)}, \frac{k}{2\tau - \chi\tau^2}\right\}. \tag{3.7}$$

As before let $c = c_0$. For the piecewise continuous function $\overline{W}(z)$, we can deduce that

$$d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) \leq 0$$

for $z \leq z_3$. When $z > z_3$, we can show

$$d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1 - \overline{W})(h\overline{U} - \overline{W}) \leq h\overline{U}F(\overline{U}),$$

where

$$F(\overline{U}) = d\gamma\tau(1-\overline{U})(1-2\overline{U}) - c\bar{\mu}\gamma(1-\overline{U}) + r(1-\gamma)(1-h\gamma\overline{U}) + d\gamma\chi\tau^2\overline{U}(1-\overline{U}).$$

One can easily conclude that $F(\overline{U})$ is concave up. Therefore, for $\overline{U} \in [0, \frac{1}{h\gamma}]$, we will prove $F(\overline{U}) < 0$ by showing that they are negative at two endpoints 0 and $\frac{1}{h\gamma}$. For the left endpoint, we have $F(0) = -\eta_1[r - (d-1)\tau + (1-k)] < 0$. For the right endpoint, by using (3.7), we have

$$F\left(\frac{1}{h\gamma}\right) = \gamma\left(1 - \frac{1}{h\gamma}\right)\left\{d\tau - c\bar{\mu} - d(2\tau - \chi\tau^2)\frac{1}{h\gamma}\right\} < 0.$$

This indicates that $d\mathcal{D}_2[\overline{W}] + c\overline{W}' + r(1-\overline{W})(h\overline{U} - \overline{W}) \le 0$ for all $z \in (-\infty, \infty)$. On the other hand, we can show that

$$Y_1(z) = \frac{\overline{W} - \overline{U}}{(1-\overline{U})\overline{U}} = \begin{cases} \dfrac{1}{\overline{U}} \le h\gamma, & \text{when } z \le z_3, \\ \dfrac{h\gamma - 1}{1 - \overline{U}} \le h\gamma, & \text{when } z > z_3. \end{cases}$$

Then in view of the conditions in the theorem, we find that $-2\tau + \chi\tau^2 + kh\gamma < 0$. By Theorem 2.8, the proof is complete. □

The above results are related to the linear speed selection. Now we start to establish the nonlinear speed selection by constructing a fast decaying lower solution for U-equation.

First we set

$$\underline{U}_1(z) = \frac{\underline{k}}{1 + e^{\mu_2(c)z}}, \quad \underline{W}_1(z) = \frac{\underline{U}_1(z)}{\underline{k}}, \tag{3.8}$$

where $c = c_0 + \varepsilon$, $\underline{k} \in (0, 1)$ is to be determined. Then we obtain the following result.

Theorem 3.8. *The minimal wave speed of (1.6) is nonlinearly selected if*

$$\frac{1}{hr}[(d+1)\tau + d\tau^2 + (1-k) + r] < 1 - 2\tau - \tau^2.$$

Proof. By the assumption, we can find $\underline{k}$ satisfying

$$\frac{1}{hr}[(d+1)\tau + d\tau^2 + (1-k) + r] < \underline{k} < 1 - 2\tau - \tau^2.$$

Via a substitution of (3.8) into (2.29) together with the condition in the theorem, we can directly prove inequality (2.29) and

$$\underline{W}_1(1 - \underline{W}_1)\left\{-[d\tau + d\tau^2 + c\mu_2(c)] + r(h\underline{k} - 1)\right\} \ge 0.$$

Here, we have used the fact that $\mu_2(c) \to \bar{\mu}$ and $e^{\mu_2(c)} + e^{-\mu_2(c)} - 2 \to \tau$ as $\varepsilon \to 0$. Therefore, the proof is complete. □

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It is worth mentioning that one can choose smooth upper solutions to obtain different conditions for the linear selection. We now set W as a solution to the following equation

$$W' = -\frac{1}{2}\bar{\mu} W(1-W).$$

The explicit expression for W is given by

$$W = \frac{1}{1+e^{\frac{1}{2}\bar{\mu}z}}.$$

Further, we set $U^{\frac{1}{2}} = W$, namely,

$$U = W^2 = \left(\frac{1}{1+e^{\frac{1}{2}\bar{\mu}z}}\right)^2.$$

Substituting it into $\mathcal{D}_2[U]$ by a straightforward calculation (the software such as Mathematica can be used), we have

$$\mathcal{D}_2[U] = \tau U(1-U^{\frac{1}{2}})(1-\frac{3}{2}U^{\frac{1}{2}}) + R(x,U), \;\; x = e^{\frac{1}{2}\bar{\mu}z},$$

where the remainder term $R(x,U)$ is given by

$$R(x,U) = \tau U(1-U^{\frac{1}{2}})U^{\frac{1}{2}} g(x).$$

Here,

$$g(x) = \frac{a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4}{1 + 2\sqrt{\tau+4}x + (\tau+6)x^2 + 2\sqrt{\tau+4}x^3 + x^4},$$

where

$$a_1 = -2a + \frac{1}{2},\; a_2 = a^2\tau - 4a - 1 + \sqrt{\tau+4},\; a_3 = 2a^2\tau + 6 - 4a - 2\sqrt{\tau+4} + \frac{\tau}{2},$$
$$a_4 = a^2\tau - 4a - \tau - 1 + \sqrt{\tau+4},\; a_5 = \frac{9}{2} - 2a - 2\sqrt{\tau+4},$$

with

$$a = \frac{\sqrt{\tau+4}-2}{\tau}.$$

If the maximum of $g(x)$, $x \in (0,+\infty)$, is denoted by Ω, i.e.,

$$\Omega = \max_{x\in(0,\infty)} g(x), \tag{3.9}$$

which is positive and always exists since the denominator of $g(x)$ is positive and the limit at infinity is finite, then the term $\mathcal{D}_2[U]$ can be estimated by

$$\mathcal{D}_2[U] \le \tau U(1-U^{\frac{1}{2}})(1-\frac{3}{2}U^{\frac{1}{2}}) + \tau\Omega U(1-U^{\frac{1}{2}})U^{\frac{1}{2}}.$$

Based on the above analysis, we have the following theorem.

Theorem 3.9. *Let Ω be defined in (3.9). The minimal wave speed of (1.6) is linearly selected if $-\frac{3}{2}\tau+1+\tau\Omega \le 0$ and one of the following conditions is satisfied:*

$$d\tau_1 - \frac{1}{2}(\tau+(1-k)) < r < \frac{1}{h}(2d\tau_1 - d\chi_1\tau_1^2), \tag{3.10}$$

or

$$\frac{1}{h}(2d\tau_1 - d\chi_1\tau_1^2) \le r < \frac{d\tau_1 + \frac{1}{2}(\tau+(1-k)) - d\chi_1\tau_1^2}{h-1}, \tag{3.11}$$

where

$$\tau_1 = \sqrt{\tau+4}-2,\ \ \chi_1 = \frac{1}{\tau_1+4+2\sqrt{\tau_1+4}}.$$

Proof. Note that

$$U' = -\bar{\mu}U(1-U^{\frac{1}{2}}),\ \text{and}\ \ W' = -\frac{1}{2}\bar{\mu}W(1-W).$$

Let $c=c_0$. One can verify that (U, W) turns to be an upper solution to system (1.6) if

$$\mathcal{D}_2[U] + cU' + U(1-k-U+kW) \le UU^{\frac{1}{2}}(1-U^{\frac{1}{2}})\left(-\frac{3}{2}\tau+1+\tau\Omega\right) \le 0$$

and

$$\begin{aligned}
&d\mathcal{D}_2[W] + cW' + r(1-W)(hU-W) \\
&\le W(1-W)\left[d\tau_1 - \frac{1}{2}c\bar{\mu} - r + (-2d\tau_1 + rh + d\chi_1\tau_1^2)W\right] \\
&\le 0,
\end{aligned}$$

which are valid by the conditions (3.10) and (3.11). The proof is thus complete. □

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4. Existence and the estimation of the critical value of h

In this section, we will study the linear/nonlinear speed selection in terms of h when other parameters are fixed. More precisely, we will investigate the existence of a threshold value h_c of h in the sense that, when h crosses from the left side of h_c to the right, the minimal wave speed selection changes from linear to nonlinear. We begin with a comparison on the parameter h as follows:

Lemma 4.1. *If the minimal wave speed of the system (1.5) is linearly selected for some $h = h_\vartheta > 1$, then it is linearly selected for all $1 < h \le h_\vartheta$.*

Proof. When $h = h_\vartheta$, we denote the corresponding solution by $(U_\vartheta, W_\vartheta)$. Then we have

$$\begin{cases} \mathcal{D}_2[U_\vartheta] + cU'_\vartheta + U_\vartheta(1 - k - U_\vartheta + kW_\vartheta) = 0, \\ cW'_\vartheta + r(1 - W_\vartheta)(h_\vartheta U_\vartheta - W_\vartheta) = 0, \\ (U_\vartheta, W_\vartheta)(-\infty) = (1, 1),\ (U_\vartheta, W_\vartheta)(+\infty) = (0, 0). \end{cases} \tag{4.1}$$

To confirm that the pair of functions $(U_\vartheta, W_\vartheta)$ is an upper solution to system (1.6) for $h < h_\vartheta$, we need to prove

$$\begin{cases} \mathcal{D}_2[U_\vartheta] + cU'_\vartheta + U_\vartheta(1 - k - U_\vartheta + kW_\vartheta) \le 0, \\ cW'_\vartheta + r(1 - W_\vartheta)(hU_\vartheta - W_\vartheta) \le 0, \\ (U_\vartheta, W_\vartheta)(-\infty) = (1, 1),\ (U_\vartheta, W_\vartheta)(+\infty) = (0, 0). \end{cases} \tag{4.2}$$

In fact, when $h < h_\vartheta$, the first equation of (4.2) is naturally satisfied since the first equation of (4.1) remains unchanged. As for the second equation of (4.2), we have

$$cW'_\vartheta + r(1 - W_\vartheta)(hU - W_\vartheta) = r(h - h_\vartheta)(1 - W_\vartheta)U_\vartheta \le 0.$$

Hence, $(U_\vartheta, W_\vartheta)$ is an upper solution. By Lemma 2.4, we know that the minimal wave speed is linearly selected for $1 < h \le h_\vartheta$. □

Lemma 4.1 allows us to define a critical value of h as

$$h_c = \sup\{h > 1 |\ \text{the minimal wave speed of system (1.6) is linearly selected}\}.$$

We have the following result.

Theorem 4.2. *The minimal wave speed of system (1.6) is linearly selected when $1 < h \le h_c$, and nonlinearly selected when $h > h_c$.*

Now, we are in a position to give an estimation for h. For this, we need the following properties of $Y_1(z)$ and $Y_2(z)$.

Lemma 4.3. *$Y_1(z, h)$ and $Y_2(z, h)$ defined in (2.21) and (2.28) are increasing with respect to h.*

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Proof. By noting that $\overline{U}$ and $\underline{U}$ are independent of h, it is sufficient to prove $\overline{W}$ and $\underline{W}$ are increasing with respect to h. This can be verified directly from (2.3). $\square$

In view of Lemma 4.3, we can define

$$h_- = \sup\{h > 1 |\text{ the inequality (2.23) is valid for all } z \in (-\infty, \infty)\},$$

$$h_+ = \inf\{h > 1 |\text{ the inequality (2.29) is valid for all } z \in (-\infty, \infty)\}.$$

It is obvious $1 \le h_- \le h_c \le h_+ < \infty$. From the previous section, we will give more estimates for them.

Theorem 4.4. *When* $0 \le d < 1 + \frac{1-k}{\tau}$, *we have*

$$h_c \ge \frac{M[(1-d)\tau + (1-k) + r]}{kr}.$$

Theorem 4.5. *When* $1 + \frac{1-k}{\tau} < d < 1 + \frac{(1-k)+r}{\tau}$, *we have*

$$h_c \ge \frac{r + (1-k) - (d-1)\tau}{r} \min\left\{\frac{d(2\tau - \chi\tau^2)}{(d-1)\tau - (1-k)}, \frac{2\tau - \chi\tau^2}{k}\right\}.$$

Theorem 4.6. *If* $2\tau + \tau^2 < 1$, *or equivalently,* $0 < \tau < -1 + \sqrt{2}$, *then we have*

$$h_c \le \frac{(d+1)\tau + d\tau^2 + (1-k) + r}{r(1 - 2\tau - \tau^2)}.$$

5. Numerical simulations

In this section, we will perform numerical simulations to manifest the results obtained in Sections 3 and 4, and also give a numerical verification of the Conjecture 1.1 we raised in the Introduction.

To proceed, we numerically compute the spreading speed of traveling waves of (1.1), named c_{num}, and then compare it with c_0 whose formula is given by (1.7). To obtain c_{num}, we solve the initial-boundary value problem of (1.1) by applying the 4th order Runge-Kutta method for ordinary differential systems, where the initial conditions are given as

$$u_j(0) = \begin{cases} 1, & 1 \le j \le N_j, \\ 0, & N_j + 1 \le j \le N_L, \end{cases} \quad \text{and } v_j(0) = \begin{cases} 0, & 1 \le j \le N_j, \\ 1, & N_j + 1 \le j \le N_L, \end{cases} \tag{5.1}$$

and the boundary conditions are

$$\begin{cases} u_1(t) - u_2(t) = u_{N_L}(t) - u_{N_L - 1}(t) = 0, \\ v_1(t) - v_2(t) = v_{N_L}(t) - v_{N_L - 1}(t) = 0 \end{cases} \tag{5.2}$$

where N_L is a large positive integer.

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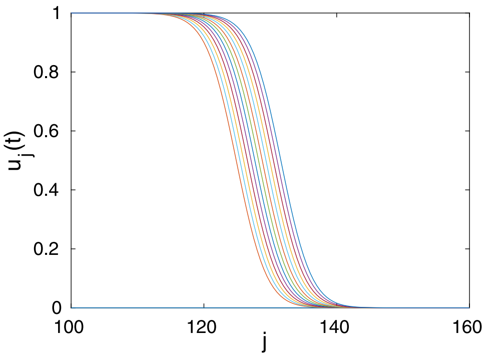

Fig. 5.1. (Color online) The solution $u_j(t)$ for $t = 200, 202, 204, \cdots, 224$. The figure is depicted when $k = 0.5$, $h = 1$, $r = 1$, and $d = 1$.

Table 5.1
Numerical demonstrations of the theorems. The table is to verify Theorems 3.1-3.8.

Theorem	k	h	d	r	c_0	c_{num}	$c_0 - c_{\text{num}}$
3.1	0.1	2	2	10	1.9068	1.9598	0.001
3.5	0.1	20	0.5	3	1.9068	1.9588	0.002
3.7	0.5	1.5	3	5	1.4417	1.4414	0.0003
3.8	0.8	2	0.5	4	0.9017	0.9561	−0.0544

Via the theory of the spreading speed (see e.g. [8]), the solution with the above initial data will evolve into a traveling wave solution of (1.1) with the minimal wave speed; that is, c_{num} is supposed to give an accurate approximation of $c_{\min}$ (see also [14]). Here Fig. 5.1 depicts the solution $u_j(t)$ at different time t. In this figure, the curves represent $u_j(t)$ and are recorded when $t = 200, 202, 204, \cdots, 224$. As we can see, those curves do manifest one property of the traveling wave, i.e., translation invariance.

We then choose the set of coefficients from Theorems 3.1, 3.5, 3.7 and 3.8 respectively to show the examples of pulled ($c_{\min} = c_0$) and pushed ($c_{\min} > c_0$) wavefronts. The results are shown in Table 5.1. It is not hard to verify that the coefficients chosen in Table 5.1 satisfy the conditions of each theorem. Let's take Theorem 3.1 as an example, i.e., the first row of data in Table 5.1. When $k = 0.1$, then $c_0 = 1.9608$ and $\bar{\mu} = 0.8074$; thus $\tau = 0.8008$ and $\chi = 0.1089$, which give $2\tau - \chi\tau^2 = 1.5317$. Clearly, our choice $hk = 0.2 < 1.5317$ and $1 + \frac{1-k}{\tau} = 2.124 > d$. In summary, (3.1) holds true under the choice of the first row of data in Table 5.1. The last column of Table 5.1 shows the error between our computed speed c_{num} and the theoretically obtained c_0. As the table shows, the first three groups of data demonstrate the linear selection since the relative error is as small as $O(10^{-4})$; this means that the numerical simulations agree with the theoretical results from Theorems 3.1, 3.5, and 3.7, respectively. For an example of a pushed wave, we take the coefficients satisfying Theorem 3.8. The result is shown in the last row of Table 5.1. As we can see, the numerically computed speed c_{num} is greater than the linear speed c_0. Actually, the coefficients chosen in this case imply that $hk = 1.6 > 1$, and we will use this set of parameters to do an extensive example to demonstrate Conjecture 1.1 later.

From Section 4, we have studied the speed selection mechanism in terms of h when other parameters are fixed; that is, there exists a critical number h_c such that (1.6) is linearly selected if $1 < h \le h_c$ and nonlinearly selected if $h > h_c$ (see details in Theorem 4.2). We also gave some

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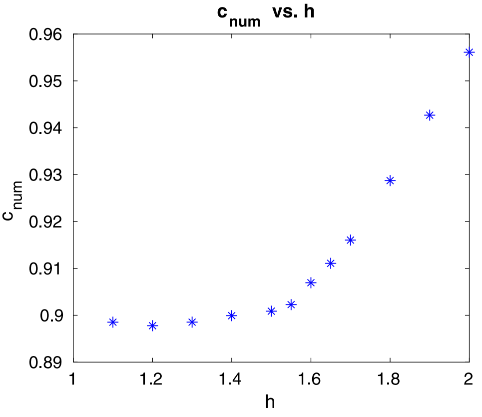

Fig. 5.2. (Color online) Linear and nonlinear selections of the minimal speed for varied values of h. The figure is depicted when $k = 0.8$, $d = 0.5$, and $r = 4$.

Table 5.2
The differences between the linear speed and the minimal speed. The table is obtained when $h = 1.5$ and $k = 0.5$.

r	$d = 0$		$d = 3$	
	c_{num}	$c_0 - c_{\text{num}}$	c_{num}	$c_0 - c_{\text{num}}$
0.1	1.4373	0.0044	1.4419	−0.0002
0.5	1.4379	0.0038	1.4382	0.0034
1.0	1.4394	0.0023	1.4368	0.0049
2.0	1.4408	0.0009	1.4401	0.0016
10.0	1.4433	−0.0016	1.4407	0.001
20.0	1.4404	0.0013	1.4396	0.0021

estimations on the value of h_c, see Theorems 4.4-4.6. Now, we apply the same numerical method on (1.1), but with fixed d, r, k and varied h. The result is shown in Fig. 5.2. The figure is drawn when $k = 0.8$, $d = 0.5$, and $r = 4$. Under this choice of coefficients, we find a lower bound of h_c as $\underline{h}_c = 1.3428$ by Theorem 4.4, and an upper bound of h_c as $\bar{h}_c = 1.9616$ by Theorem 4.6 (since $0 < \tau = 0.1939 < -1 + \sqrt{2}$). From the numerical simulation, see Fig. 5.2, we find that $h_c \simeq 1.55$, which is indeed in the interval $[\underline{h}_c, \bar{h}_c]$; thus, this is an example which demonstrates the results from Section 4.

Then we proceed to numerically verify Conjecture 1.1. For the first part, i.e., the linear selection conjecture, we fix $h = 1.5$ and $k = 0.5$, which clearly give $hk = 0.75 < 1$. Thus, we expect that, for all $r \geq 0$, the system (1.1) will be linearly selected. We take two values of d to perform the simulations, and the results are shown in Table 5.2. Due to the error induced by the scheme and Matlab itself, we are not able to obtain exactly $c_{\text{num}} = c_0$, but their difference is relatively as small as $O(10^{-3})$. As we can see, even when r is as large as 20, we still get that the speed is almost c_0. Thus, our simulations have demonstrated the linear selection part of the conjecture.

The last example is to demonstrate the existence of "r_c" when $hk > 1$. As aforementioned, in the example for Theorem 3.8, when $r = 4$, we do have a traveling wave whose minimal speed $c_{\min} \simeq c_{\text{num}} > c_0$. By extending that example, we fix $k = 0.8$, $h = 2$, $d = 0.5$, and vary the value of r, and compute c_{num} for each r. The results are shown in Fig. 5.3. As the picture shows, c is an increasing function with respect to r when $r > r_c$ and $c \simeq c_0$ when $r \leq r_c$. Moreover, it can

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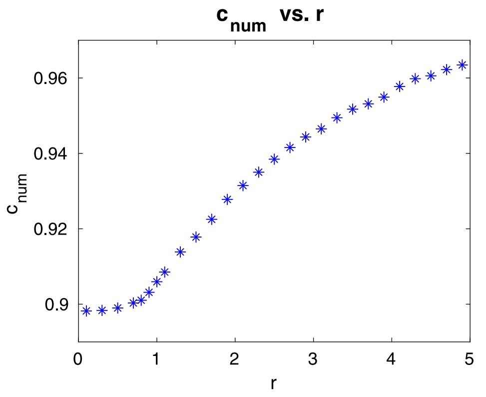

Fig. 5.3. (Color online) Linear and nonlinear selections of the minimal speed for varied values of r. The figure is depicted when $k = 0.8$, $d = 0.5$, and $h = 2$.

be seen that in our chosen case, $r_c \simeq 1$. This example combined with Table 5.2 seems to have numerically verified Conjecture 1.1.

6. Conclusion

By making use of the upper-lower solution method, we investigate the speed selection mechanism for traveling waves to the lattice Lotka-Volterra system, which can be regarded as a discrete version of the classical diffusive Lotka-Volterra competition system. New results on the linear and nonlinear speed selection are established, aiming to contribute an answer toward the new conjecture raised here for the lattice system.

For the linear speed selection, the new conditions on the parameters r, d, k, h (see Theorems 3.1, 3.5 and 3.7) extend the previous results in [11]. More precisely, we want to emphasize the following comparisons between our results and those in [11]:

(1). The comparison between A_1 in [11] and (3.1). When choosing $k = 0.1$, by the software Matlab, we can calculate $c_0 \approx 1.9608$ and $\mu_1(c_0) \approx 0.8674$. Moreover $\tau = e^{0.8674} + e^{-0.8674} - 2 \approx 0.8008$. Therefore, $2\tau - \chi\tau^2 \approx 1.5318 > 1$. This means our result is better that A_1. It is worth mentioning that, in this case, $1 + \frac{1-k}{\tau} \approx 2.1239 > 2$.

(2). The comparison between A_2 and (3.6). One can easily find that the range of d given in condition (3.6) is larger than the result in A_2. In this sense, we provide a more general result concerning the linear speed selection. Even in the region $d \in (0, 1]$, our range for r is larger than that in A_2. In fact, by taking $k = 0.1, h = 20, d = 0.5$, we can calculate the upper bound in A_2, which is given by $\frac{1-k}{hk-1} = 0.9$; while for our result in (3.6), the upper bound is approximated by $\frac{L[(1-d)\tau+(1-k)]}{h-L} \approx 4.2545 > 0.9$.

(3). By our notations, the number d^* appeared in A_3 is equal to $1 + \frac{1-k}{\tau}$. From Theorem 3.7, we have extended the range of d to a larger upper bound $d^* + \frac{r}{\tau}$, which is not contained in [11].

Most importantly, for the nonlinear speed selection, we have established a crucial Theorem 2.10, by which we successfully obtained novel results on this direction (see Theorem 3.8) that has never been studied historically.

Finally, the existence of a critical value of h, denoted by h_c, is proved by a comparison lemma coupled with the upper-lower solution method. Several estimations of h_c for the linear/nonlinear speed selection are provided, see Theorems 4.4-4.6.

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Our numerical simulations demonstrate the realizations of the linear and nonlinear speed selections. Furthermore, we seem to numerically confirm the conjecture raised in our paper.

References

[1] A. Alhasanat, C. Ou, On a conjecture raised by Yuzo Hosono, J. Dyn. Diff. Equat. 4 (2018) 1–18.

[2] A. Alhasanat, C. Ou, Minimal-speed selection of traveling waves to the Lotka-Volterra competition model, J. Differential Equations 266 (2019) 7357–7378.

[3] B. Shorrocks, I.R. Swingland, Living in a Patch Environment, Oxford Univ. Press, New York, 1990.

[4] S.N. Chow, Lattice dynamical systems, in: Dynamical Systems, in: Lecture Notes in Math., vol. 1822, Springer, Berlin, 2003, pp. 1–102.

[5] C. Conley, R. Gardner, An application of the generalized Morse index to travelling wave solutions of a competitive reaction-diffusion model, Indiana Univ. Math. J. 33 (1984) 319–343.

[6] O. Diekmann, Thresholds and traveling waves for the geographical spread of infection, J. Math. Biol. 6 (1979) 109–130.

[7] J. Fang, X. Zhao, Traveling waves for monotone semiflows with weak compactness, SIAM J. Math. Anal. 46 (6) (2014) 3678–3704.

[8] J. Fang, X. Zhao, Bistable traveling waves for monotone semiflows with applications, J. Eur. Math. Soc. 17 (2015) 2243–2288.

[9] R.A. Gardner, Existence and stability of travelling wave solutions of competition models: a degree theoretic approach, J. Differential Equations 44 (1982) 343–364.

[10] J.-S. Guo, C.-H. Wu, Traveling wave front for a two-component lattice dynamical system arising in competition models, J. Differential Equations 252 (2012) 4357–4391.

[11] J.-S. Guo, X. Liang, The minimal speed of traveling fronts for the Lotka-Volterra competition system, J. Dyn. Diff. Equat. 23 (2011) 353–363.

[12] Y. Hosono, Singular perturbation analysis of traveling waves for diffusive Lotka-Volterra competing models, Numer. Appl. Math. 2 (1989) 687–692.

[13] Y. Hosono, Traveling waves for diffusive Lotka-Volterra competition model ii: a geometric approach, Forma 10 (1995) 235–257.

[14] Y. Hosono, The minimal speed of traveling fronts for diffusive Lotka-Volterra competition model, Bull. Math. Biol. 60 (1998) 435–448.

[15] W. Huang, Problem on minimum wave speed for Lotka-Volterra reaction-diffusion competition model, J. Dyn. Diff. Equ. 22 (2010) 285–297.

[16] Y. Kan-on, Parameter dependence of propagation speed of travelling waves for competition-diffusion equations, SIAM J. Math. Anal. 26 (1995) 340–363.

[17] Y. Kan-on, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, Nonlinear Anal 28 (1997) 145–164.

[18] M.A. Lewis, B. Li, H.F. Weinberger, Spreading speed and linear determinacy for two-species competition models, J. Math. Biol. 45 (2002) 219–233.

[19] B. Li, H.F. Weinberger, M.A. Lewis, Spreading speeds as slowest wave speeds for cooperative systems, Math. Biosci. 196 (2005) 82–98.

[20] J. Mallet-Paret, Traveling Waves in Spatially Discrete Dynamical Systems of Diffusive Type. Dynamical Systems, Lecture Notes in Math., vol. 1822, Springer, Berlin, 2003, pp. 231–298.

[21] A. Okubo, P.K. Maini, M.H. Williamson, J.D. Murray, On the spatial spread of the grey squirrel in Britain, Proc. R. Soc. Lond. Ser. B, Biol. Sci. 238 (1989) 113–125.

[22] E. Renshaw, Modelling Biological Populations in Space and Time, Cambridge University Press, Cambridge, 1991.

[23] M. Tang, P. Fife, Propagating fronts for competing species equations with diffusion, Arch. Rational Mech. Anal. 73 (1980) 69–77.

[24] J.H. van Vuuren, The existence of travelling plane waves in a general class of competition-diffusion systems, IMA J. Appl. Math. 55 (1995) 135–148.

[25] H.F. Weinberger, M.A. Lewis, B. Li, Analysis of linear determinacy for spread in cooperative models, J. Math. Biol. 45 (2002) 183–218.

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