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Propagation Direction of the Traveling Wave for the Lotka–Volterra Competitive Lattice System

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Abstract
In this paper, the speed sign of the traveling wave to the bistable Lotka–Volterra competitive lattice system is investigated via the upper–lower solution method as well as the comparison principle. We provide an interval estimation for the bistable speed firstly. Two comparison principles are further established to obtain new conditions to the determinacy of the sign of the bistable speed. To our knowledge, this is the first investigation to the lattice system for the propagation direction.

Keywords Bistable lattice system · Lotka–Volterra · Traveling wave · Speed sign

Mathematics Subject Classification Primary 35K57 · 35B20 · 92D25

1 Introduction

In this paper, we study the speed sign of the bistable traveling wave for the lattice Lotka–Volterra competition system

$$\begin{cases} u_j'(t) = \mathcal{D}_2[u_j](t) + u_j(t)[1 - u_j(t) - kv_j(t)], \\ v_j'(t) = d\mathcal{D}_2[v_j](t) + rv_j(t)[1 - v_j(t) - hu_j(t)], \quad t \in R, \quad j \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $\mathcal{D}_2$ represents the second-order centre difference operator, i.e., $\mathcal{D}_2[u_j](t) = u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)$ and $\mathcal{D}_2[v_j](t) = v_{j+1}(t) + v_{j-1}(t) - 2v_j(t)$. Here, $u_j(t)$ and $v_j(t)$ represent

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the population densities of two species at niches j and time t respectively; d is the diffusion coefficient; r is the net birth rate; k, h are the competition coefficients. System (1.1) can be directly modeled for two species in a patchy environment in the competition of a common resource (see [3,10,21,23–25,27]), or it can be viewed as a discrete version of the classical Lotka–Volterra competition-diffusion system

$$\begin{cases} u_t = u_{xx} + u(1 - u - kv), \\ v_t = dv_{xx} + rv(1 - v - hu), \quad x \in R, \quad t \in R, \end{cases} \tag{1.2}$$

that has been studied in [1,2,4–7,11,13–15,17–20,26,28–30]. However, lattice dynamic system may have advantages over the continuous one in the applications in material science, image processing, pattern formation (see e.g., [3,23–25]). This motivates us to study the dynamic behavior for the system (1.1).

In this paper we assume that k and h satisfy the so-called bistable condition (or the case with strong competition)

$$k > 1 \text{ and } h > 1. \tag{1.3}$$

For further biological interpretation of this condition, we refer to [9,12,31]. For the lattice problem (1.1) in the bistable case, the existence of traveling wave was derived in [31]. We refer readers to [9] for the monotonicity and uniqueness of traveling wave solution as well as the occurrence of propagation failure phenomenon. The stability of the traveling wave was implied by the result in [12], where the bistable traveling wave for a 3-species lattice dynamic system has been studied. Despite all the success in the investigation of the existence, uniqueness and stability of the wavefront, the propagation direction of the wave has remained open for long time and it leaves a real challenge for the community of mathematical biology group. Our purpose of this paper is on this direction.

For our purpose, we first utilize a transformation $\phi_j(t) = u_j(t)$ and $\psi_j(t) = 1 - v_j(t)$ to change (1.1) into the following cooperative system

$$\begin{cases} \phi_j'(t) = \mathcal{D}_2[\phi_j](t) + \phi_j(t)[1 - k - \phi_j(t) + k\psi_j(t)], \\ \psi_j'(t) = d\mathcal{D}_2[\psi_j](t) + r[1 - \psi_j(t)][h\phi_j(t) - \psi_j(t)], \quad t \in R, \quad j \in \mathbb{Z}. \end{cases} \tag{1.4}$$

Under the condition (1.3), system (1.4) has four constant equilibria. For convenience, we denote them by

$$\mathbf{o} = (0, 0), \quad \alpha_1 = (0, 1), \quad \alpha_2 = \left(\frac{k-1}{kh-1}, \frac{h(k-1)}{kh-1}\right) \text{ and } \beta = (1, 1).$$

Through a phase plane analysis, it is easy to find that the equilibrium solutions $\mathbf{o}$ and β are stable, while the equilibrium solutions α_1 and α_2 are unstable for the following system

$$\begin{cases} u_t = u(1 - u - kv), \\ v_t = rv(1 - v - hu), \quad t \in R. \end{cases}$$

As mentioned previously, we are interested in the traveling wavefront solution to (1.4) in the form of

$$(\phi_j(t), \psi_j(t)) = (\Phi(z), \Psi(z)), \quad z = j + ct, \tag{1.5}$$

which connects $\mathbf{o}$ and β. Here, c is the wave speed. By plugging (1.5) into (1.4), we arrive at

$$\begin{cases} \mathcal{D}_2[\Phi] - c\Phi' + \Phi(1 - k - \Phi + k\Psi) = 0, \\ d\mathcal{D}_2[\Psi] - c\Psi' + r(1 - \Psi)(h\Phi - \Psi) = 0, \\ (\Phi, \Psi)(-\infty) = (0, 0), \quad (\Phi, \Psi)(\infty) = (1, 1). \end{cases} \tag{1.6}$$

Here, for the sake of simplicity, we will name the first equation of (1.6) as the Φ-equation, and the second one as the Ψ-equation.

For the bistable nonlinearity, the existence of a traveling solution to system (1.6) has been proved in [31] (Theorem 1.1 and Theorem 1.2) and can be stated as follows.

Lemma 1.1 *If the assumption* (1.3) *holds, then there exists a solution* $(c, \Phi(z), \Psi(z))$ *of* (1.6) *such that* $(\Phi, \Psi)(z)$ *is nondecreasing.*

The sign of the wave speed for the continuous system (1.2) has been studied in [8,16], but for the lattice system (1.1), to our knowledge, this could be the first work concerning this topic due to the fact that the second order centre-difference operator $\mathcal{D}_2$ appeared in (1.6) causes nontrivial difficulties. Technically, we will come up with a new method that is completely different from those in [8,16]. Based on the system parameters, suitable upper or lower solutions will be constructed to approximate the exact bistable wave. By applying the comparison principle, we will establish several novel results that can directly determine the sign of the wave speed.

The rest of this paper is arranged as follows. In Sect. 2, an interval estimate of the bistable-wave speed is established. In Sect. 3, two crucial comparison results on the sign of wave speed are proved. In Sect. 4, we focus on the establishment of explicit conditions for the direct determinacy of the speed sign. Finally, we will give a short discussion in Sect. 5.

2 Estimation of the speed of the bistable wave

In this section, we shall provide an estimation for the bistable-wave speed. For simplicity, we use $\mathcal{C}$ to denote the set of all continuous and bounded functions from R to R, and use $[\phi, \psi]_{\mathcal{C}}$ to denote the set $\{\varphi \in \mathcal{C} : \phi \leq \varphi \leq \psi\}$. It is easy to see that $\alpha_i \in [\mathbf{o}, \beta]$. When the initial phase space is restricted in $[\alpha_i, \beta]$ or $[\mathbf{o}, \alpha_i]$, $i = 1, 2$, the system (1.6) reveals a monostable property. Due to this property, it has been proved in Theorem 3.5 of [6] (also can be found in [20]) that there exists a positive constant $C^*_-(\alpha_i, \beta)$ (the left-ward spreading speed) such that when $c \geq C^*_-(\alpha_i, \beta)$, (1.6) has a nonnegative monotone traveling wavefront, which satisfies

$$(\Phi, \Psi)(-\infty) = \alpha_i, \quad (\Phi, \Psi)(\infty) = \beta.$$

Meanwhile, there also exists another positive constant $C^*_+(\mathbf{o}, \alpha_i)$(the right-ward spreading speed) such that when $c \leq -C^*_+(\mathbf{o}, \alpha_i)$, (1.6) has a nonnegative monotone traveling wavefront, which satisfies

$$(\Phi, \Psi)(-\infty) = \mathbf{o}, \quad (\Phi, \Psi)(\infty) = \alpha_i.$$

For the existence of the bistable wave, we refer to Lemma 1.1. Now, we are going to establish an estimation for the speed of the bistable traveling wave, in which the left-ward and right-ward spreading speed are involved.

Theorem 2.1 *If the speed of the bistable traveling wave solution of* (1.6) *is denoted by c, then we have*

$$-C_+^*(\boldsymbol{o}, \alpha_i) \le c \le C_-^*(\alpha_i, \beta), \quad i = 1, 2. \tag{2.1}$$

Particularly, when $i = 1$*, we have*

$$-\theta := -\min_{\mu>0}\left\{\frac{d(e^{\mu}+e^{-\mu}-2)+r}{\mu}\right\} \le c \le \min_{\mu>0}\left\{\frac{e^{\mu}+e^{-\mu}-1}{\mu}\right\} := \omega. \tag{2.2}$$

Proof We shall only prove the right inequality of (2.1) when $i = 1$, since the left inequality can be dealt with in a similar manner. To this end, in (1.4), we can choose a pair of non-decreasing functions (in j) $(\phi_j, \psi_j)(0)$ as the initial function which satisfies

$$(\phi_j, \psi_j)(0) = \begin{cases} \beta, & j \ge 1, \\ \alpha_1, & j \le 0. \end{cases} \tag{2.3}$$

On the other hand, we denote the bistable wave solution by $(\Phi, \Psi)(j+ct)$, which is the exact solution to (1.4) with the initial function $(\Phi, \Psi)(j)$. Obviously, by shifting if it is necessary, one can always suppose that

$$(\phi_j, \psi_j)(0) \ge (\Phi, \Psi)(j).$$

By the comparison principle, we conclude that

$$\phi_j(t) \ge \Phi(j+ct), \quad \psi_j(t) \ge \Psi(j+ct).$$

In addition, it is known that, for any monostable system, the asymptotic spreading speed with the initial condition as (2.3) eventually approaches the minimal wave speed $C_-^*(\alpha_1, \beta)$. Thus we claim that $c \le C_-^*(\alpha_1, \beta)$. To the contrary, if $c > C_-^*(\alpha_1, \beta)$, then at the line $j + ct = z_0$ such that $0 < \Phi(z_0) < 1$, we have

$$\phi_j(t) \ge \Phi(j+ct). \tag{2.4}$$

By letting $t \to \infty$ and noticing $\lim_{t\to\infty,\ j<-ct} \phi_j(t) = 0$ which follows from the definition of the spreading speed, the above equation (2.4) gives a contradiction. Hence, we know that $c \le C_-^*(\alpha_1, \beta)$.

To get the explicit expression for $C_-^*(\alpha_1, \beta)$, we let (Φ, Ψ) be a traveling wave solution to the system of (1.6), which connects α_1 and β. This implies $\Psi \equiv 1$, and the Φ-equation then can be changed into

$$\mathcal{D}_2[\Phi] - c\Phi' + \Phi(1-\Phi) = 0,$$

with

$$\Phi(-\infty) = 0, \quad \Phi(\infty) = 1.$$

From [6], it is well-known that

$$C_-^*(\alpha_1, \beta) = \min_{\mu>0}\left\{\frac{e^{\mu}+e^{-\mu}-1}{\mu}\right\}. \tag{2.5}$$

To get the explicit expression for $C_+^*(\mathbf{o}, \alpha_1)$, we assume (Φ, Ψ) is a traveling wave solution of (1.6), which connects $\mathbf{o}$ and α_1. This implies $\Phi \equiv 0$, and the Ψ-equation then becomes

$$d\mathcal{D}_2[\Psi] - c\Psi' - r\Psi(1-\Psi) = 0, \tag{2.6}$$

with

$$\Psi(-\infty) = 0, \quad \Psi(\infty) = 1.$$

By the transformation $\Psi = 1 - W$, (2.6) turns to be

$$d\mathcal{D}_2[W] - cW' + rW(1 - W) = 0,$$

with $W(-\infty) = 1$, $W(\infty) = 0$. Again from [6], it follows that

$$C_+^*(\mathbf{o}, \alpha_1) = \min_{\mu>0} \left\{ \frac{d(e^{\mu} + e^{-\mu} - 2) + r}{\mu} \right\}. \tag{2.7}$$

By (2.5) and (2.7), the proof is complete. □

3 Sign of the speed of the bistable wave

In this section, we study the sign of the speed of the bistable wave solution. The method of upper and lower solution will be used in the derivation, so we first give the definition of upper and lower solutions to (1.6).

Definition 3.1 If a pair of functions $(\Phi(z), \Psi(z))$ is continuous, and differentiable on R except at finite number of points z_i, $i = 1, 2, \ldots, n$, and satisfies

$$\begin{cases} \mathcal{D}_2[\Phi] - c\Phi' + \Phi(1 - k - \Phi + k\Psi) \le 0, \\ d\mathcal{D}_2[\Psi] - c\Psi' + r(1 - \Psi)(h\Phi - \Psi) \le 0, \\ (\Phi, \Psi)(-\infty) \ge (0, 0), \quad (\Phi, \Psi)(\infty) \ge (1, 1), \end{cases}$$

for all $z \ne z_i$, and $(\Phi'_-(z_i), \Psi'_-(z_i)) \ge (\Phi'_+(z_i), \Psi'_+(z_i))$ for all i, then we say that $(\Phi(z), \Psi(z))$ is an upper solution to (1.6).

Reversing all the inequality signs gives the definition of a lower solution.

The next lemma plays a critical role in reducing the coupled system (1.6) to a scalar nonlocal equation by solving abstractly the Φ-equation or the Ψ-equation.

Lemma 3.2 *Assume that c satisfies $-\theta < c < \omega$, where θ and ω are defined in (2.2). Then we have*

(1) *For any given continuous and non-decreasing function $\Phi(z)$, with $\Phi(-\infty) = 0$ and $\Phi(\infty) = a > 0$, there exists a non-decreasing function $\Psi(z)$ satisfying*

$$\begin{cases} d\mathcal{D}_2[\Psi] - c\Psi' + r(1 - \Psi)(h\Phi - \Psi) = 0 \\ \Psi(-\infty) = 0, \quad \Psi(\infty) = \min\{1, ha\}. \end{cases} \tag{3.1}$$

(2) *For any given continuous and non-decreasing function $\Psi(z)$, with $\Psi(-\infty) = 0$ and $\Psi(\infty) = 1$, there exists a non-decreasing function $\Phi(z)$ satisfying*

$$\begin{cases} \mathcal{D}_2[\Phi] - c\Phi' + \Phi(1 - k - \Phi + k\Psi) = 0 \\ \Phi(-\infty) = 0, \quad \Phi(\infty) = 1. \end{cases} \tag{3.2}$$

Proof Let $w(z) = 1 - \Psi(z)$ and $a(z) = 1 - h\Phi(z)$, then (3.1) can be rewritten as

$$\begin{cases} d\mathcal{D}_2[w](z) - cw'(z) + rw(z)(a(z) - w(z)) = 0, \\ w(-\infty) = 1, \quad w(\infty) = 1 - \min\{1, ha\}, \end{cases} \tag{3.3}$$

with $a(-\infty) = 1$ and $a(\infty) = 1 - ha$. By employing the upper–lower solution method, one can prove the existence of $w(z)$, which in turn gives the existence of Ψ. Indeed, it is easy to verify that $\overline{w}(z) = 1$ is an upper solution to (3.3).

As for the lower solution, we need to distinguish with two cases.

Case 1 If $a \le 1/h$, then it is readily to see that the function $\underline{w}(z) = 1 - ha$ can be served as a lower solution.

Case 2 If $a > 1/h$, we can assume, for sufficient small number $\epsilon > 0$, that there exists a number $z_0 \in R$ so that $a(z) \ge 1 - \epsilon$ when $z \le z_0$ since $a(-\infty) = 1$. To construct a lower solution, we first focus on the bistable wave profile equation

$$d\mathcal{D}_2[\hat{w}(z)] + \hat{c}\hat{w}'(z) + f(\hat{w}(z)) = 0, \tag{3.4}$$

where $f(\hat{w}(z))$ is a function defined as

$$f(\hat{w}(z)) = \begin{cases} r\hat{w}(-\epsilon - \hat{w}), & \hat{w} < 0, \\ r\hat{w}(1 - \epsilon - \hat{w}), & \hat{w} \ge 0. \end{cases}$$

Here, $0 < \epsilon \ll 1$ is a constant. Obviously, the system (3.4) has three equilibria $\hat{w} = 0, -\epsilon, 1 - \epsilon$. It has been proved in [6] that, there exists a speed $\hat{c} = c_\epsilon$ such that the system (3.4) has a decreasing solution satisfying

$$\hat{w}(-\infty) = 1 - \epsilon, \quad \hat{w}(\infty) = -\epsilon.$$

Letting $\epsilon \to 0$, we consider a limiting system of (3.4) as

$$d\mathcal{D}_2[\tilde{w}(z)] + \tilde{c}\tilde{w}'(z) + r\tilde{w}(z)(1 - \tilde{w}(z)) = 0. \tag{3.5}$$

It is known that equation (3.5) possesses a monotone traveling solution with

$$\tilde{c} = \min_{\mu > 0} \left\{ \frac{d(e^{\mu} + e^{-\mu} - 2) + r}{\mu} \right\} = \theta > 0.$$

Thus it is easy to verify

$$\lim_{\epsilon \to 0} \hat{c} = \theta.$$

Without loss of generality, we may assume that $\hat{w}(z) \ge 0$ as $z \le z_0$, and $\hat{w}(z) < 0$ as $z > z_0$ since any translation of $\hat{w}(z)$ is still a solution. We now claim that

$$\underline{w}(z) = \max\{0, \hat{w}(z)\}$$

is a lower solution to (3.3) as long as ϵ is small. In fact, when $\underline{w}(z) = 0$ the proof is trivial. For the case when $\underline{w}(z) = \hat{w}(z)$, we substitute it into (3.3) to obtain

$$\begin{aligned} &d\mathcal{D}_2[\underline{w}](z) - c\underline{w}'(z) + r\underline{w}(z)(a(z) - \underline{w}(z)) \\ &= -(c + \hat{c})\hat{w}'(z) + r\hat{w}(z)(a(z) - (1 - \epsilon)). \end{aligned} \tag{3.6}$$

Let $c = -\theta + \delta$ for some positive δ. We can choose ϵ sufficiently small so that $\hat{c} - \theta < \delta$ and $c + \hat{c} > 0$. The value of (3.6) is nonnegative when $z \le z_0$. Therefore, our proof of part one is complete.

For the second part, by introducing the transformation $w(z) = \Phi(-z)$, then (3.2) reduces to

$$\begin{cases} \mathcal{D}_2[w] + cw' + w(1 - k + k\Psi - w) = 0, \\ w(-\infty) = 1, \quad \Phi(\infty) = 0. \end{cases}$$

By virtue of the same analysis as in part one, one can accomplish the proof. We omit the details here. Thus the proof is complete. □

Now, we are going to construct a lower solution to the system (1.6) which is crucial to establish our main results. To begin with, we provide some notations. Denote

$$f(u, v) = u(1 - k - u + kv), \quad g(u, v) = r(1 - v)(hu - v).$$

The Jacobian matrix of (f, g) is given by

$$J = \begin{pmatrix} 1 - k - 2u + kv & ku \\ rh(1 - v) & -r(hu + 1 - 2v) \end{pmatrix}. \tag{3.7}$$

This matrix evaluated at (0, 0) and (1, 1) are given respectively by

$$J_0 = \begin{pmatrix} 1 - k & 0 \\ rh & -r \end{pmatrix}, \quad J_1 = \begin{pmatrix} -1 & k \\ 0 & -r(h - 1) \end{pmatrix}. \tag{3.8}$$

Based on (3.7) and (3.8), we further set

$$J^- = \begin{pmatrix} 1 - k + k\nu & k\nu \\ rh & r(2\nu - 1) \end{pmatrix}, \quad J^+ = \begin{pmatrix} -1 + 2\nu & k \\ rh\nu & r(1 - h + h\nu) \end{pmatrix},$$

where ν is a small positive constant. For sufficiently small ν, a direct calculation shows that $J^\pm$ has two negative eigenvalues $\lambda_1^\pm, \lambda_2^\pm$. If the principal eigenvalues of $J^\pm$ are denoted by $\lambda^\pm$, then it can be seen that

$$\lambda^\pm = \max\{\lambda_1^\pm, \lambda_2^\pm\} < 0,$$

with the corresponding eigenfunctions $\phi^\pm = (\phi_1^\pm, \phi_2^\pm)$ are positive. Hence we have

$$J^\pm \phi^\pm = \lambda^\pm \phi^\pm.$$

In addition, for small number $\nu > 0$, we have

$$\begin{aligned} J_{ij} &< J_{ij}^-, \quad \text{for} \quad 0 < u, \quad v \le \nu, \\ J_{ij} &< J_{ij}^+, \quad \text{for} \quad 0 < 1 - u, \quad 1 - v \le \nu. \end{aligned}$$

Since $(\Phi, \Psi)(-\infty) \to (0, 0)$, $(\Phi, \Psi)(\infty) \to (1, 1)$, there exists a number $R_1 = R_1(\nu) > 0$ such that

$$\begin{aligned} 0 < \Phi(z), \quad \Psi(z) < \nu, \ \text{for} \ z \le -R_1, \\ 1 - \nu < \Phi(z), \quad \Psi(z) < 1, \ \text{for} \ z \ge R_1. \end{aligned} \tag{3.9}$$

Let $\rho_i(z)$, $i = 1, 2$ be smooth positive functions satisfying

$$\begin{aligned} (\rho_1(z), \rho_2(z)) \to p\phi^-, \ \text{as} \ z \to -\infty, \\ (\rho_1(z), \rho_2(z)) \to q\phi^+, \ \text{as} \ z \to \infty. \end{aligned} \tag{3.10}$$

Here, we choose $p, q > 0$ satisfying $0 < q\phi^+ < p\phi^- \le 1$. As a result, we can make an assumption that $\rho_1(z), \rho_2(z)$ are also monotone decreasing functions satisfying $0 < \rho_i(z) \le 1$, $|\rho_i'(z)| \le 1$ with $i = 1, 2$ and $z \in R$.

Denote

$$\kappa = \min\left\{ \min_{|\xi| \le R_0} \Phi'(\xi), \ \min_{|\xi| \le R_0} \Psi'(\xi) \right\} > 0, \quad M = \max\{k, rh\}.$$

and

$$\delta_0 = \min\left\{\frac{\beta\kappa}{|c| + \beta + 2 + 2M}, \frac{\beta\kappa}{|c| + \beta + 2d + 2M}\right\}.$$

Remark 3.3 The positivity of κ has been proved in Theorem 3 of [10].

With the above notations and analysis, we are able to give the following lemma, by developing the idea in [12].

Lemma 3.4 *The following pair of functions*

$$\begin{cases} u_j^-(t) = \Phi(j + ct + \xi^- - \sigma(1 - e^{-\beta t})) - \sigma\delta e^{-\beta t}\rho_1(j + ct + \xi^-), \\ v_j^-(t) = \Psi(j + ct + \xi^- - \sigma(1 - e^{-\beta t})) - \sigma\delta e^{-\beta t}\rho_2(j + ct + \xi^-), \end{cases} \tag{3.11}$$

is a lower solution to the system (1.4), where $\beta = \frac{1}{2}\min\{-\lambda^+, -\lambda^-\}$, $\delta \in (0, \delta_0)$, $\xi^- \in R$ *and* σ *is a positive number.*

Proof For simplicity, we first set

$$\begin{aligned} N_j(t) &= (u_j^-)'(t) - \mathcal{D}_2[u_j^-](t) - u_j^-(t)[1 - k - u_j^-(t) + kv_j^-(t)], \\ M_j(t) &= (v_j^-)'(t) - d\mathcal{D}_2[v_j^-](t) - r[1 - v_j^-(t)][hu_j^-(t) - v_j^-(t)]. \end{aligned} \tag{3.12}$$

Substituting (3.11) into the equation (3.12) gives

$$\begin{aligned} N_j(t) = {} & \Phi'(\xi)(c - \sigma\beta e^{-\beta t}) - \sigma\delta e^{-\beta t}[c(\rho_1)'(\eta) - \beta\rho_1(\eta)] \\ & - \{\mathcal{D}_2[\Phi](\xi) - \sigma\delta e^{-\beta t}\mathcal{D}_2[\rho_1](\eta)\} + \Phi(1 - k - \Phi + k\Psi) \\ & - \sigma\delta e^{-\beta t}(1 - k - 2\Phi + k\Psi)\rho_1 - \sigma\delta e^{-\beta t}k\Phi\rho_2 + \sigma^2\delta^2 e^{-2\beta t}(-\rho_1 + k\rho_2)\rho_1. \end{aligned}$$

and

$$\begin{aligned} M_j(t) = {} & \Psi'(\xi)(c - \sigma\beta e^{-\beta t}) - \sigma\delta e^{-\beta t}[c(\rho_2)'(\eta) - \beta\rho_2(\eta)] \\ & - d\{\mathcal{D}_2[\Psi](\xi) - \sigma\delta e^{-\beta t}\mathcal{D}_2[\rho_2](\eta)\} + r(1 - \Psi)(h\Phi - \Psi) \\ & - \sigma\delta e^{-\beta t}rh(1 - \Psi)\rho_1 + \sigma\delta e^{-\beta t}r(1 + h\Phi - 2\Psi)\rho_2 - r\sigma^2\delta^2 e^{-2\beta t}(h\rho_1 - \rho_2)\rho_2. \end{aligned}$$

where $\xi = j + ct + \xi^- - \sigma(1 - e^{-\beta t})$ and $\eta = j + ct + \xi^-$. By virtue of the fact that (Φ, Ψ) is the exact solution, we obtain

$$\begin{aligned} N_j(t) &= -\sigma e^{-\beta t}\{\beta\Phi'(\xi) + \delta(I_1 - I_2)\}, \\ M_j(t) &= -\sigma e^{-\beta t}\{\beta\Psi'(\xi) + \delta(I_3 - I_4)\}, \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= c(\rho_1)'(\eta) - \beta\rho_1(\eta) - \mathcal{D}_2[\rho_1](\eta), \\ I_2(t) &= (1 - k - 2\Phi + k\Psi)\rho_1 + k\Phi\rho_2 - \sigma\delta e^{-\beta t}(-\rho_1 + k\rho_2)\rho_1, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I_3(t) &= c(\rho_2)'(\eta) - \beta\rho_2(\eta) - d\mathcal{D}_2[\rho_2](\eta), \\ I_4(t) &= rh(1 - \Psi)\rho_1 - r(1 + h\Phi - 2\Psi)\rho_2 + \sigma\delta e^{-\beta t}r(h\rho_1 - \rho_2)\rho_2. \end{aligned} \tag{3.14}$$

In addition, we let

$$\begin{aligned} &\Omega := \max\{3+k+\beta-2\nu, \beta+h+2d-1, \beta+rh+2d+r-2r\nu\}, \\ &\mu := \frac{\beta q}{|c|+\Omega}\min\{\phi_1^+, \phi_2^+\}. \end{aligned} \tag{3.15}$$

Then it follows from (3.10) that there exists a number $R_2 := R_2(\mu)$ so that

$$|\rho_i(z) - p\phi_i^-| < \mu, \text{ for } z < -R_2; \quad |\rho_i(z) - q\phi_i^+| < \mu, \text{ for } z > R_2, \quad i = 1, 2, \tag{3.16}$$

and

$$|(\rho_i)'(z)| < \mu, \text{ for } |z| > R_2, \quad i = 1, 2. \tag{3.17}$$

By denoting $R_0 := \max\{R_1 + \sigma, R_2 + 1\}$, we shall divide the discussions into three cases: (i) $\eta > R_0$; (ii) $\eta < -R_0$; (iii) $-R_0 \le \eta \le R_0$.

Case (i) If $\eta > R_0 \ge R_2 + 1$, bearing (3.16) and (3.17) in mind, then we have

$$\begin{aligned} I_1 &\ge -\beta q\phi_1^+ - (\beta + |c| + 2)\mu, \\ I_3 &\ge -\beta q\phi_2^+ - (\beta + |c| + 2d)\mu. \end{aligned} \tag{3.18}$$

For sufficiently small δ, it can be seen that the last terms in I_2, I_4 are dominated by the previous two terms, see (3.13) and (3.14). On the other hand, since $\eta > R_0 \ge R_1 + \sigma$, then we conclude that $\xi = \eta - \sigma(1 - e^{-\beta t}) > R_1$. Therefore, by (3.9), we obtain

$$1 - \nu < \Phi(\xi), \Psi(\xi) < 1, \text{ for } \xi > R_1.$$

As a result, I_2, I_4 can be estimated by

$$\begin{aligned} I_2(t) &\le (-1 + 2\nu)\rho_1 + k\rho_2, \\ I_4(t) &\le rh\nu\rho_1 + r(1 - h + h\nu)\rho_2. \end{aligned} \tag{3.19}$$

Furthermore, by (3.16), we get

$$\begin{aligned} I_2(t) &\le (-1 + 2\nu)\rho_1 + k\rho_2 \\ &\le q((-1 + 2\nu)\phi_1^+ + k\phi_2^+) + \mu(1 - 2\nu + k) \\ &= q\lambda^+\phi_1^+ + \mu(1 - 2\nu + k) \end{aligned}$$

and

$$\begin{aligned} I_4(t) &\le rh\nu\rho_1 + r(1 - h + h\nu)\rho_2 \\ &\le q(rh\nu\phi_1^+ + r(1 - h + h\nu)\phi_2^+) + r(h - 1)\mu \\ &= q\lambda^+\phi_2^+ + r(h - 1)\mu. \end{aligned}$$

Consequently, by the definition of β in the lemma and μ in (3.15), we have

$$\begin{aligned} N_j(t) &\le -\sigma e^{-\beta t}\{\beta\Phi'(\xi) + \delta[(-\beta - \lambda^+)q\phi_1^+ - (|c| + 3 + k + \beta - 2\nu)\mu]\} \\ &< -\sigma\delta e^{-\beta t}\{\beta q\phi_1^+ - (|c| + 3 + k + \beta - 2\nu)\mu\} \\ &\le 0 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} M_j(t) &\le -\sigma e^{-\beta t}\{\beta\Psi'(\xi) + \delta[(-\beta - \lambda^+)p^+\phi_2^+ - (\beta + |c| + 2d)\mu - r(h-1)\mu]\} \\ &< -\sigma\delta e^{-\beta t}\{\beta q\phi_2^+ - (|c| + \beta + h + 2d - 1)\mu\} \\ &\le 0. \end{aligned} \tag{3.21}$$

Case (ii) when $\eta < -R_0$, in a similar way to case (i), we can get

$$\begin{aligned} I_1 &\ge -\beta p\phi_1^- - (\beta + |c| + 2)\mu, \\ I_3 &\ge -\beta p\phi_2^- - (\beta + |c| + 2d)\mu. \end{aligned} \tag{3.22}$$

Moreover, a direct calculation leads to

$$\begin{aligned} I_2(t) &\le (1 - k + k\nu)\rho_1 + k\nu\rho_2 \\ &\le p((1 - k + k\nu)\phi_1^- + k\nu\phi_2^-) + \mu(k-1) \\ &= p\lambda^-\phi_1^- + \mu(k-1) \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} I_4(t) &\le rh\rho_1 + r(2\nu - 1)\rho_2 \\ &\le p(rh\phi_1^- + r(2\nu - 1)\phi_2^-) + r(h + 1 - 2\nu)\mu \\ &= p\lambda^-\phi_2^- + r(h + 1 - 2\nu)\mu. \end{aligned} \tag{3.24}$$

Equations (3.22), (3.23) combined with (3.24) and (3.15) yield

$$\begin{aligned} N_j(t) &\le -\sigma\delta e^{-\beta t}\{\beta p\phi_1^- - (|c| + 1 + k + \beta - 2\nu)\mu\} \le 0, \\ M_j(t) &\le -\sigma\delta e^{-\beta t}\{\beta p\phi_2^- - (|c| + \beta + rh + 2d + r - 2r\nu)\mu\} \le 0. \end{aligned} \tag{3.25}$$

Case (iii) In this case, because of $0 < \rho_1^+ \le 1$ and $|(\rho_1^+)'| \le 1$, we have $I_1 \ge -(|c| + \beta + 2)$ and $I_3 \ge -(|c| + \beta + 2d)$. Additionally, it is easy to see that

$$I_2 \le 2M, \quad M = \max\{k, rh\}.$$

Hence, we have the following two estimations

$$\begin{aligned} N_j(t) &\le -\sigma e^{-\beta t}\{\beta\Phi'(\xi) - \delta(|c| + \beta + 2 + 2M)\} \\ &< -\sigma e^{-\beta t}\{\beta\kappa - \delta(|c| + \beta + 2 + 2M)\}, \\ M_j(t) &\le -\sigma e^{-\beta t}\{\beta\Psi'(\xi) - \delta(|c| + \beta + 2d + 2M)\} \\ &< -\sigma e^{-\beta t}\{\beta\kappa - \delta(|c| + \beta + 2d + 2M)\}. \end{aligned} \tag{3.26}$$

The desired inequality $N_j(t), M_j(t) < 0$ follows directly from the definition of δ. Thus, by (3.20), (3.21), (3.25) and (3.26), we have proved that $N_j(t), M_j(t) \le 0$ on the whole line, which implies that $(u_j^-(t), v_j^-(t))$ is a lower solution. The proof is complete. □

The following two results are concerned with the comparison between the speed of an upper solution (or a lower solution) and c (the speed of the bistable traveling wave solution). To be exact, we have

Theorem 3.5 *If there exists a non-negative and non-decreasing upper solution* $(\overline{\Phi}, \overline{\Psi})(z)$ *with speed* $\overline{c} < 0$ *for the system* (1.6)*, satisfying*

$$(\overline{\Phi}, \overline{\Psi})(-\infty) < (1, 1), \quad (\overline{\Phi}, \overline{\Psi})(\infty) \ge (1, 1), \tag{3.27}$$

then

$$c \leq \overline{c} < 0.$$

Proof For the system (1.4), we can choose a pair of continuous and non-decreasing functions (ϕ, ψ) as the initial function which satisfies

$$(\phi_j, \psi_j)(0) = (0, 0), \text{ for } j < -M; \ (\phi_j, \psi_j)(0) = (1-\delta, 1-\delta), \text{ for } j > M.$$

Here M is a positive number, and $\delta \in (0, 1)$ is a sufficient small number. By shifting if it is necessary, we can always suppose that

$$\phi_j(0) \leq \overline{\Phi}(j), \quad \psi_j(0) \leq \overline{\Psi}(j).$$

By the comparison principle, we have

$$\overline{\Phi}(j + \overline{c}t) \geq \phi_j(t), \quad \overline{\Psi}(j + \overline{c}t)) \geq \psi_j(t), \tag{3.28}$$

for all $t \geq 0$. On the other hand, we have shown in Lemma (3.4) that $(u_j^-(t), v_j^-(t))$ forms a lower solution to the system (1.4). This implies

$$\phi_j(t) \geq \Phi(j + ct + \xi^- - \sigma(1 - e^{-\beta t})) - \sigma\delta e^{-\beta t}\rho_1(j + ct + \xi^-). \tag{3.29}$$

From the assumption (3.27), we know that there exists a number γ such that $\overline{\Phi}(\gamma) < 1$. To the contrary, if $c > \overline{c}$, then on the line $\gamma = j + \overline{c}t$, it follows from (3.28) and (3.29) that

$$\begin{aligned} \overline{\Phi}(\gamma) = \overline{\Phi}(j + \overline{c}t) &\geq \Phi(j + \overline{c}t + (c - \overline{c})t + \xi^- - \sigma(1 - e^{-\beta t})) \\ &\quad - \sigma\delta e^{-\beta t}\rho_1(j + ct + \xi^-) \\ &\geq \Phi(j + \overline{c}t + (c - \overline{c})t + \xi^- - \sigma(1 - e^{-\beta t})) - \sigma\delta e^{-\beta t}, \end{aligned} \tag{3.30}$$

which gives a contradiction as $\overline{\Phi}(\gamma) \geq 1$, for $t \to \infty$. Hence, we have $c \leq \overline{c}$. The proof is complete. □

Analogous to Theorem 3.5, we arrive at a similar theorem below.

Theorem 3.6 *If there exists a non-negative and non-decreasing lower solution* $(\underline{\Phi}, \underline{\Psi})(z)$ *with speed* $\underline{c} > 0$ *for the system* (1.6) *such that*

$$(\underline{\Phi}, \underline{\Psi})(-\infty) = (0, 0) < (\underline{\Phi}, \underline{\Psi})(\infty) \leq (1, 1).$$

Then we have

$$c \geq \underline{c} > 0.$$

For later use, we need to analyze the eigenvalue problem of the system (1.6). Firstly, the characteristic equation of the Φ-equation near $\mathbf{o}$ is given by

$$\Gamma_1(\mu) := (e^{\mu} + e^{-\mu} - 2) - c\mu + (1 - k) = 0.$$

By the concavity of $\Gamma_1(\mu)$ as well as $\Gamma_1(-\infty) > 0$, $\Gamma_1(\infty) > 0$ and $\Gamma_1(0) = 1 - k < 0$, it can be shown that there always exist two zeros for the function $\Gamma_1(\mu)$ for any given c: the positive zero $\mu_1(c)$, and the negative one $\mu_2(c)$.

Additionally, the characteristic equation of the Ψ-equation near β is given by

$$\Gamma_2(\mu) := d(e^{\mu} + e^{-\mu} - 2) + c\mu - r(h - 1) = 0.$$

Similar to the discussion on $\Gamma_1(\mu)$, it is easy to see that the equation $\Gamma_2(\mu) = 0$ has two different roots, which are denoted by $\mu_3(c)$ for the positive one, and by $\mu_4(c)$ for the negative one.

Based on the above analysis, now we proceed to establish some criteria for the determination of the speed sign.

First we define $(\overline{\Phi}, \overline{\Psi})(z)$ as follows. Let

$$\overline{\Phi}(z) = \frac{1}{1 + e^{-\mu_1(-\epsilon)z}}, \text{ with } 0 < \epsilon \ll 1, \tag{3.31}$$

and $\overline{\Psi}(z)$ be a solution to the Ψ-equation, whose existence is ensured by Lemma 3.2 (1).

Corollary 3.7 *The speed c of the bistable traveling wave solution of* (1.6) *satisfies* $c \le -\epsilon < 0$ *provided that*

$$-2(k-1) + kY_1(z) + \frac{(k-1)^2}{k+3+2\sqrt{k+3}} < 0, \ \textit{where } Y_1(z) = \frac{\overline{\Psi} - \overline{\Phi}}{\overline{\Phi}\left(1 - \overline{\Phi}\right)}. \tag{3.32}$$

Proof By substituting the pair of functions $(\overline{\Phi}, \overline{\Psi})(z)$ into the Φ-equation, we have

$$\begin{aligned} &\mathcal{D}_2[\overline{\Phi}] + \epsilon\overline{\Phi}' + \overline{\Phi}(1 - k - \overline{\Phi} + k\overline{\Psi}) \\ &= \overline{\Phi}^2\left(1 - \overline{\Phi}\right)\left[-2(e^{\mu_1} + e^{-\mu_1} - 2) + k\frac{\overline{\Psi} - \overline{\Phi}}{\overline{\Phi}\left(1 - \overline{\Phi}\right)} + (e^{\mu_1} + e^{-\mu_1} - 2)^2 R_1(\overline{\Phi})\right], \end{aligned}$$

where

$$R_1(\overline{\Phi}) := \frac{e^{-\mu_1 z}(1 - e^{-\mu_1 z})}{1 + e^{-\mu_1 z}(e^{\mu_1} + e^{-\mu_1}) + e^{-2\mu_1 z}}. \tag{3.33}$$

In the above derivation, the equality $(e^{\mu_1} + e^{-\mu_1} - 2) + \epsilon\mu_1 + (1-k) = 0$ and the relationship $\overline{\Phi}' = \mu_1\overline{\Phi}(1 - \overline{\Phi})$ have been used. In addition, the maximum of $R_1(\overline{\Phi})$ can be deduced directly and is given by $\chi := \frac{1}{\tau+4+2\sqrt{\tau+4}}$ where $\tau = e^{\mu_1} + e^{-\mu_1} - 2$. Moreover, as $\epsilon \to 0^+$, it can be seen that $e^{\mu_1} + e^{-\mu_1} - 2 \to k - 1$. As ϵ can be taken sufficiently small, (3.32) implies that $(\overline{\Phi}, \overline{\Psi})$ is an upper solution. As a result of Theorem 3.5, the proof is complete. □

Similarly by redefining

$$\overline{\Psi}(z) = \frac{1}{1 + e^{-\mu_3(-\epsilon)z}}, \text{ with } 0 < \epsilon \ll 1,$$

and $\overline{\Phi}(z)$ as the solution to the Φ-equation (see Lemma 3.2 (2)), we have the following corollary.

Corollary 3.8 *The speed c of the bistable traveling wave solution of* (1.6) *satisfies* $c \le -\epsilon < 0$ *provided that*

$$2(h-1) + hY_2(z) + \frac{r(h-1)^2}{d} < 0, \ \textit{where } Y_2(z) = \frac{\overline{\Phi} - \overline{\Psi}}{\overline{\Psi}\left(1 - \overline{\Psi}\right)}. \tag{3.34}$$

Proof We first work on the second-order centre difference operator. In fact, by denoting $\mu_3(-\epsilon)$ by μ_3 for short, we have

$$\begin{aligned}\mathcal{D}_2[\overline{\Psi}] &= \overline{\Psi}(z+1) + \overline{\Psi}(z+1) - 2\overline{\Psi}(z) \\ &= \frac{1}{1+e^{-\mu_3(z+1)}} + \frac{1}{1+e^{-\mu_3(z-1)}} - 2\frac{1}{1+e^{-\mu_3 z}} \\ &= \overline{\Psi}(1-\overline{\Psi})\frac{(e^{\mu_3}+e^{-\mu_3}-2)(e^{-\mu_3 z}-1)(e^{-\mu_3 z}+1)}{(1+e^{-\mu_3(z+1)})(1+e^{-\mu_3(z-1)})}.\end{aligned}$$

Due to $\overline{\Psi}' = \mu_3\overline{\Psi}(1-\overline{\Psi})$, it follows that

$$\begin{aligned}&d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi}-\overline{\Psi}) \\ &\quad = \overline{\Psi}(1-\overline{\Psi})[d(e^{\mu_3}+e^{-\mu_3}-2)(1-2\overline{\Psi}) + \epsilon\mu_3 - r + rh\frac{\overline{\Phi}}{\overline{\Psi}} \\ &\qquad + d(e^{\mu_3}+e^{-\mu_3}-2)^2(1-\overline{\Psi})R_2(\Psi)],\end{aligned}$$

where

$$R_2(\overline{\Psi}) := \frac{(1+e^{-\mu_3 z})(1-e^{-\mu_3 z})}{1+e^{-\mu_3 z}(e^{\mu_3}+e^{-\mu_3})+e^{-2\mu_3 z}}.$$

The replacement of $R_2(\overline{\Psi})$ by its maximum (which is 1) leads to

$$\begin{aligned}&d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi}-\overline{\Psi}) \\ &\quad \le \overline{\Psi}(1-\overline{\Psi})^2\left[2d(e^{\mu_3}+e^{-\mu_3}-2) + rh\frac{\overline{\Phi}-\overline{\Psi}}{\overline{\Psi}\left(1-\overline{\Psi}\right)} + d(e^{\mu_3}+e^{-\mu_3}-2)^2\right].\end{aligned} \tag{3.35}$$

As $\epsilon \to 0^+$, it can be obtained directly from $\Gamma_2(\mu_3) = 0$ that $e^{\mu_3}+e^{-\mu_3}-2 \to \frac{r(h-1)}{d}$. This, together with (3.34) and (3.35) implies that $(\overline{\Phi}, \overline{\Psi})$ is an upper solution. By Theorem 3.5, the proof is complete. □

Next, we are going to provide two corollaries from which one can derive that the speed sign is positive.

Let

$$\underline{\Phi}(z) = \frac{p}{1+e^{-\mu_1(\epsilon)z}}, \text{ and } \underline{\Psi}(z) \text{ be the function determined by Lemma 3.2 (1),}$$

where $0 < \epsilon \ll 1$ and $p \in (\frac{1}{h}, 1)$. By this choice of $(\underline{\Phi}, \underline{\Psi})$, we have

Corollary 3.9 *The speed c of the bistable traveling wave solution of* (1.6) *satisfies* $c \ge \epsilon > 0$ *provided that*

$$-2(k-1) + kY_3(z) - (k-1)^2 > 0, \text{ where } Y_3(z) = \frac{\underline{\Psi} - \underline{\Phi}\left(\frac{k-1+p}{kp}\right)}{\frac{\underline{\Phi}}{p}\left(1-\frac{\underline{\Phi}}{p}\right)}. \tag{3.36}$$

Proof A substitution of $(\underline{\Phi}, \underline{\Psi})(z)$ into the Φ-equation leads to

$$\begin{aligned}&\mathcal{D}_2[\underline{\Phi}] - \epsilon\underline{\Phi}' + \underline{\Phi}(1-k-\underline{\Phi}+k\underline{\Psi}) \\ &\quad = \frac{\underline{\Phi}^2}{p}\left(1-\frac{\underline{\Phi}}{p}\right)\left(-2(e^{\mu_1}+e^{-\mu_1}-2) + k\frac{\underline{\Psi}-\underline{\Phi}\left(\frac{k-1+p}{kp}\right)}{\frac{\underline{\Phi}}{p}\left(1-\frac{\underline{\Phi}}{p}\right)} + R_3(\underline{\Phi})\right),\end{aligned}$$

where $R_3(\underline{\Phi})$ is the same as the function $R_1(\underline{\Phi})$ in (3.33), but with $\mu_1(-\epsilon)$ replaced by $\mu_1(\epsilon)$. The fact that the minimum of $R_3(\underline{\Phi})$ is -1 combined with the relationship $\underline{\Phi}' = \mu_1\underline{\Phi}\left(1-\frac{\underline{\Phi}}{p}\right)$ shows that

$$\begin{aligned}
&\mathcal{D}_2[\underline{\Phi}] - \epsilon\underline{\Phi}' + \underline{\Phi}(1-k-\underline{\Phi}+k\underline{\Psi}) \\
&\quad > \frac{\underline{\Phi}^2}{p}\left(1-\frac{\underline{\Phi}}{p}\right) \\
&\quad \left(-2(e^{\mu_1}+e^{-\mu_1}-2) + k\frac{\underline{\Psi}-\underline{\Phi}\left(\frac{k-1+p}{kp}\right)}{\frac{\underline{\Phi}}{p}\left(1-\frac{\underline{\Phi}}{p}\right)} - (e^{\mu_1}+e^{-\mu_1}-2)^2\right).
\end{aligned} \tag{3.37}$$

The above formula (3.37), along with (3.36), implies that $(\underline{\Phi}, \underline{\Psi})(z)$ is a lower solution, as long as ϵ is sufficiently small. By Theorem 3.6, the proof is complete. □

Remark 3.10 When $p = 1$, we have

$$Y_3(z) = \frac{\underline{\Psi}-\underline{\Phi}}{\underline{\Phi}\left(1-\underline{\Phi}\right)}.$$

Let $\underline{\Psi}(z) = \frac{1}{1+e^{-\mu_3(\epsilon)z}}$, $0 < \epsilon \ll 1$, and $\underline{\Phi}(z)$ be the corresponding solution for the Φ-equation (see Lemma 3.2 (2)). Then we have

Corollary 3.11 *The speed c of the bistable traveling wave solution of* (1.6) *satisfies* $c \geq \epsilon > 0$, *provided that*

$$2(h-1) + hY_4(z) - \frac{\bar{\chi}r(h-1)^2}{d} > 0, \ \textit{where}\ Y_4(z) = \frac{\underline{\Phi}-\underline{\Psi}}{\underline{\Psi}\left(1-\underline{\Psi}\right)}$$

and

$$\bar{\chi} = \left(\frac{r(h-1)}{d} + 4 + 2\sqrt{\frac{r(h-1)}{d}+4}\right)^{-1}.$$

For the proof of Corollary 3.11, we omit it here, since it follows from the similar discussion as Theorem 3.6 and Corollary 3.9.

4 Explicit formulas for the wave speed sign

In this section, we provide explicit conditions on the determination of the direction of the bistable traveling wave via the construction of suitable upper or lower solutions to the system (1.6).

Theorem 4.1 *For fixed parameters k, h, d and r, the speed of the bistable traveling wave of* (1.6) *is negative if*

$$1 < \frac{rh}{r+d(1-k)} < \frac{k-1}{k}\left[2-\overline{\chi}(k-1)\right], \tag{4.1}$$

where $\overline{\chi} = (k+3+2\sqrt{k+3})^{-1}$.

Proof Let $\overline{\Phi}$ be the function defined in (3.31), namely, $\overline{\Phi}(z) = \frac{1}{1+e^{-\mu_1(-\epsilon)z}}$, and meanwhile redefine $\overline{\Psi}(z)$ as

$$\overline{\Psi}(z) = \min\{1, p_1\overline{\Phi}(z)\} = \begin{cases} 1, & z > z_1, \\ p_1\overline{\Phi}, & z \le z_1, \end{cases}$$

where z_1 is the unique root of $p_1\overline{\Phi}(z) = 1$ for some $p_1 > 1$. By virtue of condition (4.1), we can choose p_1 to satisfy

$$1 < \frac{rh}{r + d(1-k)} < p_1 < \frac{k-1}{k}\left[2 - (k-1)\overline{\chi}\right]. \tag{4.2}$$

When $z \ge z_1 + 1$, we have $\overline{\Psi}(z) \equiv 1$ and

$$d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) = 0.$$

When $z_1 \le z < z_1 + 1$, we have

$$d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) = d(p_1\overline{\Phi}(z-1) - 1) < 0.$$

When $z_1 - 1 \le z < z_1$, it is easy to see that $p_1\overline{\Phi}(z+1) \ge 1$, hence

$$\begin{aligned} &d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) \\ &\quad = d\left(1 + p_1\overline{\Phi}(z-1) - 2p_1\overline{\Phi}(z)\right) + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) \\ &\quad = dp_1\left(\overline{\Phi}(z+1) + \overline{\Phi}(z-1) - 2\overline{\Phi}(z)\right) + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) \\ &\qquad + d(1 - p_1\overline{\Phi}(z+1)) \\ &\quad \le dp_1\left(\overline{\Phi}(z+1) + \overline{\Phi}(z-1) - 2\overline{\Phi}(z)\right) + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}). \end{aligned} \tag{4.3}$$

When $z < z_1 - 1$, we obtain

$$\begin{aligned} &d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) \\ &\quad = dp_1\left(\overline{\Phi}(z+1) + \overline{\Phi}(z-1) - 2\overline{\Phi}(z)\right) + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}). \end{aligned} \tag{4.4}$$

Thus, when $z < z_1$, (4.3) and (4.4) enable us to deal with the Ψ-equation in a unified way. More precisely, we get

$$\begin{aligned} &d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) \\ &\le \overline{\Phi}(1-\overline{\Phi})\left\{dp_1\tau(1-2\overline{\Phi}) + \epsilon p_1\mu_1 + r(h-p_1)\frac{1-p_1\overline{\Phi}}{1-\overline{\Phi}} + dp_1\tau^2\chi\overline{\Phi}\right\} \\ &:= \overline{\Phi}(1-\overline{\Phi})F(\overline{\Phi}). \end{aligned}$$

Here $\tau = e^{\mu_1} + e^{-\mu_1} - 2$, and $\chi = (\tau + 4 + 2\sqrt{\tau+4})^{-1}$. By noticing $\tau \to k-1$, as $\epsilon \to 0^+$, we rewrite $F(\Phi)$ in the form of

$$F(\overline{\Phi}) = dp_1(k-1)(1-2\overline{\Phi}) + r(h-p_1)\frac{1-p_1\overline{\Phi}}{1-\overline{\Phi}} + dp_1(k-1)^2\overline{\chi}\overline{\Phi}.$$

It is easy to check that $F''(\Phi) > 0$. Furthermore, by (4.2), one can show immediately that $F(1/p_1) = d(k-1)(p_1 - 2 + (k-1)\chi) < 0$ and $F(0) = dp_1(k-1) + r(h-p_1) < 0$. Hence, we have

$$d\mathcal{D}_2[\overline{\Psi}] + \epsilon\overline{\Psi}' + r(1-\overline{\Psi})(h\overline{\Phi} - \overline{\Psi}) < 0, \text{ for } z \in R. \tag{4.5}$$

For the Φ-equation, by the estimation

$$Y_1(z) = \begin{cases} \frac{1}{\overline{\Phi}} \le p_1, & \text{when } z > z_1, \\ \frac{p_1 - 1}{1 - \overline{\Phi}} \le p_1, & \text{when } z \le z_1, \end{cases}$$

and the condition (4.2), we conclude that $-2(k-1) + kp_1 + \overline{\chi}(k-1)^2 \le 0$. This together with (4.5) ensures that $(\overline{\Phi}, \overline{\Psi})$ is an upper solution to the system (1.6). Then by Corollary 3.7, we finish the proof. □

Since the sign of the speed of bistable wavefront to be negative is dependent on the choice of upper solutions, we may construct different upper solutions to produce different sufficient conditions. For example, we let

$$\Psi(z) = \frac{1}{1 + e^{-\frac{1}{2}\mu_1(-\epsilon)z}}, \text{ where } 0 < \epsilon \ll 1,$$

which is a solution to the following differential equation

$$\Psi' = \frac{1}{2}\mu_1 \Psi(1 - \Psi).$$

Here, we use $\mu_1 := \mu_1(-\epsilon)$ for short. We further set $\Phi = \Psi^2$, which satisfies

$$\Phi' = \mu_1 \Phi \left(1 - \Phi^{\frac{1}{2}}\right), \quad \Phi(-\infty) = 0, \quad \Phi(\infty) = 1.$$

Now, we are going to give an estimation of $\mathcal{D}_2[\Phi]$. To this end, we first rewrite $\mathcal{D}_2[\Phi]$ as

$$\mathcal{D}_2[\Phi] = \tau \Phi \left(1 - \Phi^{\frac{1}{2}}\right)\left(1 - \frac{3}{2}\Phi^{\frac{1}{2}}\right) + R(x, \Phi),$$

where

$$R(x, \Phi) = \tau \Phi (1 - \Phi^{\frac{1}{2}}) \Phi^{\frac{1}{2}} g(x),$$

with

$$g(x) = \frac{a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4}{1 + 2\sqrt{\tau + 4}x + (\tau + 6)x^2 + 2\sqrt{\tau + 4}x^3 + x^4},$$

where $a = \frac{\sqrt{\tau+4} - 2}{\tau}$, $e^{\frac{1}{2}\mu_1 z} = x$ and

$$a_1 = -2a + \frac{1}{2}, \quad a_2 = a^2\tau - 4a - 1 + \sqrt{\tau + 4}, \quad a_3 = 2a^2\tau + 6 - 4a - 2\sqrt{\tau + 4} + \frac{\tau}{2},$$
$$a_4 = a^2\tau - 4a - \tau - 1 + \sqrt{\tau + 4}, \quad a_5 = \frac{9}{2} - 2a - 2\sqrt{\tau + 4}.$$

The maximum of $g(x)$, $x \in (0, +\infty)$ can be deduced through a direct calculation and if it is denoted by Ω, then $\mathcal{D}_2[\Phi]$ can be estimated by

$$\mathcal{D}_2[\Phi] \le \tau \Phi \left(1 - \Phi^{\frac{1}{2}}\right)\left(1 - \frac{3}{2}\Phi^{\frac{1}{2}}\right) + \tau \Omega \Phi \left(1 - \Phi^{\frac{1}{2}}\right)\Phi^{\frac{1}{2}}. \tag{4.6}$$

On the basis of the above analysis, we arrive at the following theorem.

Theorem 4.2 *For fixed parameters k, h, d and r, the speed of the bistable traveling wave connecting $\mathbf{o}$ and β is negative provided that $5 - 3k + 2(k-1)\overline{\Omega} \le 0$ (here, $\overline{\Omega}$ is the limit of Ω as $\epsilon \to 0^+$), and one of the following conditions is satisfied,*

$$d\delta < r < \frac{d\delta(\sqrt{\delta+4}+2)^2}{h(\delta+4+4\sqrt{\delta+4})}, \tag{4.7}$$

or

$$\frac{d\delta(\sqrt{\delta+4}+2)^2}{h(\delta+4+4\sqrt{\delta+4})} \le r < \frac{d\delta(4+2\sqrt{\delta+4})}{(h-1)(\delta+4+2\sqrt{\delta+4})}, \tag{4.8}$$

where

$$\delta = \sqrt{k+3} - 2.$$

Proof By virtue of (4.6), substituting (Φ, Ψ) into the Φ-equation and the Ψ-equation respectively leads to

$$\begin{aligned} &\mathcal{D}_2[\Phi] - \epsilon\Phi' + \Phi(1-k-\Phi+k\Psi) \\ &\le \Phi\Phi^{\frac{1}{2}}(1-\Phi^{\frac{1}{2}})\left(-\frac{3}{2}\tau - \epsilon\mu_1\Phi^{-\frac{1}{2}} + 1 + \tau\Omega\right), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &d\mathcal{D}_2[\Psi] - \epsilon\Psi' + r(1-\Psi)(h\Phi - \Psi) \\ &\le \Psi(1-\Psi)\left[d\tau_1 + \frac{1}{2}\epsilon\mu_1 - r + (-2d\tau_1 + rh + d\chi_1\tau_1^2)\Psi\right], \end{aligned} \tag{4.10}$$

where

$$\tau_1 = \sqrt{\tau+4} - 2, \quad \chi_1 = \frac{1}{\tau_1 + 4 + 2\sqrt{\tau_1+4}}.$$

Let $\epsilon \to 0^+$, we have $\tau \to k-1$. The right expression of (4.9) is negative follows naturally from the assumption, while the last expression of (4.10) is negative follows from (4.7) and (4.8). This implies that the pair of function (Φ, Ψ) is an upper solution to system (1.6). By Theorem 3.5 and Corollary 3.7, the proof is thus complete. □

Next, we shall apply Corollary 3.9 to establish sufficient conditions for the speed of the bistable traveling wave solution to be positive.

Theorem 4.3 *For fixed parameters k, h, d and r, the speed of the bistable traveling wave of (1.6) is positive provided that*

$$\begin{aligned} 1 &< \frac{k}{1 - \frac{r(h-1)}{d} - \frac{r^2(h-1)^2}{d^2}} \\ &< \min\left\{\frac{2d + r(h-1)}{d + r(h-1)}, \frac{1}{h}\left[2(h-1) - \frac{\bar{\chi}r(h-1)^2}{d}\right]\right\}, \end{aligned} \tag{4.11}$$

where

$$\bar{\chi} = \left(\frac{r(h-1)}{d} + 4 + 2\sqrt{\frac{r(h-1)}{d} + 4}\right)^{-1}.$$

Proof In view of the condition (4.11), one can take the constant p_2 to satisfy

$$1 < \frac{k}{1 - \frac{r(h-1)}{d} - \frac{r^2(h-1)^2}{d^2}} < p_2 < \min\left\{\frac{2d + r(h-1)}{d + r(h-1)}, \frac{1}{h}\left[2(h-1) - \frac{\bar{\chi} r(h-1)^2}{d}\right]\right\}. \tag{4.12}$$

As in Corollary 3.11, we choose $\underline{\Psi}(z) = \frac{1}{1+e^{-\mu_3(\epsilon)z}}$ with $0 < \epsilon \ll 1$, and redefine $\underline{\Phi}$ by

$$\underline{\Phi} = \begin{cases} 0, & \text{when } z \le z_2, \\ 1 - p_2\underline{\Psi}(-z), & \text{when } z > z_2, \end{cases}$$

where z_2 is the root of the equation $1 - p_2\underline{\Psi}(-z) = 0$. By the definition of $\underline{\Psi}(z)$, the following two relations are not hard to verify, one is $\underline{\Psi}(z) = 1 - \underline{\Psi}(-z)$, and another one is

$$\underline{\Psi}'(z) = \mu_3\underline{\Psi}(z)(1 - \underline{\Psi}(z)).$$

Moreover, we have the estimation for the term $\mathcal{D}_2[\underline{\Psi}]$, that is

$$\mathcal{D}_2[\underline{\Psi}] > \tilde{\tau}\underline{\Psi}(1 - \underline{\Psi})\left(1 - 2\underline{\Psi} - \tilde{\tau}\underline{\Psi}\right), \text{ for all } z \in R.$$

Here $\tilde{\tau} = e^{\mu_3} + e^{-\mu_3} - 2$.

Because $\underline{\Phi}$ is a piecewise function, we need to divide our discussion into four cases. When $z \le z_2 - 1$, $\underline{\Phi} \equiv 0$, hence

$$\mathcal{D}_2[\underline{\Phi}] - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi}) = 0.$$

When $z_2 - 1 < z \le z_2$, we obtain

$$\mathcal{D}_2[\underline{\Phi}] - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi}) = 1 - p_2\underline{\Psi}(-(z+1)) > 0.$$

When $z_2 < z \le z_2 + 1$, we have

$$\begin{aligned}&\mathcal{D}_2[\underline{\Phi}] - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi})\\ &= \underline{\Phi}(z+1) - 2\underline{\Phi}(z) - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi})\\ &\ge \underline{\Phi}(z+1) + (1 - p_2\underline{\Psi}(-(z-1))) - 2\underline{\Phi}(z) - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi}).\end{aligned}$$

When $z > z_2 + 1$, we get

$$\begin{aligned}&\mathcal{D}_2[\underline{\Phi}] - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi})\\ &= \underline{\Phi}(z+1) + (1 - p_2\underline{\Psi}(-(z-1))) - 2\underline{\Phi}(z) - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi}).\end{aligned}$$

Therefore, we can handle the case $z_2 < z \le z_2 + 1$ and $z > z_2 + 1$ in a unified way. Furthermore, when $z > z_2$, we have $\underline{\Phi} = 1 - p_2 + p_2\underline{\Psi}$. Substituting it into the Φ-equation gives

$$\begin{aligned}&\mathcal{D}_2[\underline{\Phi}] - \epsilon\underline{\Phi}' + \underline{\Phi}(1 - k - \underline{\Phi} + k\underline{\Psi})\\ &\ge (1 - \underline{\Psi})\left[\tilde{\tau}p_2\underline{\Psi}(1 - 2\underline{\Psi}) - \epsilon\mu_3 p_2\underline{\Psi} + (p_2 - k)(1 - p_2 + p_2\underline{\Psi}) - \tilde{\tau}^2 p_2\underline{\Psi}^2\right]\\ &:= (1 - \underline{\Psi})G(\underline{\Psi}).\end{aligned}$$

Since $\tilde{\tau}$ tends to $\frac{r(h-1)}{d}$ for sufficient small ϵ, it is easy to calculate that

$$G(1) = \left(1 - \frac{r(h-1)}{d} - \frac{r^2(h-1)^2}{d^2}\right)p_2 - k$$

and

$$G\left(\frac{p_2-1}{p_2}\right)=(p_2-1)\frac{r(h-1)}{d}\left(\frac{2-p_2}{p_2}-\frac{r(h-1)(p_2-1)}{dp_2}\right).$$

By making use of (4.12), one can show that $G(1)>0$ and $G(\frac{p_2-1}{p_2})>0$. This together with $G''(\Psi)<0$ implies

$$\mathcal{D}_2[\underline{\Phi}]-\epsilon\underline{\Phi}'+\underline{\Phi}(1-k-\underline{\Phi}+k\underline{\Psi})>0,\ \text{ for } z>z_2.$$

When $z>z_2$, we deal with the Ψ-equation in a similar way. Indeed, we have

$$\begin{aligned}&d\mathcal{D}_2[\underline{\Psi}]-\epsilon\underline{\Psi}'+r(1-\underline{\Psi})(h\underline{\Psi}-\underline{\Psi})\\&\geq r\underline{\Psi}(1-\underline{\Psi})^2\left[2(h-1)+h\frac{\underline{\Phi}-\underline{\Psi}}{\underline{\Psi}\left(1-\underline{\Psi}\right)}-\frac{\bar{\chi}r(h-1)^2}{d}\right]\\&\geq 0\end{aligned}\tag{4.13}$$

due to the right inequality of (4.12), that is, $p_2<\frac{1}{h}\left[2(h-1)-\frac{\bar{\chi}r(h-1)^2}{d}\right]$. For another case $z\leq z_2$, and the same inequality for the Ψ-equation holds by a similar derivation to (4.13). The above discussion implies that $(\underline{\Phi},\underline{\Psi})$ is a lower solution to (1.6). Thus, the proof is complete by Theorem 3.6 and Corollary 3.11. □

Theorem 4.4 *For fixed parameters k,h,d and r, the speed of the bistable traveling wave is positive provided that*

$$\frac{r+d(k-1)+\overline{\chi}d(k-1)^2}{hr}<3-2k-(k-1)^2.\tag{4.14}$$

Proof Here, to construct a lower solution, we choose

$$\underline{\Phi}=\frac{p_3}{1+e^{-\mu_1(\epsilon)z}},\quad \underline{\Psi}=\frac{\underline{\Phi}}{p_3},\tag{4.15}$$

where $0<\epsilon\ll 1$. Additionally, according to (4.14), one can take p_3 so that

$$\frac{r+d(k-1)+\overline{\chi}d(k-1)^2}{hr}<p_3<3-2k-(k-1)^2.\tag{4.16}$$

We first substitute (4.15) into (3.36) to obtain

$$-2(k-1)+kY_3(z)-(k-1)^2=-2(k-1)+\frac{1-p_3}{1-\frac{\underline{\Phi}}{p_3}}-(k-1)^2.$$

Notice $0<\underline{\Phi}<p_3$ and use the right inequality of (4.16) to get

$$-2(k-1)+kY_3(z)-(k-1)^2\geq -2(k-1)+1-p_3-(k-1)^2>0.\tag{4.17}$$

Inequality (4.17) indicates that

$$\mathcal{D}_2[\underline{\Phi}]-\epsilon\underline{\Phi}'+\underline{\Phi}(1-k-\underline{\Phi}+k\underline{\Psi})>0.\tag{4.18}$$

In addition, a substitution of (4.15) into the Ψ-equation together with an estimation of $\mathcal{D}_2[\underline{\Psi}]$ gives

$$\begin{aligned}&d\mathcal{D}_2[\underline{\Psi}]-\epsilon\underline{\Psi}'+r(1-\underline{\Psi})(h\underline{\Phi}-\underline{\Psi})\\&\geq \underline{\Psi}(1-\underline{\Psi})[d\tau(1-2\underline{\Psi})-\epsilon\mu_1-r+rhp_3-\chi d\tau^2\underline{\Psi}].\end{aligned}$$

Since $\tau \to k-1$ as $\epsilon \to 0^+$, it is sufficient to prove

$$\begin{aligned} & d\mathcal{D}_2[\underline{\Psi}] - \epsilon\underline{\Psi}' + r(1-\underline{\Psi})(h\underline{\Phi} - \underline{\Psi}) \\ & \geq \underline{\Psi}(1-\underline{\Psi})(-r - d(k-1) - \overline{\chi}d(k-1)^2 + rhp_3) \\ & \geq 0, \end{aligned} \tag{4.19}$$

which is true by the first inequality of (4.16). Equations (4.18) and (4.19) imply that $(\underline{\Phi}, \underline{\Psi})$ is a lower solution to (1.6). By Theorem 3.6 and Corollary 3.9, the proof is complete. □

5 Conclusion and discussion

The sign of the bistable wavefront speed to the lattice (discrete) system (1.1) was investigated in this paper via the upper–lower solution method. Although we employed the idea developed in [1,2] to reduce the coupled system (1.6) to a scalar non-local equation by way of abstractly solving the Ψ-equation and the Φ-equation (the latter is not required in the above two mentioned references, but we need it), the existence of the second-order difference operator makes the overall analysis quite different and even difficult than those in [1,2]. We gave an interval estimation for the bistable speed firstly, then two comparison principles were finally established to produce new conditions on the determinacy of the sign of the bistable speed. To our knowledge, this was the first investigation to the lattice system (1.1) for the propagation direction.

In the continuous version of system (1.2), by re-scaling the original variables t and x and reversing the roles of u and v, one can find that there exists a symmetry property for the bistable wave speed as

$$c(d, r, k, h) = -\sqrt{drc}\left(\frac{1}{d}, \frac{1}{r}, h, k\right),$$

which implies we can obtain dual conditions for the speed sign by replacing d, r, k and h by $\frac{1}{d}, \frac{1}{r}, h$ and k, see [22]. However, such an interesting symmetry property does not hold for the discrete system (1.1). To see this, it follows from (4.1) that the range of k, h, r and d is given equivalently by

$$k \in \left(2.1328, 1 + \frac{r}{d}\right), \quad h \in (1, 2), \quad r > d. \tag{5.1}$$

Here, the number 2.1328 is the approximation of the unique root of the equation

$$\frac{k-1}{k}\left(2 - \frac{k-1}{k+3+2\sqrt{k+3}}\right) = 1.$$

Hence, for fixed r, d and k satisfying (5.1), the species U fails to win the competition provided that its competitiveness h is relatively weaker and is bounded by

$$h < \frac{k-1}{k}\left(2 - \frac{k-1}{k+3+2\sqrt{k+3}}\right)\left(1 + \frac{d(1-k)}{r}\right).$$

From (4.11), we can find that, when $h = 5$, $r = 0.5$ and $d = 9$, the species U wins the competition provided that $k < 1.0877$. This result cannot be derived by the symmetry.

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