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Ahmad Alhasanat & Chunhua Ou

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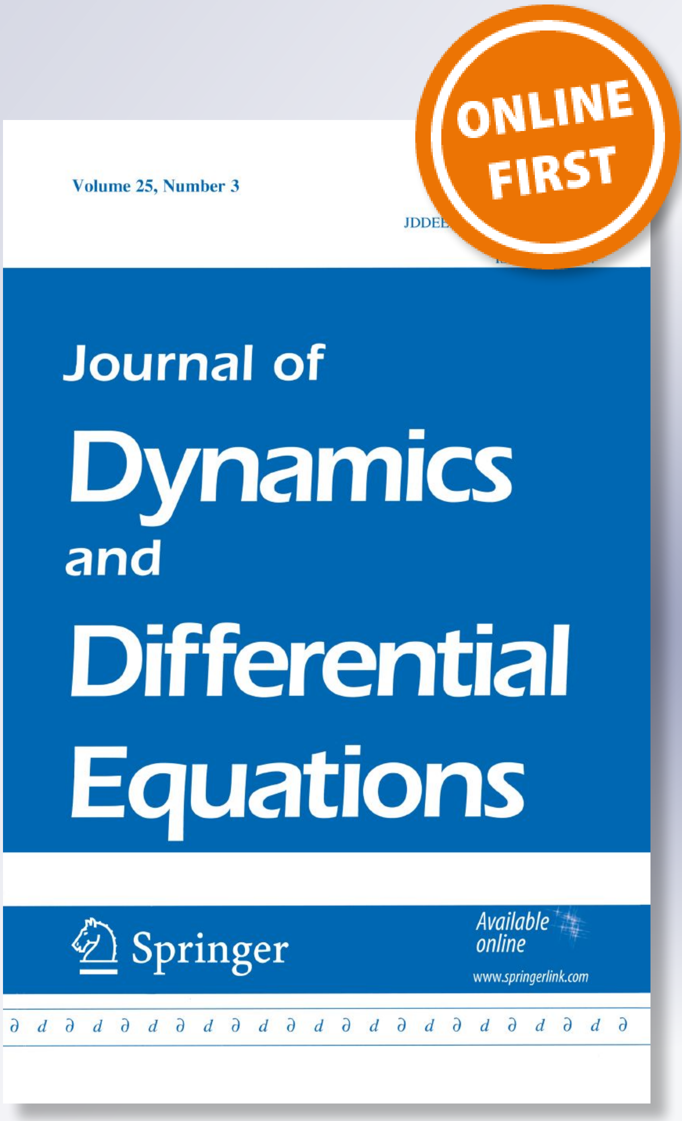

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On a Conjecture Raised by Yuzo Hosono

Ahmad Alhasanat[1] · **Chunhua Ou**[1]

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Abstract In this paper, we study the speed selection mechanism for traveling wave solutions to a two-species Lotka–Volterra competition model. After transforming the partial differential equations into a cooperative system, the speed selection mechanism (linear vs. nonlinear) is investigated for the new system. Hosono conjectured that there is a critical value r_c of the birth rate so that the speed selection mechanism changes only at this value. In the absence of diffusion for the second species, we obtain the speed selection mechanism and successfully prove a modified version of the Hosono's conjecture. Estimation of the critical value is given and some new conditions for linear or nonlinear selection are established.

Keywords Lotka–Volterra · Traveling waves · Speed selection

Mathematics Subject Classification 35K40 · 35K57 · 92D25

1 Introduction

Consider the diffusive Lotka–Volterra competition model

$$\begin{cases} \phi_t = d_1\phi_{xx} + r_1\phi(1 - b_1\phi - a_1\psi), \\ \psi_t = d_2\psi_{xx} + r_2\psi(1 - a_2\phi - b_2\psi), \end{cases}$$

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✉ Chunhua Ou
ou@mun.ca

Ahmad Alhasanat
A.Alhasanat@mun.ca

1 Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada

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with the initial data

$$\phi(x,0)=\phi_0(x)\geq 0, \qquad \psi(x,0)=\psi_0(x)\geq 0, \quad \forall x\in\mathbb{R}.$$

Here $\phi(x,t)$ and $\psi(x,t)$ are the population densities of the first and the second species at time t and location x, respectively; d_1 and d_2 are the diffusion coefficients; r_1 and r_2 are the net birth rates; a_1 and a_2 are the competition coefficients; $1/b_1$ and $1/b_2$ are the carrying capacities of two species. All of these parameters are assumed to be nonnegative. Biologically, the model is used to study the logistic growth of two species population under competition. Originally, Okubo et al. [16] used this model to describe the interaction between the externally introduced gray squirrels and the indigenous red squirrels in Britain.

Non-dimensionalizing the problem by

$$\begin{aligned}
&\sqrt{r_1/d_1}\,x\to x, \quad r_1 t\to t,\\
&b_1\phi(x,t)=\tilde{\phi}(x,t), \quad b_2\psi(x,t)=\tilde{\psi}(x,t),\\
&d=\frac{d_2}{d_1}, \quad r=\frac{r_2}{r_1}, \quad \frac{a_1}{b_2}\to a_1, \quad \frac{a_2}{b_1}\to a_2,
\end{aligned}$$

gives a new system

$$\begin{cases}\tilde{\phi}_t=\tilde{\phi}_{xx}+\tilde{\phi}(1-\tilde{\phi}-a_1\tilde{\psi}),\\ \tilde{\psi}_t=d\tilde{\psi}_{xx}+r\tilde{\psi}(1-a_2\tilde{\phi}-\tilde{\psi}).\end{cases}$$

A change of variable $u=\tilde{\phi}$ and $v=1-\tilde{\psi}$ transforms the above model into a cooperative system

$$\begin{cases}u_t=u_{xx}+u(1-a_1-u+a_1v),\\ v_t=dv_{xx}+r(1-v)(a_2u-v),\end{cases} \tag{1.1}$$

with the initial data

$$u(x,0)=u_0(x)=b_1\phi_0(x), \qquad v(x,0)=v_0(x)=1-b_2\psi_0(x), \quad \forall x\in\mathbb{R}.$$

Throughout this paper, we assume that a_1 and a_2 satisfy the condition

$$0<a_1<1<a_2 \tag{1.2}$$

that arose in many previous studies. In [16], (1.2) means that the gray squirrels out-competes the reds. For biological interpretation of this condition, see also [4–6,11,22].

The cooperative system (1.1), under the condition (1.2), has only three equilibria in the region $\{(u,v)|0\leq u\leq 1, 0\leq v\leq 1\}$, which are $e_0=(0,0)$, $e_1=(1,1)$, and $e_2=(0,1)$. It is easy to see that e_0 is an unstable and e_1 is a stable equilibrium to the following ordinary differential system

$$\begin{cases}u'=u(1-a_1-u+a_1v),\\ v'=r(1-v)(a_2u-v),\end{cases}$$

A traveling wave solution to the system (1.1) that connects e_1 and e_0 is a special solution in the form

$$u(x,t)=U(z), \ v(x,t)=V(z), \ \ z=x-ct,$$

for some constant $c\geq 0$, with the conditions

$$(U,V)(-\infty)=e_1, \quad (U,V)(\infty)=e_0.$$

Here, (U, V) is called the wavefront, z is the wave variable, and c is the wave speed. Substituting this into the system (1.1) leads to an ordinary differential system

$$\begin{cases} -cU' = U'' + U(1 - a_1 - U + a_1 V), \\ -cV' = dV'' + r(1 - V)(a_2 U - V), \\ (U, V)(-\infty) = e_1, \quad (U, V)(\infty) = e_0, \end{cases} \tag{1.3}$$

where prime denotes the derivative with respect to the wave variable z. Results in [9,11,12,20] proved that there exists a constant $c_{\min} \geq 0$ so that the system has a positive traveling wave solution if and only if $c \geq c_{\min}$. In other words, $c_{\min}$ can be expressed as

$$c_{\min} = \inf\{c : (1.3) \text{ has a positive solution } (U, V)\}.$$

Standard linearization analysis near the equilibrium point e_0 shows that the necessary condition for the existence of a traveling wave solution is

$$c \geq c_0 = 2\sqrt{1 - a_1}. \tag{1.4}$$

The value of c_0 is the minimal wave speed for the linear system with non-negative traveling wave solutions. Based on the relation between the two speed values $c_{\min}$ and c_0, we have the following definition.

Definition 1 If $c_{\min} = c_0$, then we say that the minimal wave speed is linearly selected; otherwise, if $c_{\min} > c_0$, we say that the minimal wave speed is nonlinearly selected.

The problem of speed selection (linear and nonlinear) has been of a great interest in biological and mathematical studies, see e.g. [3–8,10,11,13,18,19,21,22]. In literature, linear speed selection for the system (1.1) was studied in [2,5,7,10,11,15,16]. Particularly, in [5], it was proved that the linear speed selection is realized if

$$d = 0 \text{ and } (a_1 a_2 - 1)r \leq (1 - a_1). \tag{1.5}$$

Lewis et al. [10] applied the results in [22] and proved that the minimal wave speed for (1.3) is linearly selected when the condition

$$d \leq 2 \text{ and } (a_1 a_2 - 1)r \leq (2 - d)(1 - a_1) \tag{1.6}$$

holds. By constructing an upper and a lower solutions, Huang [7] extended the above result by proving that the linear speed selection is realized without the restriction $d \leq 2$ but with the condition

$$\frac{(2 - d)(1 - a_1) + r}{r a_2} \geq \max\left\{a_1, \frac{d - 2}{2|d - 1|}\right\}. \tag{1.7}$$

These two conditions [(1.6) and (1.7)] are equivalent when $d \leq 2$, and are similar to (1.5) when $d = 0$.

We should mention that, in 1998, Hosono in [6] studied the speed selection problem numerically and found that the wave speed is not always linearly selected. Based on his numerical simulations, he raised the following conjecture.

Hosono's conjecture *If $a_1 a_2 \leq 1$, then $c_{\min} = c_0$ for all $r > 0$. If $a_1 a_2 > 1$, then there exits a positive number r_c such that $c_{\min} = c_0$ for $0 < r \leq r_c$, and $c_{\min} > c_0$ for $r > r_c$.*

This conjecture has been outstanding for almost 20 years and it is still open now. The purpose of this paper is to work on the Hosono's conjecture for the special case when $d = 0$ in (1.3). Indeed, when $d = 0$ and $a_1 a_2 \leq 1$, the first part of the conjecture holds true by (1.6) and our analysis in Remark 3.11 also confirms this part, while for $d > 2$, it is still unsolved. However, we find that the second part of the conjecture is not completely correct, since the critical number r_c could be infinite even though $a_1 a_2 > 1$ is true. Therefore we provide a modified version of this conjecture when $d = 0$ and prove it rigorously. Our first result is the following theorem.

Theorem 1.1 *Suppose $d = 0$ in (1.1). There exists r_c, $0 \le r_c \le \infty$, such that*

(1) *If $r \le r_c$, the minimal wave speed is linearly selected.*
(2) *If $r > r_c$, the minimal wave speed is nonlinearly selected.*

We further give some estimates of r_c. This successfully leads to some explicit, new and important conditions for both linear or nonlinear speed selection mechanisms. In [7], Huang strongly believes that the condition (1.6) is necessary and sufficient for the linear speed selection. Our results are against this claim.

We should emphasize that we will use the upper–lower solution method coupled with comparison principle to prove our result. The method originates from Diekmann [1] with two classical constructions of upper and lower solutions that have been extensively applied in the research of traveling wave solutions. We will construct a new and smooth upper solution to analyze the linear speed selection and a new lower solution to analyze the nonlinear speed selection. We find that these new types of solutions approximate more accurately to the true traveling waves, and this not only improves previous explicit results on the linear selection, but also provides some new results on the nonlinear selection that was thought to be very difficult in study.

The rest of the paper is organized as follows. We analyze the traveling wave solution to (1.1), when $d = 0$, near the equilibrium point e_0 in Sect. 2. By applying the upper–lower solution method, we study the speed selection mechanisms and prove the modified Hosono's conjecture, Theorem 1.1, in Sect. 3. In Sect. 4, we estimate the critical value r_c and give explicit conditions for the speed selection. Conclusions are presented in Sect. 5, and Sect. 6 is an "Appendix" where the upper–lower solution technique is illustrated for our model.

2 Local Analysis of the Wave Profiles Near e_0

By letting $d = 0$ in (1.3), we get

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) = 0, \\ cV' + r(1 - V)(a_2 U - V) = 0, \\ (U, V)(-\infty) = e_1, \quad (U, V)(\infty) = e_0. \end{cases} \tag{2.1}$$

To understand the stable manifold of the above system near e_0, we linearize it around e_0 to have a linear system (still denote it as (U, V))

$$\begin{cases} U'' + cU' + (1 - a_1)U = 0, \\ cV' + r(a_2 U - V) = 0. \end{cases} \tag{2.2}$$

This is a constant-coefficient system and the first equation is de-coupled from the second one. For the first equation, assume $U(z) = C_1 e^{-\mu z}$, with constants C_1 and μ. This leads to

$$\mu^2 - c\mu + 1 - a_1 = 0,$$

which implies μ as

$$\mu_1 = \mu_1(c) = \frac{c - \sqrt{c^2 - 4(1-a_1)}}{2} \quad \text{or} \quad \mu_2 = \mu_2(c) = \frac{c + \sqrt{c^2 - 4(1-a_1)}}{2}. \qquad (2.3)$$

For shortform, here we denote μ_1 and μ_2 as $\mu_1(c)$ and $\mu_2(c)$, respectively. For $c > c_0$, where $c_0 = \sqrt{1-a_1}$ is defined in (1.4), it follows that $\mu_1 < \mu_2$ and μ_1 is an increasing function of c, while μ_2 is a decreasing function of c. Furthermore, any positive solution U can be expressed as

$$U(z) = C_1 e^{-\mu_1 z} + C_2 e^{-\mu_2 z}$$

for either positive C_1, or $C_1 = 0$, $C_2 > 0$. Hence, as $z \to \infty$, from the second equation in (2.2), we have

$$V(z) \sim C_1 \frac{r a_2}{c\mu_1 + r} e^{-\mu_1 z}$$

when $C_1 > 0$, or

$$V(z) \sim C_2 \frac{r a_2}{c\mu_2 + r} e^{-\mu_2 z}$$

when $C_1 = 0$, $C_2 > 0$.

3 The Speed Selection

In this section we will study the speed selection of (2.1). The method used is the upper–lower solution pair coupled with the comparison technique, see the "Appendix" section for details. Due to $d = 0$, V in the second nonlinear equation can be solved explicitly in terms of U. Indeed, define first

$$y(z) = \frac{V(z)}{1 - V(z)} \quad \text{and} \quad \mu(z) = \exp\left(\frac{r}{c}\int_0^z (a_2 U(t) - 1)dt\right).$$

From the second equation in (2.1), the differential equation of $y(z)$ is given by

$$y' + \frac{r}{c}(a_2 U - 1)y = -\frac{r}{c} a_2 U,$$

with the boundary conditions

$$y(-\infty) = \infty, \quad y(\infty) = 0.$$

Multiplying both sides of the above equation by $\mu(z)$ and integrating over $[z, \infty)$ give the formula of $y(z)$ as

$$y(z) = \frac{r a_2}{c\mu(z)} \int_z^\infty \mu(s) U(s) ds.$$

This yields a formula for $V(z)$ as

$$V(z) = \frac{y(z)}{1 + y(z)} = \frac{r a_2 \int_z^\infty \mu(s) U(s) ds}{c\mu(z) + r a_2 \int_z^\infty \mu(s) U(s) ds} := H(U)(z). \qquad (3.1)$$

By using this formula, (2.1) reduces to a non-local equation

$$\begin{cases} L_1(U,V) := U'' + cU' + U(1 - a_1 - U + a_1 V) = 0, \\ U(-\infty) = 1, \ U(\infty) = 0, \end{cases} \tag{3.2}$$

where V is given in (3.1).

Remark 3.1 V is a continuous function of c. When $c \to c_0$, V tends to

$$V_{c_0}(z) = \frac{ra_2 \int_z^\infty \mu(s)U(s)ds}{c_0\mu(z) + ra_2 \int_z^\infty \mu(s)U(s)ds}.$$

As we can see, even though we already obtain the formula (3.1) for the second equation of (2.1), the analysis of wavefronts to the nonlocal Eq. (3.2) is still complicated and challenging. For later use, instead of the exact formula (3.1), we provide an estimate for V in terms of U. This estimate is given by the following lemma.

Lemma 3.2 *For any continuous and decreasing function $U(z)$ satisfying $U(-\infty) = 1, U(\infty) = 0$, assume that $V(z)$, with $V(-\infty) = 1, V(\infty) = 0$, is the solution of the second equation of* (2.1). *Then we have $V(z) \le a_2 U(z)$.*

Proof Note that $0 \le V(z) \le 1$ with $V(-\infty) = 1$ and $V(\infty) = 0$. In the meantime, we have $a_2 U(-\infty) = a_2 > 1$ and $U(z)$ is a decreasing function for $z \in \mathbb{R}$. From these facts, there exists a first point z^* so that $a_2 U(z^*) = 1$ and $a_2 U(z) > V(z), \ \forall z < z^*$. Assume by contradiction there exists a point $\bar{z}$, $z^* < \bar{z} < \infty$ so that $a_2 U(\bar{z}) < V(\bar{z})$. From the formula of $V'(z)$,

$$V'(z) = -\frac{r}{c}(1 - V(z))(a_2 U(z) - V(z)),$$

$V(z)$ is increasing in the right neighborhood of $\bar{z}$, that is, for small $\delta > 0$, $V(\bar{z} + \delta) > V(\bar{z})$. But since $U(z)$ is a decreasing function, it follows that $V(\bar{z}) > a_2 U(\bar{z}) \ge a_2 U(\bar{z} + \delta)$, and $V(\bar{z} + \delta) > a_2 U(\bar{z} + \delta)$. This implies that $V(z)$ is greater than $a_2 U(z)$ and, hence by the differential equation, $V(z)$ is increasing for all $z > \bar{z}$, which contradicts the fact that $V(\infty) = 0$. The proof is complete. □

By using the upper–lower solution technique, we shall prove the existence of a threshold value of r, in the sense that the speed selection changes from linear to nonlinear when r increases and crosses this threshold value. For this purpose, we first provide a routine choice of the lower solution and then prove a comparison lemma on the linear selection.

For $c = c_0 + \epsilon_1$, where ϵ_1 is a sufficiently small positive number, define a continuous function $\underline{U}(z)$ as

$$\underline{U}(z) = \begin{cases} e^{-\mu_1 z}(1 - Me^{-\epsilon_2 z}), & z \ge z_1, \\ 0, & z < z_1, \end{cases}$$

where μ_1 is defined in (2.3), $0 < \epsilon_2 \ll 1$, M is a positive constant to be determined, and $z_1 = \frac{1}{\epsilon_2} \log M$. Let $\underline{V}(z) = H(\underline{U})(z)$. We can obtain the following lemma.

Lemma 3.3 *When $c = c_0 + \epsilon_1$, the pair of functions $(\underline{U}(z), \underline{V}(z))$ is a lower solution to the system* (2.1).

Proof Since $\underline{V}$ is the exact solution to the V-equation when $U(z) = \underline{U}(z)$. This automatically gives

$$c\underline{V}' + r(1 - \underline{V})(a_2\underline{U} - \underline{V}) = 0, \quad \forall z \in \mathbb{R}.$$

For the U-equation, when $z \le z_1$, we have

$$\underline{U}'' + c\underline{U}' + \underline{U}(1 - a_1 - \underline{U} + a_1\underline{V}) = 0.$$

When $z > z_1$, it follows that

$$\begin{aligned} L_1(\underline{U}, \underline{V}) =& \underline{U}'' + c\underline{U}' + \underline{U}(1 - a_1 - \underline{U} + a_1\underline{V}) \\ =& e^{-\mu z}\left\{\mu_1^2 - c\mu_1 + 1 - a_1\right\} \\ &- Me^{-(\mu+\epsilon_2)z}\left\{(\mu_1 + \epsilon_2)^2 - c(\mu_1 + \epsilon_2) + 1 - a_1\right\} \\ &- e^{-2\mu_1 z}\left(1 - Me^{-\epsilon_2 z}\right)^2 + a_1\zeta_1\underline{V}e^{-\mu_1 z}\left(1 - Me^{-\epsilon_2 z}\right). \end{aligned}$$

In view of definition of μ_1, the first term vanishes and, for sufficiently small ϵ_2, the second term is positive. We choose M sufficiently large so that $z_1 > 0$ and the second term dominates the third one. The last term is positive. Hence, $L_1(\underline{U}, \underline{V}) \ge 0$. □

The following lemma provides a comparison principle for the linear selection.

Lemma 3.4 *For the system (2.1), if the wave speed is linearly selected when $r = r_\beta$, for some $r_\beta > 0$, then it is linearly selected for all $r < r_\beta$.*

Proof Let $(U_\beta, V_\beta)(z)$ be the solution of the system (2.1) when $r = r_\beta$, that is,

$$\begin{cases} U_\beta'' + cU_\beta' + U_\beta(1 - a_1 - U_\beta + a_1 V_\beta) = 0, \\ cV_\beta' + r_\beta(1 - V_\beta)(a_2 U_\beta - V_\beta) = 0, \\ (U_\beta, V_\beta)(-\infty) = e_1, \quad (U_\beta, V_\beta)(\infty) = e_0. \end{cases} \tag{3.3}$$

We want to show that $(U_\beta, V_\beta)(z)$ is an upper solution to the system with $r < r_\beta$, i.e.,

$$\begin{cases} U_\beta'' + cU_\beta' + U_\beta(1 - a_1 - U_\beta + a_1 V_\beta) \le 0, \\ cV_\beta' + r(1 - V_\beta)(a_2 U_\beta - V_\beta) \le 0. \end{cases}$$

The first inequality is naturally satisfied from (3.3). For the second inequality, add and subtract $r_\beta(1 - V_\beta)(a_2 U_\beta - V_\beta)$ to the left-hand side to get

$$\begin{aligned} & cV_\beta' + r(1 - V_\beta)(a_2 U_\beta - V_\beta) \\ & = cV_\beta' + r_\beta(1 - V_\beta)(a_2 U_\beta - V_\beta) + (r - r_\beta)(1 - V_\beta)(a_2 U_\beta - V_\beta) \\ & = (r - r_\beta)(1 - V_\beta)(a_2 U_\beta - V_\beta) \\ & \le 0. \end{aligned}$$

Here, we have used the fact that $V_\beta(z) \le a_2 U_\beta(z)$, $\forall z \in \mathbb{R}$, obtained by Lemma 3.2. Applying Theorem 6.2 with the upper solution $(U_\beta, V_\beta)(z)$ and the lower solution defined in Lemma 3.3, we conclude that the wave speed is linearly selected for $r < r_\beta$. □

From this lemma, we define a critical value of r as

$$r_c = \sup\{\, r \mid \text{the linear speed selection of the system (2.1) is realized}\}. \tag{3.4}$$

Clearly $0 \le r_c \le \infty$ and the following result holds true.

Theorem 3.5 *The minimal wave speed of the system (2.1) is linearly selected for all $r \le r_c$, and nonlinearly selected for $r > r_c$.*

Remark 3.6 This theorem is the main result Theorem 1.1 which we emphasize in the Introduction section. If $r_c = 0$ then the interval $0 < r \leq r_c$ is empty. This means that the nonlinear speed selection is realized for all r. Similarly, when $r_c = \infty$ we mean that the linear speed selection is realized for all r.

To estimate the critical value r_c, we now proceed to construct a novel upper solution to Eq. (3.2), which in turn, with the exact formula of $V(z)$, is an upper solution to the two-equation system (2.1). Again let $c = c_0 + \epsilon_1$, where ϵ_1 is a sufficiently small positive number. Take also $\overline{k} = 1 + \epsilon_1$. Define a continuous monotone function $\overline{U}(z)$ as

$$\overline{U} = \frac{\overline{k}}{1 + Ae^{\mu_1 z}}, \quad \text{and let} \quad \overline{V} = H(\overline{U}), \tag{3.5}$$

where A ia a positive constant and μ_1 is defined in (2.3). Finding the derivatives $\overline{U}'$ and $\overline{U}''$ with the awareness of $\overline{U}' = -\mu \overline{U}\left(1 - \frac{\overline{U}}{\overline{k}}\right)$, and substituting them into (3.2) yield

$$\begin{aligned} L_1(\overline{U}, \overline{V}) = \overline{U}\left(1 - \frac{\overline{U}}{\overline{k}}\right) \\ \left\{(\mu_1^2 - c\mu_1 + 1 - a_1) + \frac{\overline{U}}{\overline{k}}\left(-2\mu_1^2 + a_1 \frac{\overline{V} - \overline{U}\left(\frac{a_1 - 1 + \overline{k}}{a_1 \overline{k}}\right)}{\left(1 - \frac{\overline{U}}{\overline{k}}\right)\frac{\overline{U}}{\overline{k}}}\right)\right\}. \end{aligned} \tag{3.6}$$

The formula of $\mu_1(c)$ gives $\mu_1 = \sqrt{1 - a_1} + \delta_1(\epsilon_1)$, with $\delta_1(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$. By the theory of upper–lower solutions in the "Appendix" section, it is easy to see that, for $\epsilon_1 \ll 1$, the pair of functions $(\overline{U}(z), \overline{V}(z))$ is an upper solution to the system (2.1) when

$$-2(1 - a_1) + a_1 Y_1(z) < 0, \ z \in \mathbb{R}, \quad \text{where} \ Y_1(z) = \frac{\overline{V} - \overline{U}}{(1 - \overline{U})\overline{U}}. \tag{3.7}$$

In the following lemmas we want to prove the boundedness of $Y_1(z)$ and its monotonicity with respect to the parameter r.

Lemma 3.7 *The function $Y_1(z)$ is bounded above for all $z \in \mathbb{R}$.*

Proof Since $Y_1(z)$ is continuous in $\mathbb{R}$, it is enough to show that $\lim_{z \to \pm\infty} Y_1(z) < \infty$. Note that, as $z \to -\infty$, we have

$$\begin{aligned} \mu(z) &\sim D_1 \exp\left(\frac{r}{c}(a_2 - 1)z\right), \qquad D_1 = \exp\left(\int_0^{-\infty} \frac{r}{c}(\overline{U} - 1)dt\right), \\ y(z) &\sim D_1^{-1} D_2 \exp\left(-\frac{r}{c}(a_2 - 1)z\right), \qquad D_2 = \exp\left(\int_{-\infty}^{\infty} \frac{ra_2}{c}\mu(s)\overline{U}(s)ds\right), \\ \overline{V}(z) &\sim 1 - D_1 D_2^{-1} \exp\left(\frac{r}{c}(a_2 - 1)z\right), \\ \overline{U}(z) &\sim 1 - A\exp(\mu_1 z). \end{aligned}$$

This gives

$$\lim_{z\to-\infty} Y_1(z) = \lim_{z\to-\infty} \frac{Ae^{\mu_1 z} - D_1 D_2^{-1} e^{\frac{r}{c}(a_2-1)z}}{Ae^{\mu_1 z}}$$
$$= \begin{cases} 1 & , \text{ when } r(a_2-1) > c\mu_1 \\ D_3 & , \text{ when } r(a_2-1) = c\mu_1 \\ -\infty & , \text{ when } r(a_2-1) < c\mu_1 \end{cases}$$

where $D_3 = 1 - D_1 D_2^{-1} A^{-1} < 1$. For the limit when $z \to \infty$, we also have

$$\lim_{z\to\infty} \frac{\overline{V} - \overline{U}}{(1-\overline{U})\overline{U}} = \lim_{z\to\infty} \left(\frac{y(z)}{\overline{U}(z)} - 1 \right) = \lim_{z\to\infty} \frac{ra_2 \int_z^\infty \mu(s)\overline{U}(a)ds}{c\mu(z)\overline{U}(z)} - 1.$$

By making use of the L'Hospital's rule, it follows that

$$\lim_{z\to\infty} \frac{\overline{V} - \overline{U}}{(1-\overline{U})\overline{U}} = \frac{r(a_2-1) - c\mu_1}{r + c\mu_1}.$$

This implies that $Y_1(z)$ is bounded above. □

Lemma 3.8 *The function $Y_1(z)$ is non-decreasing with respect to r.*

Proof Since $\overline{U}(z)$ is independent of r, it is enough to show that $\overline{V}(z)$ is non-decreasing with respect to r. We prove this in the following two steps:

Step 1 We claim here $a_2\overline{U}(z) \geq \overline{V}(z)$, $\forall z \in \mathbb{R}$. The proof is similar to the proof of Lemma 3.2 and is omitted here.

Step 2 Let $\tau = z/r$ and $(\overline{U}, \overline{V})(z) = (\tilde{U}, \tilde{V})(\tau)$. Substituting into the $\overline{V}'(z)$ formula gives

$$\tilde{V}_\tau = -\frac{1}{c}(1-\tilde{V})(a_2\tilde{U} - \tilde{V}).$$

From step 1, $\tilde{V}(\tau)$ is a non-increasing function in τ. Since τ is decreasing in r, then $\tilde{V}(\tau)$ (hence $\overline{V}(z)$) is a non-decreasing function in r. The lemma is proved. □

By the above lemmas, we can define

$$r_- = \sup\{ r \geq 0 \mid \text{the inequality (3.7) holds for } c = c_0 \text{ and } \forall z \in \mathbb{R}\}. \quad (3.8)$$

Hence, the following lemma is true.

Lemma 3.9 *For $c = c_0 + \epsilon_1$ and $r < r_-$, where ϵ_1 is a sufficiently small positive number and r_- is defined in (3.8), the pair of functions $(\overline{U}(z), \overline{V}(z))$, defined in (3.5), is an upper solution to the system (2.1) with $(\overline{U}, \overline{V})(-\infty) = (\overline{k}, 1)$ and $(\overline{U}, \overline{V})(\infty) = (0, 0)$.*

Now, we are ready to state our result for the linear speed selection.

Theorem 3.10 *The linear speed selection of the system (2.1) is realized when $r \leq r_-$.*

Proof When $r < r_-$, by choosing $(\overline{U}, \overline{V})(z)$ and $(\underline{U}, \underline{V})(z)$ in the above two lemmas (Lemmas 3.9 and 3.3) and using Theorem 6.2, we conclude that the system (2.1) has a traveling wave solution $(U, V)(x - ct)$ with $(U, V)(-\infty) = (1, 1)$ and $(U, V)(\infty) = (0, 0)$ for any $c = c_0 + \epsilon_1 > c_0$. This implies the linear speed selection of the system (2.1). When r_- is finite and $r = r_-$, a limiting argument (a sequence $\{r_n\}$ with a limit r_-) can show the linear selection of the wave speed. This completes the proof. □

Remark 3.11 We can use the classical exponential function $\left(1, \frac{ra_2}{c\mu+r}\right) e^{-\mu_1 z}$ as an upper solution to the system (2.1). The linear selection is realized when

$$r \le r_0 := \begin{cases} \infty, & a_1 a_2 \le 1, \\ \dfrac{2(1-a_1)}{a_1 a_2 - 1}, & a_1 a_2 > 1, \end{cases}$$

which agrees with the condition (1.5) (and (1.6) when $d = 0$). This is also found in [17]. We will see that our choice of upper solution (3.5) gives some better and new results.

To see the novel contribution of our upper solution to the linear selection, we will show that the condition (1.6) is not necessary for the linear speed selection when $d = 0$. Indeed, the following remark shows that $r_- > r_0$ when $a_1 a_2 > 1$.

Remark 3.12 We give a counterexample with $r_- > r_0$ to show the non-necessity of the condition (1.6). Let $d = 0, a_1 = 0.5, a_2 = 3, r = 4, \overline{k} = 1.001, c = c_0 + 0.001$, and $A = 1$. Then $r_0 = 2$,

$$\overline{U}(z) = \frac{1.001}{1 + e^{0.6310z}}, \qquad \mu(z) = \exp\left(2.8264 \int_0^z (3\,\overline{U}(z) - 1)dt\right),$$
$$y(z) = \frac{8.4793}{\mu(z)} \int_z^\infty \mu(s)\overline{U}(s)ds, \qquad \overline{V}(z) = \frac{y(z)}{1 + y(z)},$$

and

$$-2(1-a_1) + a_1 \frac{\overline{V} - \overline{U}}{(1-\overline{U})\overline{U}} = -1 + 0.5Y_1(z) := Y_0(z).$$

Using MATLAB, we plot the graph of $Y_0(z)$. Figure 1 shows that $Y_0(z) < 0$ for all $z \in \mathbb{R}$. This implies that the wave speed is linearly selected for $r < 4$. The result is better than the previous one in Remark 3.11 that only gives the linear selection for $r \in (-\infty, 2]$. In other words, we have

$$r_0 = 2 < r_-.$$

From the result in Theorem 3.10, it is obvious to see that r_- is a lower bound of r_c, that is $r_- \le r_c$. A natural question to ask is whether the speed selection mechanism changes to nonlinear selection at some value of r ($\ge r_-$). To investigate this, we proceed to find a value of r so that the nonlinear speed selection is realized when r is greater than this value.

Lemma 3.13 *For $c_1 > c_0$, assume that there exists a lower monotonic solution $(\underline{U}, \underline{V})$ to the system (2.1), with $(0, 0) \le (\underline{U}, \underline{V}) < (1, 1)$, satisfying $(\underline{U}, \underline{V})(z) \sim (\zeta_1, \zeta_2)e^{-\mu_2 z}$ for some $(\zeta_1, \zeta_2) > (0, 0)$ as $z \to \infty$, where μ_2 is defined in (2.3) and $z = x - c_1 t$, i.e., $(\underline{U}, \underline{V})(z)$ has the faster decay rate near infinity. Then no traveling wave solution to (2.1) exists with speed $c \in [c_0, c_1)$.*

Proof By the assumption, it follows that $(\underline{U}, \underline{V})(x - c_1 t)$ is a lower solution to the following partial differential equation

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = r(1 - v)(a_2 u - v), \end{cases} \tag{3.9}$$

with the initial conditions

$$u(x, 0) = \underline{U}(x) \text{ and } v(x, 0) = \underline{V}(x).$$

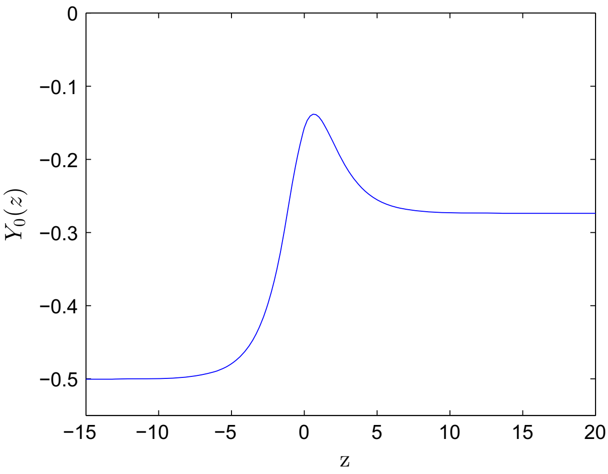

Fig. 1 Graph of $Y_0(z)$ defined in Remark 3.12

Assume to the contrary, for some $c \in [c_0, c_1)$, there exists a monotonic and positive traveling wave solution $(U, V)(x - ct)$ to the system (3.9), with the initial condition

$$u(x, 0) = U(x) \;\text{ and }\; v(x, 0) = V(x).$$

The local analysis of this solution near e_0 can be easily carried out, see e.g., Sect. 2. By the monotonicity of μ_1 and μ_2 in terms of c, we can always assume (by shifting if necessary) $(\underline{U}, \underline{V})(x) \le (U, V)(x)$ for all $x \in (-\infty, \infty)$. Since $(\underline{U}, \underline{V})(x - c_1 t)$ is a lower solution to the system (3.9) and by comparison, we have

$$\begin{aligned} \underline{U}(x - c_1 t) &\le U(x - ct), \\ \underline{V}(x - c_1 t) &\le V(x - ct), \end{aligned} \tag{3.10}$$

for all $(x, t) \in (\mathbb{R}, \mathbb{R}^+)$. On the other hand, fix $z = x - c_1 t$. Then $\underline{U}(z) > 0$ is fixed, and we have

$$U(x - ct) = U(z + (c_1 - c)t) \sim U(\infty) = 0 \text{ as } t \to \infty.$$

By (3.10), this implies that $\underline{U}(z) \le 0$, which is a contradiction. The proof is complete. □

By this lemma, we will find an upper bound of r_c by a suitable choice of a lower solution. Define

$$\underline{U}_1 = \frac{\underline{k}}{1 + Be^{\mu_2 z}} \;\text{ and }\; \underline{V}_1 = H(\underline{U}_1), \tag{3.11}$$

where B ia a positive constant, μ_2 is defined in (2.3) and $0 < \underline{k} < 1$. Similar as previous analysis we find

$$L_1(\underline{U}_1, \underline{V}_1) = \underline{U}_1 \left(1 - \frac{\underline{U}_1}{\underline{k}}\right) \left\{ (\mu_2^2 - c\mu_2 + 1 - a_1) + \frac{\underline{U}_1}{\underline{k}} \left(-2\mu_2^2 + a_1 \frac{\underline{V}_1 - \underline{U}_1 \left(\frac{a_1 - 1 + \underline{k}}{a_1 \underline{k}}\right)}{\left(1 - \frac{\underline{U}_1}{\underline{k}}\right) \frac{\underline{U}_1}{\underline{k}}} \right) \right\}.$$

The pair of functions $(\underline{U}_1(z), \underline{V}_1(z))$ is a lower solution to (2.1) when

$$-2\mu_2^2 + a_1 Y_2(z) > 0, \ \ z \in \mathbb{R}, \tag{3.12}$$

where

$$Y_2(z) = \frac{\underline{V}_1 - \underline{U}_1\left(\frac{a_1 - 1 + \underline{k}}{a_1 \underline{k}}\right)}{\left(1 - \frac{\underline{U}_1}{\underline{k}}\right)\frac{\underline{U}_1}{\underline{k}}}. \tag{3.13}$$

It is easy to find $\lim_{z\to-\infty} Y_2(z) = \infty$, for $0 < \underline{k} < 1$. The same argument as that in the proof of Lemma 3.7 can yield that $\lim_{z\to\infty} Y_2(z)$ is finite. Hence, the minimum value of $Y_2(z)$ is defined. In view of the monotonicity of $Y_1(z)$ with respect to r in Lemma 3.8, the result is true for $Y_2(z)$ as well. Then we can define

$$r_+ = \inf\{\, r \geq 0 \mid \text{the inequality (3.12) holds for some } c > c_0\}. \tag{3.14}$$

Hence, $(\underline{U}_1, \underline{V}_1)(z)$ is a lower solution to (2.1) when $r > r_+$. Then by Lemma 3.13, the following result is true.

Theorem 3.14 *The nonlinear speed selection of the system* (2.1) *is realized when* $r > r_+$.

Remark 3.15 By Lemma 3.13, for the nonlinear selection we only need to construct a lower solution with decaying behavior $(\underline{U}, \underline{V})(z) \sim (\underline{\zeta}_1, \underline{\zeta}_2)e^{-\mu_2 z}$ for some $(\underline{\zeta}_1, \underline{\zeta}_2) > (0, 0)$ as $z \to \infty$ where μ_2 is the faster decay coefficient defined in (2.3). Our new lower solution in (3.11) particularly plays this role. For given values of a_1, a_2 and r, the nonlinear determinacy can be easily obtained by checking the validity of (3.12) in view of the software of MATLAB. For further explicit (analytic) formula on the nonlinear selection, we refer to Theorem 4.5 in Sect. 4.

Remark 3.16 Similarly as in Remark 3.12, examples can be easily constructed to demonstrate the computation of the value of r_+ so that the nonlinear selection exists for $r > r_+$.

By the above analysis, we can use formulas of r_- and r_+ defined in (3.8) and (3.14) to estimate the value of r_c, defined in (3.4), and get a general estimation as

$$r_- \leq r_c \leq r_+.$$

Further estimates on them are presented in the next section.

4 Further Estimation of r_c

The extreme values of $Y_1(z)$ and $Y_2(z)$ cannot be easily found due to the complicated formula $V = H(U)$. For this reason, we will establish some upper and lower solutions for the V-equation instead of using the exact formula. This will lead to some new and explicit results on the linear and nonlinear speed selection.

Theorem 4.1 *When* $a_1 a_2 \leq 2(1 - a_1)$, *the minimal wave speed of the system* (2.1) *is linearly selected for all* $r \geq 0$, *that is,* $r_c = \infty$.

Proof We use the same function $\overline{U}(z)$ defined in (3.5), and re-define $\overline{V}(z)$ as

$$\overline{V}(z) = \min\{1, a_2 \overline{U}(z)\} = \begin{cases} 1, & z \leq z_2, \\ a_2 \overline{U}(z), & z > z_2, \end{cases}$$

where z_2 satisfies $a_2\overline{U}(z_2) = 1$. We want to show that $(\overline{U}(z), \overline{V}(z))$ forms an upper solution. Indeed, for the V-equation, when $z \le z_2$, it gives $c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = 0$, and when $z > z_2$, we have

$$c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = -a_2 c\mu_1 \overline{U}(1-\overline{U}) \le 0.$$

Same formulas as those in (3.6) and (3.7) hold true, and an estimate of $Y_1(z)$ is given by

$$Y_1(z) = \begin{cases} \dfrac{1}{\overline{U}} \le a_2, & \text{when } z \le z_2, \\ \dfrac{a_2 - 1}{1 - \overline{U}} \le a_2, & \text{when } z > z_2. \end{cases}$$

Then we have $-2(1-a_1) + a_1 Y_1(z) \le -2(1-a_1) + a_1 a_2 \le 0$ for all r. By a similar argument as that in the proof of Theorem 3.10, we conclude that the result is true. □

From Remark 3.11, $a_1 a_2 \le 1$ implies that $r_c = \infty$. We combine this and the above theorem to have the following corollary.

Corollary 4.2 *The condition* $a_1 a_2 \le \max\{1, 2(1-a_1)\}$ *implies the linear speed selection for* (2.1).

By another choice of the upper solution, we have the following theorem.

Theorem 4.3 *When* $a_1 \le 2/3$ *and* $a_1 a_2 > 2(1-a_1)$*, the minimal wave speed of the system* (2.1) *is linearly selected for all*

$$r \le \frac{4(1-a_1)^2}{a_1 a_2 - 2(1-a_1)}, \quad \textit{that is,} \quad r_c \ge \frac{4(1-a_1)^2}{a_1 a_2 - 2(1-a_1)}.$$

Proof Here we choose $\overline{V}(z)$ as

$$\overline{V}(z) = \min\left\{1, \frac{2(1-a_1)}{a_1}\overline{U}(z)\right\} = \begin{cases} 1, & z \le z_3, \\ \dfrac{2(1-a_1)}{a_1}\overline{U}(z), & z > z_3, \end{cases}$$

where z_3 satisfies $2(1-a_1)\overline{U}(z_3) = a_1$. When $z \le z_3$, we have $c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = 0$, and when $z > z_3$, we have

$$\begin{aligned} &c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) \\ &\quad = -\frac{2c(1-a_1)}{a_1}\left\{-\mu_1\overline{U}(1-\overline{U})\right\} + r\left(1 - \frac{2(1-a_1)}{a_1}\overline{U}\right)\left(a_2\overline{U} - \frac{2(1-a_1)}{a_1}\overline{U}\right). \end{aligned}$$

Since $a_1 \le 2/3$, the inequality $1 - \frac{2(1-a_1)}{a_1}\overline{U} \le 1 - \overline{U}$ is true. Hence, it follows

$$\begin{aligned} &c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) \\ &\quad \le \frac{2(1-a_1)}{a_1}\overline{U}(1-\overline{U})\left\{-c\mu_1 + r\left(\frac{a_1 a_2}{2(1-a_1)} - 1\right)\right\} \\ &\quad \le 0, \end{aligned}$$

provided that

$$r < \frac{4(1-a_1)^2}{a_1 a_2 - 2(1-a_1)} \quad \text{and} \quad c = c_0 + \epsilon_1,$$

for small ϵ_1. Also for this choice of $(\overline{U},\overline{V})$, we can easily verify $Y_1(z) \le \dfrac{2(1-a_1)}{a_1}$. Then it gives $-2(1-a_1)+a_1Y_1(z) \le 0$. This means that $(\overline{U}(z),\overline{V}(z))$ is an upper solution and the proof is complete. □

Again, from Remark 3.11, when $a_1a_2 > 1$, it follows that

$$r_c \ge \frac{2(1-a_1)}{a_1a_2-1}.$$

Define $M =: \max\{1, 2(1-a_1)\}$. If $a_1 \le 1/2 < 2/3$, then $M = 2(1-a_1)$. In this case, by Theorem 4.3 we have showed that, for $a_1a_2 > M$,

$$r_c \ge \frac{4(1-a_1)^2}{a_1a_2-2(1-a_1)} = \frac{2M(1-a_1)}{a_1a_2-M}.$$

Thus we have the following extension.

Corollary 4.4 *When $a_1a_2 > M =: \max\{1, 2(1-a_1)\}$, the minimal wave speed of the system (2.1) is linearly selected for all*

$$r \le \frac{2M(1-a_1)}{a_1a_2-M}, \quad \textit{that is,} \quad r_c \ge \frac{2M(1-a_1)}{a_1a_2-M}.$$

Theorem 4.5 *If there exists $\eta < 1$ so that $\eta \ge (2/a_1)\max\{1-a_1, 1/a_2\}$, then the minimal wave speed of the system (2.1) is nonlinearly selected for all*

$$r > \frac{2(1-a_1)\eta}{(1-\eta)^2}, \quad \textit{that is,} \quad r_c \le \frac{2(1-a_1)\eta}{(1-\eta)^2}.$$

Proof Let $\underline{U}_1(z)$ be the same function defined in (3.11), and re-define $\underline{V}_1(z)$ as

$$\underline{V}_1(z) = \min\{\eta, \eta a_2\underline{U}_1(z)\} = \begin{cases} \eta, & z \le z_4, \\ \eta a_2\underline{U}_1(z), & z > z_4, \end{cases}$$

where z_4 satisfies $a_2\underline{U}_1(z_4) = 1$. When $z \le z_4$, due to $a_2\underline{U}(z) \ge 1$, we have

$$c\underline{V}_1' + r(1-\underline{V}_1)(a_2\underline{U}_1 - \underline{V}_1) = r(1-\eta)(a_2\underline{U}_1 - \eta) \ge 0.$$

For the region $z > z_4$, we obtain

$$\begin{aligned} c\underline{V}_1' + r(1-\underline{V}_1)(a_2\underline{U}_1 - \underline{V}_1) &= -\eta a_2 c\mu_2\underline{U}_1(1-\underline{U}_1) + r(1-\eta a_2\underline{U}_1)(a_2\underline{U}_1 - \eta a_2\underline{U}_1) \\ &\ge -\eta a_2 c\mu_2\underline{U}_1(1-\underline{U}_1) + r(1-\eta)(1-\eta)a_2\underline{U}_1 \\ &\ge \eta a_2\underline{U}_1\left\{-c\mu_2 + \frac{r}{\eta}(1-\eta)^2\right\} \\ &\ge 0, \end{aligned}$$

provided that $r > \frac{2(1-a_1)\eta}{(1-\eta)^2}$ and $c = c_0 + \epsilon_1$, for some small ϵ_1. On the other hand, since $\eta a_1 a_2 \ge 2$, we can fix the value of $\underline{k}$ in the formula of $\underline{U}_1(z)$ so that the following inequality

$$\frac{1-a_1}{\eta a_1a_2-1} \le \underline{k} \le 1-a_1$$

holds true. Obviously, the same formula as that in (3.13) is still true with the new choice of $\underline{V}_1(z)$. By the choice of $\underline{k}$ and since $0 \le \underline{U}_1 \le \underline{k} < 1$, we have

$$-2(1-a_1) + a_1Y_2(z) = -2(1-a_1) + a_1\frac{\underline{V}_1 - \underline{U}_1\left(\frac{a_1-1+\underline{k}}{a_1\underline{k}}\right)}{\left(1-\frac{\underline{U}_1}{\underline{k}}\right)\frac{\underline{U}_1}{\underline{k}}}$$

$$\geq \begin{cases} -2(1-a_1)+a_1\eta-(a_1-1+\underline{k}), & z \leq z_4 \\ -2(1-a_1)+a_1a_2\eta\underline{k}-(a_1-1+\underline{k}), & z > z_4 \end{cases}$$
$$\geq 0.$$

Hence, by Lemma 3.13, we conclude that the minimal wave speed is nonlinearly selected. □

Remark 4.6 By Theorem 4.5, we can provide an example for the nonlinear selection. Assume that $a_1 = 0.8, a_2 = 5$. It gives that the minimal wave speed of the system is nonlinearly selected when $r > 0.8$.

Remark 4.7 We can include the case $a_1 = 0$ in the condition (1.2), where the speed selection can be studied directly. In this case, $c_0 = 2\sqrt{1-a_1} = 2$, and the system (2.1) reads

$$\begin{cases} U'' + cU' + U(1-U) = 0, \\ cV' + r(1-V)(a_2U - V) = 0, \\ (U, V)(-\infty) = e_1, (U, V)(\infty) = e_0. \end{cases}$$

The first equation is the well-known Fisher equation. It has a positive and monotonic traveling wave solution for all $c \geq 2$. Using its solution in the formula $V = H(U)$ shows that the system has a positive solution for any $c \geq 2$. Hence, the minimal wave speed is linearly selected.

5 Conclusions

The speed selection mechanisms (linear and nonlinear) for traveling waves to a two-species Lotka–Volterra competition model (1.1) are investigated when $d = 0$ and $0 \leq a_1 < 1 < a_2$. New types of the upper/lower solutions are constructed. We prove a modified version of Hosono's conjecture, and further estimates of the critical value r_c are provided.

The linear determinacy in the condition (1.6) with $d = 0$, has been extended to the condition

$$d = 0 \text{ and } (a_1a_2 - M)r \leq M(2-d)(1-a_1),$$

where $M = \max\{1, 2(1-a_1)\}$. It extends the results in [5,7], when $d = 0$, as well. This together with a counterexample shows that they are sufficient but not necessary for the linear speed selection. Our result also indicates that the wave speed is linearly selected when $a_1a_2 > 1$ for all values of r, provided that an extra condition on a_1 and a_2 is satisfied. This shows the failure of Hosono's conjecture for the existence of finite r_c when $a_1a_2 > 1$.

By our analysis, some new results on nonlinear speed selection are also established, see e.g. Theorem 4.5.

The speed selection mechanism when $d > 0$ is challenging and will be addressed in a separated paper.

6 Appendix: Upper–Lower Solution Method

A useful method to prove the existence of monotone traveling wave solution is the upper–lower solution technique originated in Diekmann [1]. Here we illustrate the main idea. By transforming the system (2.1) to a system of integral equations, we can define a monotone iteration scheme in terms of the integral system. By construction an upper and a lower solutions to the system and using the iteration scheme, we can give the existence of traveling wave solutions.

Let α be a sufficiently large positive number so that

$$\alpha U + U(1 - a_1 - U + V) := F(U, V)$$

and

$$\alpha V + r(1 - V)(a_2 U - V) := G(U, V)$$

are monotone in U and V, respectively. Equations in (2.1) are equivalent to

$$\begin{cases} U'' + cU' - \alpha U = -F(U, V), \\ \qquad cV' - \alpha V = -G(U, V). \end{cases} \tag{6.1}$$

Define constants $\lambda_1^{\pm}$ as

$$\lambda_1^- = \frac{-c - \sqrt{c^2 + 4\alpha}}{2} < 0 \text{ and } \lambda_1^+ = \frac{-c + \sqrt{c^2 + 4\alpha}}{2} > 0.$$

By applying the variation-of-parameter method to the first equation in the system (6.1), and the first order differential equation theory to the second equation, the system can be written in the form

$$\begin{cases} U(z) = T_1(U, V)(z), \\ V(z) = T_2(U, V)(z), \end{cases} \tag{6.2}$$

where

$$T_1(U, V)(z) = \frac{1}{\lambda_1^+ - \lambda_1^-} \left\{ \int_{-\infty}^{z} e^{\lambda_1^-(z-s)} F(U, V)(s) ds + \int_{z}^{\infty} e^{\lambda_1^+(z-s)} F(U, V)(s) ds \right\},$$
$$T_2(U, V)(z) = \frac{1}{c} \int_{z}^{\infty} e^{\frac{\alpha}{c}(z-s)} G(U, V)(s) ds.$$

Definition 2 A pair of continuous functions $(U(z), V(z))$ is an upper (a lower) solution to the integral equations system (6.2) if

$$\begin{cases} U(z) \geq (\leq)\ T_1(U, V)(z), \\ V(z) \geq (\leq)\ T_2(U, V)(z). \end{cases}$$

Lemma 6.1 *A continuous function $(U, V)(z)$ which is differentiable on $\mathbb{R}$ except at finite number of points z_i, $i = 1, \ldots, n$, and satisfies*

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) \leq 0, \\ cV' + r(1 - V)(a_2 U - V) \leq 0 \end{cases}$$

for $z \neq z_i$, and $U'(z_i^-) \geq U'(z_i^+)$, for all z_i, is an upper solution to the integral equations system (6.2). The same result is true for the lower solution by reversing the inequalities.

Proof We give the proof for the upper solution where the same argument can be applied for the lower solution. From

$$U'' + cU' - \alpha U + F(U, V) \leq 0$$
$$cV' - \alpha V + G(U, V) \leq 0,$$

we have

$$
\begin{aligned}
T_1(U,V)(z) &= \frac{1}{\lambda_1^+ - \lambda_1^-}\left\{\int_{-\infty}^{z} e^{\lambda_1^-(z-s)}F(U,V)(s)ds + \int_{z}^{\infty} e^{\lambda_1^+(z-s)}F(U,V)(s)ds\right\}\\
&\le \frac{-1}{\lambda_1^+ - \lambda_1^-}\left\{\int_{-\infty}^{z} e^{\lambda_1^-(z-s)}(U'' + cU' - \alpha U)(s)ds\right.\\
&\quad \left. + \int_{z}^{\infty} e^{\lambda_1^+(z-s)}(U'' + cU' - \alpha U)(s)ds\right\}.
\end{aligned}
$$

Simple computations as that in [14, proof of Lemma 2.5] yield

$$T_1(U,V)(z) \le U(z).$$

Similarly $T_2(U,V) \le V(z)$. This implies that $(U,V)(z)$ is an upper solution to the system (6.2). □

The existence of an upper and a lower solution to the system (6.2) will give the existence of the actual traveling wave solution. Indeed, for our problem, we assume that the following hypothesis is true.

Hypothesis 1 *There exists a monotone non-increasing upper solution* $(\overline{U},\overline{V})(z)$ *and a non-zero lower solution* $(\underline{U},\underline{V})(z)$ *to the system* (6.2) *with the following properties:*

(1) $(\underline{U},\underline{V})(z) \le (\overline{U},\overline{V})(z)$, *for all* $z \in \mathbb{R}$,
(2) $(\overline{U},\overline{V})(+\infty) = e_0, \quad (\overline{U},\overline{V})(-\infty) = (\overline{k}_1,\overline{k}_2)$,
(3) $(\underline{U},\underline{V})(+\infty) = e_0, \quad (\underline{U},\underline{V})(-\infty) = (\underline{k}_1,\underline{k}_2)$,

for $e_0 \le (\underline{k}_1,\underline{k}_2) \le e_1$ *and* $(\overline{k}_1,\overline{k}_2) \ge e_1 = (1,1)$ *so that no equilibrium solution to* (2.1) *exists in the set* $\{(U,V)|e_1 < (U,V) \le (\overline{k}_1,\overline{k}_2)\}$. □

From the integral system, we define an iteration scheme as

$$
\begin{cases}
(U_0,V_0) = (\overline{U},\overline{V}),\\
U_{n+1} = T_1(U_n,V_n), \quad n = 0,1,2,\ldots,\\
V_{n+1} = T_2(U_n,V_n), \quad n = 0,1,2,\ldots,
\end{cases}
\tag{6.3}
$$

and arrive at the following result by the upper–lower solution method, see e.g. [1].

Theorem 6.2 *If Hypothesis* 1 *holds, then the iteration* (6.3) *converges to a non-increasing function* $(U,V)(z)$*, which is a solution to the system* (2.1) *with* $(U,V)(-\infty) = e_1$ *and* $(U,V)(\infty) = e_0$*. Moreover,* $(\underline{U},\underline{V})(z) \le (U,V)(z) \le (\overline{U},\overline{V})(z)$ *for all* $z \in \mathbb{R}$.

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