

SPEED SELECTION FOR REACTION DIFFUSION EQUATIONS IN HETEROGENEOUS ENVIRONMENTS

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ABSTRACT. In this paper, we study the spreading (the minimal) speed selection for reaction diffusion equations in heterogeneous periodic habitats. The key feature of the nonlinear selection is unveiled. Using comparison principles and upper(lower) solution techniques, new and practical criteria for determining the minimal speed are provided in this work.

1. INTRODUCTION

In this paper, we are concerned with wavefront propagation phenomena for reaction-advection-diffusion equations in periodic media of the type

$$u_t = \Delta u + q(x) \cdot \nabla u + f(x, u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N \quad (1)$$

where $N \geq 1$ is the space dimension.

To give the description of the advection term $q(x)$ and the reaction $f(x, u)$, we first let L_1, \dots, L_N be N positive real numbers and recall the definitions of a periodicity cell and an L -periodic flow: The set

$$\mathcal{C} = \{x \in \mathbb{R}^N \text{ such that } x_1 \in (0, L_1), \dots, x_N \in (0, L_N)\}$$

is called the periodicity cell of \mathbb{R}^N . A field $w : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is said to be L -periodic if $w(x_1 + k_1, \dots, x_N + k_N) = w(x_1, \dots, x_N)$ almost everywhere in \mathbb{R}^N and for all $k = (k_1, \dots, k_N) \in L_1\mathbb{Z} \times \dots \times L_N\mathbb{Z}$.

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In this work, the underlying advection $q(x) = (q_1(x), \dots, q_N(x))$ is a $C^{1,\alpha}(\mathbb{R}^N)$, for some $\alpha > 0$, vector field satisfying

$$\begin{cases} q \text{ is } L\text{-periodic with respect to } x, \\ \nabla \cdot q \equiv 0 \text{ in } \mathbb{R}^N. \end{cases} \quad (2)$$

The nonlinearity $f = f(x, u)$ in (1) is a function defined in $\mathbb{R}^N \times [0, 1]$, such that

$$f \geq 0, f \text{ is } L\text{-periodic with respect to } x \text{ and of class } C^{1,\alpha}(\mathbb{R}^N \times [0, 1]), \quad (3)$$

and

$$\begin{cases} \forall x \in \mathbb{R}^N, & f(x, 0) = f(x, 1) = 0, \\ \exists \rho \in (0, 1), \forall x \in \mathbb{R}^N, \forall 1 - \rho \leq s \leq s' \leq 1, & f(x, s) \geq f(x, s'), \\ \forall s \in (0, 1), \exists x \in \mathbb{R}^N \text{ such that } & f(x, s) > 0. \end{cases} \quad (4)$$

We are interested in the spreading speed (or the minimal speed) of the pulsating traveling fronts which were introduced in Berestycki, Hamel [1] and Xin [9] as follows:

Definition 1.1. *Let $e = (e^1, \dots, e^N)$ be an arbitrarily unit direction in \mathbb{R}^N . A function $u = u(t, x)$ is called a pulsating traveling front propagating in the direction of e , with an effective speed $c \neq 0$, if u is a classical solution of*

$$\begin{aligned} u_t &= \Delta u + q(x) \cdot \nabla u + f(x, u), t \in \mathbb{R}, x \in \mathbb{R}^N, \\ \forall k \in L_1\mathbb{Z} \times \dots \times L_N\mathbb{Z}, \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, & u(t + \frac{k \cdot e}{c}, x) = u(t, x + k), \\ \forall t \in \mathbb{R}, \lim_{x \cdot e \rightarrow -\infty} u(t, x) &= 0 \text{ and } \lim_{x \cdot e \rightarrow +\infty} u(t, x) = 1, \\ 0 &\leq u \leq 1, \end{aligned} \quad (5)$$

where the above limits hold locally in t and uniformly in the directions of \mathbb{R}^N that are orthogonal to e .

Alternatively, upon using a traveling wave variable $s := x \cdot e + ct$ and plugging the ansatz

$$u(t, x) := \phi(x \cdot e + ct, x) = \phi(s, x)$$

in (5), we obtain that the pulsating traveling wave ϕ is L -periodic in x and it satisfies the equation

$$\Delta_x \phi + \phi_{ss} + 2e \cdot \nabla_x \phi_s + q \cdot \nabla_x \phi + (q \cdot e - c)\phi_s + f(x, \phi(s, x)) = 0, \quad (6)$$

for all $(s, x) \in \mathbb{R} \times \mathbb{R}^N$. Furthermore, defining L_c to be the operator

$$L_c \phi := \Delta_x \phi + \phi_{ss} + 2e \cdot \nabla_x \phi_s + q \cdot \nabla_x \phi + (q \cdot e - c)\phi_s \text{ in } \mathbb{R} \times \mathbb{R}^N, \quad (7)$$

we can see that a pulsating traveling front $\phi(s, x)$ satisfies the wave profile equation

$$L_c\phi + f(x, \phi) = 0, \quad (8)$$

subject to the limiting boundary conditions

$$\lim_{s \rightarrow -\infty} \phi(s, x) = 0 \text{ and } \lim_{s \rightarrow +\infty} \phi(s, x) = 1, \quad (9)$$

uniformly in $x \in \mathbb{R}^N$.

The existence of traveling fronts for this class of equations is well studied by now. For full details, we refer the reader to [1]. We will summarize the results of [1] that are relevant to our present work in the following theorem:

Theorem A (Berestycki, Hamel [1]). *Let e be any unit vector in \mathbb{R}^N . Assume that q satisfies (2) and let f be a nonlinearity satisfying (3)-(4). Then, there exists $c^* > 0$ such that the problem (8)-(9) has no solution (c, ϕ) if $c < c^*$ while, for each $c \geq c^*$, it has a pulsating traveling front solution (c, ϕ) such that ϕ is increasing in s .*

Although the above theorem provides the existence of fronts and a threshold c^* , the exact formula for c^* is unknown for many nonlinearities satisfying the general conditions (3)-(4) which we have in present work. Further clarifications will be laid out in the next few paragraphs. To estimate the speed c^* , given in Theorem A above, we assume that f is differentiable with respect to u at $u = 0$ and set

$$\eta(x) = f'_u(x, 0) > 0, \quad x \in \mathbb{R}^N. \quad (10)$$

The linearized version of (8) then reads

$$L_c\phi + \eta(x)\phi = 0. \quad (11)$$

As done in Berestycki, Hamel, Nadirashvili [3] and Hamel [4], we can obtain the minimal *linear* speed, which we denote by c_0 , as

$$c_0(e) = c_0^{q,f}(e) = \min_{\lambda > 0} \frac{k(\lambda)}{\lambda}, \quad (12)$$

where $k(\lambda) = k_{e,q,\eta,\lambda}$ is the principal eigenvalue of the elliptic operator $L_{e,q,\eta,\lambda}$ defined by

$$L_{e,q,\eta,\lambda}\Psi := \Delta\Psi + 2\lambda e \cdot \nabla\Psi + q \cdot \nabla\Psi + [\lambda^2 + \lambda q \cdot e + \eta]\Psi, \quad (13)$$

acting on the space

$$E = \{ \Psi \in C^2(\overline{\Omega}), \Psi \text{ is } L\text{-periodic with respect to } x \}.$$

A detailed study of the properties of $k(\lambda)$ is done in [1] and [3]. In particular, [1] shows that $\lambda \mapsto k(\lambda)$ is a convex function. Assuming that

$$k(0) > 0, \quad (14)$$

it follows that there exists a unique finite $\lambda = \bar{\mu}$ so that

$$c_0(e) = \frac{k(\bar{\mu})}{\bar{\mu}}. \quad (15)$$

In particular, for $c > c_0$, the equation

$$c\lambda = k(\lambda) \quad (16)$$

has two solutions $\lambda = \mu_1(c)$ and $\lambda = \mu_2(c)$, with $\mu_1(c) < \mu_2(c)$. When $c = c_0$, we have $\mu_1(c) = \mu_2(c) = \bar{\mu}$. Moreover, appealing to the convexity of the function $k(\lambda)$, we get that $\mu_1(c)$ is *decreasing* in c and $\mu_2(c)$ is *increasing* in c .

We comment now on the influence of the nonlinearity f on the speeds c and c^* . In the particular case where the nonlinearity f satisfies the KPP condition

$$f(x, u) \leq \eta(x)u, \quad (17)$$

the minimal speed c^* in Theorem A is exactly equal to c_0 in (12) (see [3, 4]). However, for a more general nonlinearity function f which only satisfies conditions (3) and (4), it is still unknown how the minimal speed is determined. We know that $c^* \geq c_0$ always holds (provided that f satisfies the conditions of Theorem A). The primary purpose of this paper is to investigate this matter. We will prove that, when the minimal speed is greater than the linear speed c_0 , the corresponding wave front (*pushed front*) decays with a faster rate; this solves the conjecture in [4, page, 363]. Based on the upper-lower solution method, we will provide an easy-to-use approach to determine when the minimal speed is selected linearly or nonlinearly. In the case of nonlinear selection, we will show a way to give a lower or an upper bound estimate of the minimal speed. Before going further we recall the following terminology which is used in literature (see [8] for instance)

Definition 1.2 (Linear and nonlinear selection mechanisms). *Under the assumptions of Theorem A, we call the case $c^* = c_0$ the linear selection mechanism and the case $c^* > c_0$ the nonlinear selection mechanism.*

The rest of this paper is organized as follows. In Section 2, we will show our main results. Section 3 serves as an application of our theorems.

2. PUSHED WAVEFRONT

For a given wavefront ϕ satisfying (8) with $c > c_0$, a straightforward derivation of the characteristics of the linear part of the wave profile proposes that either

$$\phi(s, x) \sim C_1 \Psi_{\mu_1}(x) e^{\mu_1(c)s}, \quad C_1 > 0, \quad (18)$$

or

$$\phi(s, x) \sim C_2 \Psi_{\mu_2}(x) e^{\mu_2(c)s}, \quad C_2 > 0 \quad (19)$$

as $s \rightarrow -\infty$, where $\Psi_{\mu_i}(x)$, $i = 1, 2$, is the eigenfunction corresponding to the principal eigenvalue $k(\mu_i)$ defined in (13). For a rigorous proof of this property, we refer the reader to [4].

Alternatively, when linearizing the first equation of (5) at $u = 0$, we obtain the linear partial differential equation

$$u_t = \Delta u + q(x) \cdot \nabla u + \eta(x)u, \text{ where } \eta(x) = \partial_u f(x, 0). \quad (20)$$

The above equation defines a linear semiflow $M(u_0) = u(t, x, u_0)$, where u_0 is the initial data. Obviously, we have

$$M(\Psi_{\mu_i}(x)e^{\mu_i(c)x \cdot e}) = \Psi_{\mu_i}(x)e^{\mu_i(c)[x \cdot e + ct]}, \quad i = 1, 2. \quad (21)$$

2.1. Fast decay nature of the pushed wavefront.

Theorem 2.1 (Necessary and sufficient condition). *Assume that (14) holds and let $\phi_{c^*}(s, x)$ be the wavefront of (8) with the speed c^* (the minimal speed). Consider the linear speed c_0 defined in (15). The following results hold:*

(i) *If there exists a speed $c = \bar{c} > c_0$ so that (8) has a non-decreasing traveling wave solution $\phi_{\bar{c}}(s, x)$, connecting 0 to 1, with the following behavior*

$$\phi_{\bar{c}}(s, x) \sim C\Psi_{\mu_2(\bar{c})}(x)e^{\mu_2(\bar{c})s} \text{ as } s \rightarrow -\infty \quad (22)$$

for some positive constant C , where μ_2 is defined in (13), then we have $c^ = \bar{c} > c_0$. In other words, the minimal speed is nonlinearly selected.*

(ii) *If the spreading speed c^* is nonlinearly selected (i.e. $c^* > c_0$), then the wave front $\phi_{c^*}(s, x)$ has the fast decay behavior defined in (19), i.e.,*

$$\phi_{c^*}(s, x) \sim C_2\Psi_{\mu_2}(x)e^{\mu_2(c^*)s}, \text{ as } s \rightarrow -\infty, \text{ for some } C_2 > 0. \quad (23)$$

Proof. (i). We first prove part one. Suppose that there is a traveling wave with speed $c = c' < \bar{c}$. Then by Theorem 1.5(a) in [4], it yields that (22) is not true. This contradiction implies that the minimal wave speed is nonlinearly selected.

(ii). For the second part, instead of the definition of wavefront in (8)-(9), we can alternatively re-phrase the definition of a traveling wave in terms of semiflow as in Liang and Zhao [7]. Assume that $Q(u_0) = u(t, x, u_0)$ is the solution semiflow induced by (1) with the initial function $u_0(x)$ to be continuous, nonnegative and bounded. A traveling wave solution $\phi(s, x)$, with $\phi(-\infty, x) = 0$, $\phi(\infty, x) = 1$, should satisfy

$$Q[\phi(x \cdot e, x)] = \phi(x \cdot e + ct, x). \quad (24)$$

Due to the Laplacian operator in the equation, one can easily get that the semiflow Q is compact and strongly positive.

We assume that the minimal speed c^* is nonlinearly selected; that is, $c^* > c_0$. We proceed to show that at the speed $c = c^*$, the traveling wave $W_{c^*}(s, x)$ satisfies

$$W_{c^*}(s, x) \sim C\Psi_{\mu_2(c^*)}(x)e^{\mu_2(c^*)s} \text{ as } s \rightarrow -\infty \quad (25)$$

for some constant C . By (18) and (19), assume to the contrary that

$$W_{c^*}(s, x) \sim C_3 \Psi_{\mu_1(c^*)}(x) e^{\mu_1(c^*)s} \quad \text{as } s \rightarrow -\infty, \quad (26)$$

for some positive constant C_3 and eigenvector $\Psi_{\mu_1(c^*)}$. We will prove that the operator Q has a traveling wave $W_c(x \cdot e, x)$ satisfying

$$Q(W_c) = W_c(x \cdot e + ct, x) \quad \text{or} \quad T_{ct}Q(W_c) = W_c, \quad (27)$$

for some speed $c = c^* - \delta$, where $T_{ct}W(s) = W(x - ct)$ is the right-shifting operator, and δ is a sufficiently small and positive number. Hence, c^* is not the minimal speed and this will lead to a contradiction.

Indeed, under assumption (26), we define

$$\bar{W} = W_{c^*}(s, x)\omega(s, x), \quad \text{where } \omega(s, x) = \frac{1}{1 + \frac{\Psi_{\mu_1(c)}(x)}{\Psi_{\mu_1(c^*)}(x)} \delta e^{-(\mu_1(c) - \mu_1(c^*))s}}. \quad (28)$$

As δ is sufficiently small, \bar{W} is close to W_{c^*} but with a different decaying rate at $-\infty$. We will show the existence of a solution to (27) when δ is small. In (27), set

$$W_c = \bar{W} + V \quad (29)$$

and the equation for V is given by

$$T_{ct}Q(\bar{W} + V) = \bar{W} + V, \quad (30)$$

where $V = V(s, x)$ is a function to be determined. After a simple calculation, it follows that

$$V = T_{c^*t}M(W_{c^*})V + F_0 + M_\delta V + F_{high}(V), \quad (31)$$

where

$$F_0 = T_{ct}Q(\bar{W}) - \bar{W}, \quad (32)$$

$$M_\delta V = [T_{ct}M(\bar{W}) - T_{c^*t}M(W_{c^*})] V \quad (33)$$

and

$$F_{high}(V) = T_{ct}Q(\bar{W} + V) - T_{ct}Q(\bar{W}) - T_{ct}M(\bar{W})V. \quad (34)$$

Here $M(\bar{W})$ is the Fréchet derivative of Q around the function \bar{W} . With a simple estimate, it follows $M_\delta V = O(\delta)V$, $F_0 = O(\delta)$, satisfying

$$F_0 = o(e^{\mu_1(c^*)s}) \quad \text{as } s \rightarrow -\infty.$$

For a solution to (31), first we recall that for V in the space C_0 , where

$$C_0 = \{u \in C(\mathbb{R} \times [0, L], \mathbb{R}) : u(\pm\infty, x) = 0\},$$

$M(W_{c^*})$ is defined by

$$M(W_{c^*})[V] = \lim_{\rho \rightarrow 0} \frac{Q[W_{c^*} + \rho V] - Q[W_{c^*}]}{\rho}.$$

Obviously the operator $T_{c^*t}M(W_{c^*})$ is compact and strongly positive, with a principal eigenvalue $\lambda = 1$ and the corresponding eigenvector $\bar{v} = W'_{c^*}$. It is not difficult to see that W'_{c^*} shares the same decaying behavior as that of W_{c^*} , i.e.,

$$W'_{c^*} \sim C(x)e^{\mu_1(c^*)s} \quad \text{as } s \rightarrow -\infty \quad (35)$$

for some periodic function $C(x)$, where W'_{c^*} represents the first derivative of $W_{c^*}(s, x)$ with respect to s .

To omit this eigenvector \bar{v} , we further construct a weighted space \mathcal{V} as

$$\mathcal{V} = \{v \in C_0 : ve^{-\mu_1(c)s} = o(1) \text{ as } s \rightarrow -\infty, \}$$

where $c = c^* - \delta$. Therefore, we see that the eigenvector $\bar{v} = W'_{c^*}$ is not in \mathcal{V} , which implies that $T_{c^*t}M(W_{c^*})$ has no eigenvalue $\lambda = 1$ in \mathcal{V} . Since the operator $T_{c^*t}M(W_{c^*})$ is compact and strongly positive in \mathcal{V} , it follows that $T_{c^*t}M(W_{c^*}) - I$ has a bounded inverse in \mathcal{V} , where I is the identity operator. Using the inverse function theorem in the space \mathcal{V} , we obtain that there exists a small positive number δ_0 so that problem (31) has a solution V for any $\delta \in [0, \delta_0)$. Back to (29), it follows that we have a solution W_c for $c = c^* - \delta$. The positivity of W_c can be guaranteed by the choice of a sufficiently small δ . This completes the proof. \square

Remark 2.1. *In the proof of the first part of Theorem 2.1, we have made use of the result Theorem 1.5 of [4]. Actually we can give a direct and easy proof in the case when the positive equilibrium is exponentially stable. Suppose that (22) is true. We proceed to prove that (8) has no traveling waves for any c in (c_0, \bar{c}) . To the contrary, suppose that for some $c \in (c_0, \bar{c})$, there exists a traveling wave $W_c(x \cdot e, x)$ satisfying either (18) or (19). In view of the monotonicity of $\mu_1(c)$ and $\mu_2(c)$ in c , we get $W_c(s, x) > \phi_{\bar{c}}(s, x)$ for s near $-\infty$. To find the behavior of this solution near ∞ , let $\bar{k}(-\gamma)$ be the principal eigenvalue of the linear operator $L_{c,q,\eta,-\gamma}$ defined in (13) (but with η replaced by $\partial_u f(x, 1)$). By linearizing equation (8) at 1, we obtain a characteristic equation $-\gamma c - \bar{k}(-\gamma) = 0$. We may assume*

$$\bar{k}(0) < 0. \quad (36)$$

Based on the convexity of $\bar{k}(-\gamma)$, it follows that there exists a unique positive γ as the solution of the characteristic equation and γ is a decreasing function in c for $c \geq c_0$. This implies

$$W_c \sim 1 - \Psi_\gamma(x)e^{-\gamma x} \quad (37)$$

for some positive γ and positive function $\Psi_\gamma(x)$. In view of the monotonicity of γ in c , this further yields that $\phi_{\bar{c}}(s, x) \ll W_c(s, x)$ for s near ∞ . Therefore, it is always possible to make a shift of distance ξ_0 for the variable s in $W_c(s, x)$ such that

$$\bar{W}_c(x \cdot e, x) = W_c(x \cdot e + \xi_0, x) > \phi_{\bar{c}}(x \cdot e, x).$$

The monotonicity of the map Q implies

$$\bar{W}_c(x \cdot e + ct, x) = Q(\bar{W}_c(x \cdot e, x)) \geq Q(\phi_{\bar{c}}(x \cdot e, x)) = \phi_{\bar{c}}(x \cdot e + \bar{c}t, x) \quad (38)$$

for $x \in (-\infty, \infty)$. On the other hand, on the line $x \cdot e + t\bar{c} = z_0$ for some fixed value z_0 , it follows that $\phi_{\bar{c}}(x \cdot e + \bar{c}t, x) = \phi_{\bar{c}}(z_0, x) > 0$ and

$$\bar{W}_c(x \cdot e + ct, x) = \bar{W}_c(z_0 - t(\bar{c} - c), x) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

which contradicts (38). As such, there exist no traveling waves for Q for $c \in (c_0, \bar{c})$. It follows now, from Theorem A, that we cannot have traveling waves with speed $c = c_0$. Thus, the first part of Theorem 2.1 is proved.

Remark 2.2. In the degenerate case when $\bar{k}(0) = 0$, we speculate that the above idea and argument still work, as long as we can show that the traveling wave solution is non-increasing in c , as $s \rightarrow \infty$. This could be done by constructing upper solution $\hat{\phi} = 1$ and lower solution $\phi = \phi_{\bar{c}}$ for (8) for $\bar{c} > c, s \geq s_0$, where s_0 is a given constant. The uniqueness of the wavefront (up to translation) may be used. We will leave this to interested readers.

Remark 2.3. The second part of our Theorem 2.1 confirms the conjecture in [4, page, 363].

Remark 2.4. Although we have unveiled the important feature of the pushed wavefront in Theorem 2.1, practically we cannot establish criteria by Theorem 2.1 for speed selection, since exact traveling wave formulas are unknown. Next we will develop certain easy-to-apply formulas to determine the speed selection mechanism, based on constructions of upper or lower solutions that may approximate the exact traveling waves to some extent. The establishment of these criteria don't rely on Theorem 2.1 and it can be of independent interest to readers.

2.2. Linear selection.

Theorem 2.2 (Linear Selection). *Let c_0 be defined in (12). Further assume that there exists a continuous and positive function $U(s, x)$ satisfying*

$$L_{c_0}U + f(x, U) \leq 0, \quad (39)$$

and

$$\liminf_{x \rightarrow \infty} U(s, x) > 0, \quad \lim_{s \rightarrow -\infty} U(s, x) = 0. \quad (40)$$

Then the linear selection is realized.

Proof. Similar to what is done in [5] and [7], we can define the leftward spreading speed c^* as

$$c^* := \sup\{c : \lim_{i \rightarrow -\infty, i \in \mathbb{Z}} a(c; iL + \theta) = 1, \theta \in [0, L]\} \quad (41)$$

where

$$a(c; x) = \lim_{n \rightarrow \infty} a_n(c; x).$$

In our setting, for a given real number c , the sequence of functions $\{a_n\}_{n=0}^{\infty}$ is defined as

$$a_0(c; x) = \phi(x), \quad a_{n+1}(c; x) = R_c[a_n(c; \cdot)](x), \quad (42)$$

and

$$R_c[a_n](x) = \max\{\phi(x), T_c[Q_1[a_n]](x)\}, \quad (43)$$

where $\phi(x)$ is non-decreasing function that satisfies

$$\phi(x) = 0 \quad \text{for } x \leq 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} (\phi(x) - \omega) = 0,$$

$0 < \omega < 1$, and Q_1 is the solution semiflow Q_t at $t = 1$. Here, the limit in (41) is obtained by splitting the variable x interval-by-interval with each interval length as L . c^* is independent of the choice of ϕ , see [5, 7]. Therefore, we can let $\phi(\infty)$ be small so that the upper solution U (or a shift of U if needed) satisfies

$$a_0(c_0; x) \leq U(x \cdot e, x) \quad (44)$$

for all $x \in (-\infty, \infty)$. From (43), (42), (40) and (39), by induction, it follows that

$$a_{n+1}(c_0; x) \leq U(x \cdot e, x), \quad n \geq 0.$$

Thus $a(c_0; -\infty) = 0$. By (41), we have $c^* \leq c_0$. Therefore, we arrive at $c^* = c_0$ by Theorem A, and the linear selection is realized. \square

Corollary 2.1. *Suppose that $f(x, u) \leq f'(x, 0)u$. Then the linear selection is realized.*

Proof. For $c = c_0$, one can easily verify that $U = e^{\bar{\mu}s} \Psi_{\bar{\mu}}(x)$ is an upper solution of the wave profile equation, where $\bar{\mu}$ is defined in (15). \square

Corollary 2.2. *Let*

$$\bar{\phi}(s, x) := \frac{\Psi(x)}{\Psi(x) + e^{-\mu_1 s}} \quad (45)$$

where Ψ is the principal eigenfunction of (13) corresponding to $\lambda = \mu_1 = \bar{\mu}$ and the principal eigenvalue $k(\mu_1) = \mu_1 c$ for $c = c_0$. Then the minimal speed is linearly selected if

$$-2\mu_1^2 \bar{\phi} \Psi - 2\bar{\phi} \frac{|\nabla \Psi|^2}{\Psi} - 4\mu_1 \bar{\phi} e \cdot \nabla \Psi + \frac{\Psi f(x, \bar{\phi})}{\bar{\phi}(1 - \bar{\phi})} - \eta(x) \Psi \leq 0 \quad (46)$$

Proof. We note that

$$\bar{\phi}_s(s, x) = \mu_1 \bar{\phi} (1 - \bar{\phi}) \quad \text{and} \quad \bar{\phi}_{ss}(s, x) = \mu_1^2 \bar{\phi} (1 - \bar{\phi}) (1 - 2\bar{\phi}) \quad (47)$$

and

$$1 - \bar{\phi}(s, x) = \frac{e^{-\mu_1 s}}{\Psi(x) + e^{-\mu_1 s}} \quad \text{for all } (s, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Also,

$$\nabla_x \bar{\phi}(s, x) = \frac{e^{-\mu_1 s} \nabla \Psi(x)}{(\Psi(x) + e^{-\mu_1 s})^2} = \bar{\phi} (1 - \bar{\phi}) \frac{\nabla \Psi(x)}{\Psi(x)},$$

which leads to

$$\nabla_x \bar{\phi}_z = \mu_1 \bar{\phi} (1 - \bar{\phi}) (1 - 2\bar{\phi}) \frac{\nabla \Psi}{\Psi}.$$

Moreover,

$$\begin{aligned} \Delta_x \bar{\phi} &= (\nabla \bar{\phi} - 2\bar{\phi} \nabla_x \bar{\phi}) \cdot \frac{\nabla \Psi}{\Psi} + \bar{\phi} (1 - \bar{\phi}) \frac{\Delta \Psi}{\Psi} \frac{|\nabla \Psi|^2}{|\Psi|^2} \\ &= \bar{\phi} (1 - \bar{\phi}) (1 - 2\bar{\phi}) \frac{|\nabla \Psi|^2}{|\Psi|^2} + \bar{\phi} (1 - \bar{\phi}) \frac{\Delta \Psi}{\Psi} + \bar{\phi} (1 - \bar{\phi}) \frac{|\nabla \Psi|^2}{|\Psi|^2}. \end{aligned}$$

Now, we substitute the above quantities in $L_c \bar{\phi} + f(x, \bar{\phi})$ to obtain

$$\begin{aligned} &L_c \bar{\phi} + f(x, \bar{\phi}) \\ &= \frac{\bar{\phi}(1-\bar{\phi})}{\Psi} \left\{ \mu_1^2 (1-2\bar{\phi}) \Psi - 2\bar{\phi} \frac{|\nabla \Psi|^2}{\Psi} + \Delta \Psi \right. \\ &\quad \left. + 2\mu_1 (1-2\bar{\phi}) e \cdot \nabla \Psi + q \cdot \nabla \Psi + \mu_1 q \cdot e \Psi - c\mu_1 \Psi + \frac{\Psi f(x, \bar{\phi})}{\bar{\phi}(1-\bar{\phi})} \right\} \\ &= \frac{\bar{\phi}(1-\bar{\phi})}{\Psi} \left\{ k(\mu_1) \Psi - c\mu_1 \Psi - 2\mu_1^2 \bar{\phi} \Psi - 2\bar{\phi} \frac{|\nabla \Psi|^2}{\Psi} \right. \\ &\quad \left. - 4\mu_1 \bar{\phi} e \cdot \nabla \Psi + \frac{\Psi f(x, \bar{\phi})}{\bar{\phi}(1-\bar{\phi})} - \eta(x) \Psi \right\} \\ &= \frac{\bar{\phi}(1-\bar{\phi})}{\Psi} \left\{ -2\mu_1^2 \bar{\phi} \Psi - 2\bar{\phi} \frac{|\nabla \Psi|^2}{\Psi} - 4\mu_1 \bar{\phi} e \cdot \nabla \Psi + \frac{\Psi f(x, \bar{\phi})}{\bar{\phi}(1-\bar{\phi})} - \eta(x) \Psi \right\}. \end{aligned} \quad (48)$$

The last line in the above equation simplifies the previous one since $k(\mu_1) - c\mu_1 = 0$. Therefore, $\bar{\phi}$ is an upper solution for $c = c_0$. This completes the proof. \square

2.3. Nonlinear selection.

Theorem 2.3 (Nonlinear selection). *For $c_1 > c_0$, suppose that there exists a function $V(s, x)$ satisfying*

$$0 < V(s, x) < 1, \quad \limsup_{s \rightarrow \infty} V(s, x) < 1, \quad V(s, x) = \Psi_{\mu_2(c_1)}(x) e^{\mu_2(c_1)s} \quad \text{as } s \rightarrow -\infty \quad (49)$$

and

$$L_{c_1}V + f(x, V) \geq 0, \quad (50)$$

where $\mu_2(c_1)$ is defined in (16). Then, $c^* \geq c_1$ and no traveling waves exist for $c \in [c_0, c_1)$. In other words, the nonlinear selection is realized.

Proof. The proof is similar to that of Remark 2.1 and it is omitted. With the second condition in (49), there is no need of the condition $\bar{k}(0) < 0$. \square

Corollary 2.3. For $c = c_0 + \varepsilon$, where ε is a sufficiently small number, let

$$\underline{\phi}(s, x) := \frac{\Psi(x)}{\Psi(x) + e^{-\mu_2 s}}. \quad (51)$$

If

$$-2\mu_2^2 \underline{\phi} \Psi - 2\underline{\phi} \frac{|\nabla \Psi|^2}{\Psi} - 4\mu_2 \underline{\phi} e \cdot \nabla \Psi + \frac{\Psi f(x, \underline{\phi})}{\underline{\phi}(1-\underline{\phi})} - \eta(x)\Psi > 0, \quad (52)$$

then nonlinear selection is realized.

Proof. Similar computations to the ones performed on $\bar{\phi}$, above, yield that

$$\begin{aligned} L_c \underline{\phi} + f(x, \underline{\phi}) &= \frac{\phi(1-\phi)}{\Psi} \left\{ \mu_2^2(1-2\underline{\phi})\Psi - 2\underline{\phi} \frac{|\nabla \Psi|^2}{\Psi} + \Delta \Psi + 2\mu_2(1-2\underline{\phi})e \cdot \nabla \Psi \right. \\ &\quad \left. + q \cdot \nabla \Psi + \mu_2 q \cdot e \Psi - c\mu_2 \Psi + \frac{\Psi f(x, \underline{\phi})}{\underline{\phi}(1-\underline{\phi})} \right\} \\ &= \frac{\phi(1-\phi)}{\Psi} \left\{ k(\mu_2)\Psi - c\mu_2 \Psi - 2\mu_2^2 \underline{\phi} \Psi - 2\underline{\phi} \frac{|\nabla \Psi|^2}{\Psi} \right. \\ &\quad \left. - 4\mu_2 \underline{\phi} e \cdot \nabla \Psi + \frac{\Psi f(x, \underline{\phi})}{\underline{\phi}(1-\underline{\phi})} - \eta(x)\Psi \right\} \\ &= \frac{\phi(1-\phi)}{\Psi} \left\{ -2\mu_2^2 \underline{\phi} \Psi - 2\underline{\phi} \frac{|\nabla \Psi|^2}{\Psi} - 4\mu_2 \underline{\phi} e \cdot \nabla \Psi + \frac{\Psi f(x, \underline{\phi})}{\underline{\phi}(1-\underline{\phi})} - \eta(x)\Psi \right\}, \end{aligned}$$

since $k(\mu_2) - c\mu_2 = 0$. Hence, the result directly follows from Theorem 2.3 by taking $V = (1 - \eta)\underline{\phi}$ for a sufficiently small η . \square

Theorem 2.3 gives the lower estimate for the minimal speed. We can also provide an upper estimate for the minimal speed when the nonlinear selection is realized.

Theorem 2.4 (upper bound for the minimal speed). For $c_2 > c_0$, suppose that there exists a function $V_2(s, x)$ satisfying

$$0 < V_2(s, x) < 1, \quad \limsup_{s \rightarrow \infty} V_2(s, x) \leq 1, \quad V_2(s, x) = \Psi_{\mu_2(c_2)}(x) e^{\mu_2(c_2)s} \quad \text{as } s \rightarrow -\infty, \quad (53)$$

and

$$L_{c_1}V + f(x, V) \leq 0, \quad (54)$$

where $\mu_2(c_2)$ is defined in (16). Then, $c^* \leq c_2$.

Proof. The proof follows from the comparison principal and it is similar to that of Theorem 2.2, as long as we choose the initial function $\phi(x)$ in (42) less than $V_2(x \cdot e, x) = V(x \cdot e + ct, x)|_{t=0}$. \square

3. APPLICATION

We consider a simple case where $N = 1$ and the advection q is a constant. Now equation (1) reads

$$u_t = u_{xx} + qu_x + f(x, u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}. \quad (55)$$

Here f is a modified KPP-Fisher function with the Allee effect

$$f(x, u) = u(1 - u)(1 + a(x)u), \quad (56)$$

where $a(x)$ is a positive periodic function. Since $\eta(x) = 1$, we get $\Psi = 1$ and it is easy to work out the linear speed as

$$c_0 = q + 2. \quad (57)$$

For any $c > q + 2$, from (16), we have the equation

$$\lambda^2 + (q - c)\lambda + 1 = 0 \quad (58)$$

with two solutions

$$\lambda = \mu_1(c) = \frac{c - q - \sqrt{(c - q)^2 - 4}}{2}, \quad \lambda = \mu_2(c) = \frac{c - q + \sqrt{(c - q)^2 - 4}}{2}. \quad (59)$$

When $c = c_0$, we have $\mu_1(c_0) = \mu_2(c_0) = 1$.

Applying Corollaries 2.2 and 2.3, we obtain that

$$c_{\min} = q + 2 \quad \text{if } a(x) \leq 2, \quad \text{for all } x$$

and

$$c_{\min} > q + 2 \quad \text{if } a(x) > 2 \quad \text{for all } x.$$

In the case where $a(x) > 2$ for all x , let

$$m = \min a(x), \quad M = \max a(x). \quad (60)$$

Then it can be derived that the minimal speed satisfies

$$q + \sqrt{\frac{m}{2}} + \sqrt{\frac{2}{m}} < c_{\min} < q + \sqrt{\frac{M}{2}} + \sqrt{\frac{2}{M}}. \quad (61)$$

This provides upper and lower estimates for the minimal speed.

4. SUMMARY

In this paper, we studied the speed selection for reaction diffusion equations in heterogeneous environments. The key feature of the nonlinear selection of the minimal speed was unveiled. We theoretically proved that the spreading speed (or the minimal speed) is linearly selected if we can find an upper solution with the linear speed; it is nonlinearly selected if we can find a lower solution with the faster decay rate, for some speed that is greater than the linear speed. In application, the bound of the minimal speed was provided for the case of nonlinear selection.

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REFERENCES

- [1] H. Berestycki, F. Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math., **55** (2002), pp 949-1032.
- [2] H. Berestycki, F. Hamel and G. Nadin, *Asymptotic spreading in heterogeneous diffusive excitable media*, J. Funct. Anal., 255 (2008), no. 9, 2146-2189.
- [3] H. Berestycki, F. Hamel, N.Nadirashvili, *The Speed of Propagation for KPP Type Problems (Periodic Framework)*, J. Eur. Math. Soc., **7** (2005), pp 173-213.
- [4] F. Hamel, *Qualitative properties of monostable pulsating fronts: exponential decay and monotonicity*, J. Math. Pures Appl., (9) 89 (2008), no. 4, 355-399.
- [5] J. Fang, X. Yu and X. Zhao, *Traveling waves and spreading speeds for time-space periodic monotone systems*, J. Funct. Anal. 272 (2017), no. 10, 4222-4262.
- [6] J. Fang and X. Zhao, *Bistable traveling waves for monotone semiflows with applications*. J. Eur. Math. Soc., (JEMS) 17 (2015), no. 9, 2243-2288.
- [7] X. Liang and X. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*. Comm. Pure Appl. Math., 60 (2007), no. 1, 1-40.
- [8] M. Lucia, C. B. Muratov, and M. Novaga. *Linear vs. nonlinear selection for the propagation speed of the solutions of scalar reaction-diffusion equations invading an unstable equilibrium*. *Communications on Pure and Applied Mathematics*, 57(5):616-636, 2004.
- [9] J.X. Xin, *Analysis and modeling of front propagation in heterogeneous media*, SIAM Review, **42** (2000), pp 161-230.