Propagation Direction of the Bistable Traveling Wavefront for Delayed Non-local Reaction Diffusion Equations

by

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Abstract

For delayed non-local reaction diffusion equations arising from population biology, selection mechanisms of the speed sign for the bistable traveling wavefront have not been found. In this paper, based on the theory of asymptotic speeds of spread for monotone semiflows, we firstly provide an interval of values of wave speed and a novel general condition for determining the speed sign by applying the comparison principle and the globally asymptotic stability of the bistable traveling wave. Moreover, through constructing novel upper/lower solutions, we give explicit conditions for the speed sign to be positive or negative. The obtained results are efficiently applied to three classical forms of the kernel functions.

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1 Introduction

We consider the following non-local delayed diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - du + \int_{-\infty}^{\infty} b(u(y, t-\tau))f(x-y)dy \tag{1.1}$$

that models the interaction of diffusion and time delay for a single population evolution. Here $D > 0$ denotes the diffusion rate, $d > 0$ is the death rate of the adult population, $\tau > 0$ measures the maturation time for a specific species, b represents the birth function and f is a disperse kernel function satisfying

$$f(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x)dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} |xf(x)|dx < \infty.$$

Besides the examples of kernel functions in Section 3, other important options for f can be found in [3]. For the derivation of this model and the biological interpretation of all functions and parameters, we refer to the references [1] and [8]-[12].

As in the reference [8], we assume that the birth function exhibits the so-called bistable non-linearity, that is, the equilibrium equation $du = b(u)$ has three solutions

$$u_1 = 0, \;\; u_2 = \alpha, \;\; u_3 = K, \quad 0 < \alpha < K \tag{1.2}$$

and the birth function has the property

$$b'(x) \geq 0 \;\; \text{for} \;\; x \in [0, K], \tag{1.3}$$

and

$$d > b'(0), \quad d < b'(\alpha), \quad d > b'(K). \tag{1.4}$$

By a traveling wavefront to (1.1), we mean a specific form of solution with $u(x,t) = U(x+ct)$, where c is the wave speed. Let $z = x + ct$. By (1.1) we then obtain the following wave profile equation

$$DU''(z) - cU'(z) - dU(z) + \int_{-\infty}^{\infty} b(U(z - c\tau - y))f(y)dy = 0 \tag{1.5}$$

subject to the boundary conditions

$$U(-\infty) = 0, \quad U(\infty) = K. \tag{1.6}$$

The existence, uniqueness and stability of traveling wavefront to (1.5)-(1.6) were investigated in [8]. In this paper we are concerned with the sign of the wave speed c that has never been studied before due to the difficulty in finding an explicit formula of c. One knows that the sign of the wave speed c indicates the propagation direction of traveling waves. Biologically, it implies the competitive result between the two states $u = 0$ and $u = K$ in the PDE model (1.1). Thus, this research is of great theoretical and practical significance.

For a traditional bistable model with the Allee effect, we refer to the Nagumo's equation

$$u_t = u_{xx} + au(1-u)(u-\alpha), \;\; a > 0, \;\; 0 < \alpha < 1, \tag{1.7}$$

which arises in the study of electrical impulses along nerves in the field of mathematical biology [2]. Its wavefront is governed by the following equation

$$U'' - cU' + aU(1-U)(U-a) = 0, \ \ U(-\infty) = 0, \ \ U(\infty) = 1. \tag{1.8}$$

The wave speed can be directly derived as

$$c = \frac{a\int_0^1 U(1-U)(U-a)dU}{\int_{-\infty}^{\infty} (U'(z))^2 \, dz} \tag{1.9}$$

so that the sign of c is determined by the integral $\int_0^1 U(1-U)(U-a)dU$. As to the value of c, although the denominator in (1.9) is difficult to work out, by using a further analytical technique, an explicit formula for the speed c of (1.8) is given by

$$c = \sqrt{\frac{a}{2}}(1-2\alpha),$$

which indicates that the wave speed is positive if $\alpha < \frac{1}{2}$ and negative if $\alpha > \frac{1}{2}$, see [2, 5] for the details.

Despite the success in obtaining the speed formula for the model (1.7), the determination of the speed sign of traveling wavefront for (1.1) has been outstanding for long decades. As such, in this paper, we are motivated to study this problem. Our new method is to construct subtle upper or lower solutions to approximate the traveling wavefront as well as its speed. we first establish an interval in which the wave speed attains its value. Next, general conditions for the positive or the negative wave speed are derived, respectively. Furthermore, we give particular conditions for determining the direction of wave propagation. Finally, the above results are applied to three classical forms of the kernel functions so that explicit conditions are obtained to guarantee that the speed sign is positive or negative. Furthermore, the idea can be significantly extended to handle a variety of bi-stable nonlinear systems.

The paper is organized as follows. In Section 2, we present the main results and Section 3 is devoted to three applications. Finally, we summarize the whole results in Section 4.

2 Sign of the Wave Speed.

We first recall the result obtained in [8] on the existence, uniqueness and stability of wavefront to (1.1).

Proposition 2.1. *[8] Assume that (1.3) and (1.4) hold. Equation (1.5)-(1.6) has exactly one traveling wavefront $U(z)$ with speed $c = \bar{c}$ satisfying $0 \le U \le K$ and $|\bar{c}| \le C$ for some positive constant C. The unique traveling wave $U(z)$ is strictly increasing and globally asymptotically stable with phase shift in the sense that there exists $\gamma > 0$ such that for any $\varphi \in [0, K]_C$ with*

$$\lim_{x\to\infty}\inf \min_{s\in[-\tau,0]} \varphi(s,x) > \alpha, \quad \lim_{x\to-\infty}\sup \max_{s\in[-\tau,0]} \varphi(s,x) < \alpha, \tag{2.1}$$

the solution $u(t,x,\varphi)$ of (1.1) with $u(s,x,\varphi) = \varphi(s,x)$, $s \in [-\tau, 0]$, $x \in (-\infty, \infty)$, satisfies

$$|u(t,x,\varphi) - U(x + \bar{c}t + \xi_0)| \le Me^{-\gamma t}, \quad t \ge 0, \ \ x \in (-\infty, \infty) \tag{2.2}$$

for some positive real number M and $\xi_0 \in (-\infty, \infty)$.

The above proposition mentions that the speed $\bar{c}$ satisfies $|\bar{c}| \leq C$ for a large constant C. However, they didn't provide any practical estimation on this C. To improve this, our first result is to get an interval estimate of the bistable speed $\bar{c}$. To this purpose, let $[\phi, \psi]_{\mathcal{C}}$ be the set $\{\omega \in \mathcal{C} : \phi \leq \omega \leq \psi\}$ and $\mathcal{C}$ denotes the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$. We recall that the bistable system implies two mono-stable inside-systems restricted to the phase sets $[\alpha, K]_C$ and $[0, \alpha]_C$, respectively. Let $c^*_-(\alpha, K)$ denote the leftward spreading speed of the semi-flow induced by (1.1) for the phase space confined to the set $[\alpha, K]_C$. For the solution semi-flow associated with system (1.1), $c^*_-(\alpha, K)$ is well defined and it is equal to the minimal speed of the traveling wave solutions (1.5) subject to boundary conditions

$$U(-\infty) = \alpha, \quad U(\infty) = K, \tag{2.3}$$

see [4]-[7] and [13] for details. Furthermore, we can have an estimate

$$c^*_-(\alpha, K) \geq c^0_-(\alpha, K),$$

where $c^0_-(\alpha, K)$ is the linear speed of (1.1) with b replaced by its linearization at $u = \alpha$, i.e., by $b'(\alpha)u(t - \tau, y)$. Similarly, let $c^*_+(0, \alpha)$ be the rightward spreading speed of (1.1) with the phase space confined to $[0, \alpha]_C$, and then it is equal to the minimal speed of the traveling wave solutions (1.5) subject to boundary conditions

$$U(-\infty) = 0, \quad U(\infty) = \alpha.$$

Also, we have

$$c^*_+(0, \alpha) \geq c^0_+(0, \alpha),$$

where $c^0_+(0, \alpha)$ is the rightward spreading speed of the linearized system of (1.1) at $u = \alpha$. As a remark, it can be easy to get the result

$$c^0_-(\alpha, K) = c^0_+(0, \alpha).$$

To proceed, we can obtain the following result.

Theorem 2.1. *Assume that (1.3) and (1.4) are true. The wave speed $\bar{c}$ of the traveling wave $U(z)$ to (1.5)-(1.6) satisfies the following estimate*

$$-c^*_+(0, \alpha) \leq \bar{c} \leq c^*_-(\alpha, K).$$

Proof. We only show $\bar{c} \leq c^*_-(\alpha, K)$, since the proof of $-c^*_+(0, \alpha) \leq \bar{c}$ is similar and omitted here.

Consider the mono-stable wave located in $[\alpha, K]_C$. It is easy to know that (1.5) subject to (2.3) has an increasing traveling wave $U_1(z)$ with the minimal speed $c = c^*_-(\alpha, K)$, which can be immediately derived from the general result in [6]. For (1.1), through choosing the initial data $\varphi(s, x)$ satisfying (2.1) and

$$\varphi(s, x) \leq U_1(x + c^*_-(\alpha, K)s), \quad s \in [-\tau, 0],$$

we have, by the comparison principle, that

$$u(t,x,\varphi) \le U_1(x + c_-^*(\alpha, K)t), \quad \text{for} \;\; t \ge 0.$$

Applying the inequality (2.2) in Proposition 2.1, we know that

$$U(x + \bar{c}t + \xi_0) - Me^{-\gamma t} \le U_1(x + c_-^*(\alpha, K)t) \tag{2.4}$$

for all $t \ge 0$, where U is the bistable traveling wave in Proposition 2.1. When t is sufficiently large, this naturally gives

$$\bar{c} \le c_-^*(\alpha, K).$$

Indeed, assume to the contrary that

$$\bar{c} > c_-^*(\alpha, K).$$

Fix a value $z = x + c_-^*(\alpha, K)t$. By sending $t \to \infty$, from (2.4), we can arrive at $K \le U_1(z)$, which is a contradiction. The proof is complete. ∎

This theorem provides only an interval estimate for the value of the bistable wave speed. The following theorem presents general conditions for the sign of the bistable wave speed to be positive or negative.

Theorem 2.2. *Suppose that (1.3) and (1.4) are satisfied. If there exists a function $\bar{U}$ such that it is an upper solution to (1.5) with speed $c = -\varepsilon$, and satisfies*

$$0 < \bar{U} < K, \qquad \textit{and} \qquad \liminf_{z \to \infty} \bar{U}(z) > \alpha,$$

then the speed $\bar{c}$ of (1.5)-(1.6) is negative. Similarly, if there exists a lower solution $\underline{U}$ of (1.5) with speed $c = \varepsilon$, satisfying

$$0 < \underline{U} < K, \qquad \textit{and} \qquad \limsup_{z \to -\infty} \underline{U}(z) < \alpha,$$

then the speed $\bar{c}$ of (1.5)-(1.6) is positive. Here ε is a small positve number,

Proof. Due to the fact that the two results share a similar argument, we will prove only the first part.

For the model (1.1), we take the initial data satisfying (2.1) and

$$\varphi(s,x) \le \bar{U}(x - \varepsilon s), \quad s \in [-\tau, 0].$$

Since $\bar{U}(x - \varepsilon t)$ is an upper solution of the model, by applying the comparison principle we have

$$u(t,x,\varphi) \le \bar{U}(x - \varepsilon t), \quad t \ge 0.$$

This and the inequality (2.2) in Proposition 2.1 yield

$$U(x + \bar{c}t + \xi_0) - Me^{-\gamma t} \le u(t,x,\varphi) \le \bar{U}(x - \varepsilon t) \tag{2.5}$$

for all $t \ge 0$. When t is sufficiently large, (2.5) gives

$$\bar{c} \le -\varepsilon,$$

that is, the speed $\bar{c}$ is negative. The proof is complete. ∎

In order to further study the sign determinacy of $\bar{c}$, we begin with denoting the characteristic equation of (1.5) by

$$F(\mu) = D\mu^2 - c\mu - d + b'(0)e^{-c\tau\mu}k(\mu) = 0, \tag{2.6}$$

which is obtained by linearizing (1.5) at $U = 0$. Here

$$k(\mu) = \int_{-\infty}^{\infty} e^{-\mu y} f(y)dy$$

and $f(y)$ is assumed to make the above integral absolutely convergent. From (2.6), it follows that

$$F(0) = -d + b'(0) < 0, \quad F(+\infty) = +\infty, \quad \text{and} \quad F''(\mu) > 0.$$

This means that, for each $c \in (-\infty, \infty)$, there exists a unique positive number $\mu = \mu_1(c) > 0$ such that $F(\mu_1) = 0$. Obviously, $\mu_1(c)$ is a continuous function of the wave speed c. In particular, if $c = 0$, then

$$\bar{\mu} = \mu_1(0) \tag{2.7}$$

is the unique positive solution of the equation

$$D\bar{\mu}^2 - d + b'(0)k(\bar{\mu}) = 0. \tag{2.8}$$

The following corollary establishes explicit conditions for the determination of the sign of the bistable wave speed of (1.5)-(1.6).

Corollary 2.1. *Suppose that (1.3) and (1.4) hold. Let*

$$\Psi(z) = -2D\bar{\mu}^2 + \frac{\int_{-\infty}^{\infty} b(\Phi(z-y))f(y)dy - b'(0)k(\bar{\mu})\Phi(1 - \frac{\Phi}{K_1}) - d\frac{\Phi^2}{K_1}}{\frac{\Phi^2}{K_1}(1 - \frac{\Phi}{K_1})}, \tag{2.9}$$

where

$$\Phi(z) = \frac{1}{\frac{1}{K_1} + e^{-\bar{\mu}z}}$$

for some positive number K_1 satisfying $\alpha < K_1 \le K$. Then we have the following results:

- *if $\Psi(z) < 0$ for all $z \in (-\infty, \infty)$, then sign$(\bar{c}) = -1$;*
- *if $\Psi(z) > 0$ for all $z \in (-\infty, \infty)$, then sign$(\bar{c}) = 1$;*
- *if $\Psi(z) \equiv 0$ for all $z \in (-\infty, \infty)$, then $\bar{c} = 0$,*

where $\bar{c}$ is the speed of the traveling wave solution $U(z)$ to (1.5)-(1.6) defined in Proposition 2.1, and $\overline{\mu}$ is defined by (2.7)-(2.8).

Proof. We prove only the first result because the proofs of the other two are similar.

Let $c = -\varepsilon$ and $\mu_1(c)$ be the solution of (2.6). Then by the choice of function

$$\phi = \frac{1}{\frac{1}{K_1} + e^{-\mu_1(c)z}},$$

it is easy to get

$$\phi' = \mu_1\phi(1-\frac{\phi}{K_1}), \quad \phi'' = \mu_1^2\phi(1-\frac{\phi}{K_1})(1-\frac{2\phi}{K_1}).$$

We shall verify that ϕ is an upper solution to (1.5). To this end, substituting ϕ into (1.5), we have that the left-hand side is equal to

$$\phi\left(1-\frac{\phi}{K_1}\right)\left[F(\mu_1)+\frac{\phi}{K_1}\Theta\right], \tag{2.10}$$

where

$$\Theta = -2D\mu_1^2 + \frac{\int_{-\infty}^{\infty} b(\phi(z-c\tau-y))f(y)dy - b'(0)k(\mu_1)e^{-c\tau\mu_1}\phi\left(1-\frac{\phi}{K_1}\right) - \frac{d\phi^2}{K_1}}{\frac{\phi^2}{K_1}(1-\frac{\phi}{K_1})}.$$

Note that $F(\mu_1)=0$ and $0<\phi<K_1$. Thus, the sign of the formula (2.10) just depends on Θ. It is easy to see that

$$\mu_1 \sim \bar{\mu}, \quad \phi \to \Phi \quad \text{as} \quad \varepsilon \to 0.$$

Then, we conclude that ϕ is an upper solution to (1.5) if $\Psi(z)<0$ for all $z\in(-\infty,\infty)$. By Theorem 2.2, we get

$$\text{sign}(\bar{c}) = -1.$$

The proof is complete. ∎

Remark 2.1. The testing function Φ in the above corollary satisfies the following differential equation

$$\Phi' = \bar{\mu}\Phi(1-\frac{\Phi}{K_1}), \ \Phi(-\infty)=0, \Phi(\infty)=K_1. \tag{2.11}$$

It gives a connection between 0 and K_1 and is natural to be a candidate for a smooth upper(lower) solution. However, this choice is not unique or optimal; other choices are possible.

Remark 2.2. The result in this corollary implies that $\Psi(z)$ can be regarded as the measurement of the competition ability of two states 0 and K.

3 Applications

In this section, we apply the general result obtained in the previous section to three kinds of specific kernel functions. New explicit conditions will be given for determining the propagation direction of the bistable traveling wave to (1.1). We fix the diffusion coefficient as $D=1$ and take the birth function by

$$b(u) = \begin{cases} pu^2e^{-u}, & u<2, \\ 4pe^{-2}, & u\geq 2, \end{cases} \tag{3.1}$$

where $p>de$ so that the equilibrium equation $du=b(u)$ has three solutions

$$u=0, \ \alpha, \quad K \quad \text{with} \quad 0<\alpha<1<K.$$

Obviously, the birth function b is non-decreasing and the conditions (1.3) and (1.4) are satisfied. In the case when $K > 2$, i.e., $p > \frac{1}{2}de^2$, there exists a constant $K_1, 2 < K_1 < K$, such that

$$pK_1^2 e^{-K_1} = dK_1. \tag{3.2}$$

In other words, when $p > \frac{1}{2}de^2$, there exists K_1 such that $2 < K_1 < K$; while for the case $p < \frac{1}{2}de^2$, it implies $1 < K_1 = K < 2$. Moreover, by a straightforward computation, we have $\bar{\mu} = \sqrt{d}$. Next, we shall demonstrate three applications to determine the speed direction.

3.1 The discrete delay case

If we take the kernel function as $f(x) = \delta(x)$ where δ is the dirac delta function, then (1.5) reduces to

$$cU' = U'' - dU + b(U(z - c\tau)) \tag{3.3}$$

subject to

$$U(-\infty) = 0, \quad U(\infty) = K. \tag{3.4}$$

By applying Corollary 2.1, we can obtain the following theorem.

Theorem 3.1. *The following results are true:*

(1) *if $K < \ln 3$, then the wave speed of (3.3)-(3.4) is negative. In this case, the bistable traveling wave propagates to the right, which implies that the stable state* 0 *wins the competition.*

(2) *if $K_1 > 2$, then the wave speed of (3.3)-(3.4) is positive. Thus, the bistable traveling wave spreads to the left, which indicates that the stable state K has competitive advantages.*

Proof. We first prove the result in (1). Let

$$\Phi(z) = \frac{1}{\frac{1}{K} + e^{-\sqrt{d}z}}. \tag{3.5}$$

From (2.9) in Corollary 2.1, it follows that

$$\Psi(z) = -2d + dK\frac{e^{K-\Phi} - 1}{K - \Phi}.$$

Here we have made use of the fact that $pK^2e^{-K} = dK$. Using the property of the function $\frac{e^x-1}{x}$, we conclude that

$$\max_{z\in(-\infty,\infty)} \Psi(z) = -2d + d(e^K - 1).$$

Therefore, when $K < \ln 3$, we get $\max_{z\in(-\infty,\infty)} \Psi(z) < 0$. Corollary 2.1 implies that the wave speed of (3.3)-(3.4) is negative.

For the second case when $K > K_1 > 2$, i.e., $p > \frac{1}{2}de^2$, we set

$$\Phi(z) = \frac{1}{\frac{1}{K_1} + e^{-\sqrt{d}z}} \tag{3.6}$$

where $K_1 > 2$ is the solution of (3.2). Therefore by comparison, we have

$$\Psi(z) \geq -2d + dK_1\frac{e^{K_1-\Phi} - 1}{K_1 - \Phi} \geq d(K_1 - 2) > 0,$$

which gives the desired result by Corollary 2.1. ∎

Remark 3.1. When $\int_0^K (b(u) - du)du = 0$, the wave speed is always zero, regardless of the value τ.

3.2 The infinitely distributed delay case

Now the kernel function is defined as an exponential distribution in the form of

$$f(x) = \begin{cases} 0, & x \in [0, +\infty), \\ m\sqrt{d}e^{m\sqrt{d}x}, & x \in (-\infty, 0), \end{cases} \tag{3.7}$$

where $m > 2$ is a positive integer. Then the system (1.5) reads

$$U''(z) - cU'(z) - dU(z) + m\sqrt{d} \int_{-\infty}^{0} b(U(z - c\tau - y))e^{m\sqrt{dy}}dy = 0 \tag{3.8}$$

subject to the boundary condition (3.4).

We now have the following theorem.

Theorem 3.2. *The following statement is true:*

(1) *assume* $K \leq 2$ *(i.e.,* $de < p \leq \frac{1}{2}de^2$ *) and*

$$K \frac{mK^m e^K \Phi^{m-2} (K - \Phi)^{-m} \int_\Phi^K t^{1-m} \left(1 - \frac{t}{K}\right)^{m-1} e^{-t}dt - 1}{K - \Phi} < 2 \tag{3.9}$$

for all $\Phi \in (0, K)$*. Then the wave speed of (3.8) with (3.4) is negative, which implies that the bistable traveling wave moves to the right direction, and thus the stable state* 0 *wins the competition.*

(2) *assume that* $K_1 > 2$ *exists for (3.2) (i.e.,* $p > \frac{1}{2}de^2$*) and*

$$K_1 \frac{mK_1^m e^{K_1} \Phi^{m-2} (K_1 - \Phi)^{-m} \int_\Phi^{K_1} t^{1-m} \left(1 - \frac{t}{K_1}\right)^{m-1} e^{-t}dt - 1}{K_1 - \Phi} > 2 \tag{3.10}$$

for all $\Phi \in (0, K_1)$*. Then the wave speed of (3.8) with (3.4) is positive, which implies that the bistable traveling wave moves to the left direction, and thus the stable state* K *wins the competition.*

Proof. (1). Similarly as before, let Φ be defined in (3.5). First we have

$$\int_{-\infty}^{\infty} b(\Phi(z - y))f(y)dy \quad = mp\sqrt{d}e^{m\sqrt{d}z} \int_z^{\infty} \Phi^2(s)e^{-\Phi(s)}e^{-m\sqrt{d}s}ds. \tag{3.11}$$

In view of

$$e^{-m\sqrt{d}s} = \Phi^{-m}(s) \left(1 - \frac{\Phi(s)}{K}\right)^m \quad \text{and} \quad pK^2e^{-K} = dK,$$

we get

$$\begin{aligned} \Psi(z) &= -2d + \frac{mpe^{m\sqrt{d}z} \int_{\Phi(z)}^K t^{1-m} \left(1 - \frac{t}{K}\right)^{m-1} e^{-t}dt - d\frac{\Phi^2}{K}}{\frac{\Phi^2}{K}\left(1 - \frac{\Phi}{K}\right)} \\ &= -2d + dK \frac{mK^m e^K \Phi^{m-2} (K - \Phi)^{-m} \int_{\Phi(z)}^K t^{1-m} \left(1 - \frac{t}{K}\right)^{m-1} e^{-t}dt - 1}{K - \Phi}. \end{aligned} \tag{3.12}$$

Thus by Corollary 2.1, the first part is proved. The proof of the second part is similar to that of the previous theorem by using the fact $b(\Phi) \geq p\Phi^2 e^{-\Phi}$. ∎

Remark 3.2. We conjecture that the maximal value of Ψ is attained when $\Phi \to 0$ and its minimal value is attained when $\Phi \to K$. A direct computation gives

$$\max_{z\in(-\infty,\infty)} \Psi(z) = -2d + d\Big(\frac{me^K}{m-2} - 1\Big). \tag{3.13}$$

Therefore the wave speed is negative if

$$-2d + d\Big(\frac{me^K}{m-2} - 1\Big) < 0.$$

Similarly for the minimal value, we assume $h(t) = t^{1-m}e^{-t}$ and have a Taylor expansion at K_1 as

$$h(t) = h(K_1) + h'(K_1)(t - K_1) + \cdots = e^{-K_1}K_1^{1-m} - e^{-K_1}K_1^{-m}(1 - m - K_1)(K_1 - t) + \cdots.$$

This gives

$$\min_{z\in(-\infty,\infty)} \Psi(z) = -2d + d\frac{m(K_1 + m - 1)}{m+1},$$

and the wave speed is positive if

$$-2d + d\frac{m(K_1 + m - 1)}{m+1} > 0.$$

3.3 The even distributed case

Let the kernel function follow a uniform distribution in an interval, that is,

$$f(x) = \begin{cases} \dfrac{1}{2a}, & x \in (-a, a), \\ 0, & \text{others}, \end{cases} \tag{3.14}$$

where a is a positive integer. Now the system (1.5) is transformed into

$$U''(z) - cU'(z) - dU(z) + \frac{1}{2a}\int_{-a}^{a} b(U(z - c\tau - y))dy = 0 \tag{3.15}$$

with the boundary condition (3.4).

Then, we get the result below.

Theorem 3.3. *The following statement holds:*

(1) *assume* $K \leq 2$ *(i.e.,* $de < p \leq \frac{1}{2}de^2$ *) and*

$$K\frac{\frac{Ke^K}{2a\sqrt{d}\Phi^2(z)}\int_{\Phi(z-a)}^{\Phi(z+a)} \frac{x}{K-x}e^{-x}dx - 1}{K - \Phi(z)} < 2, \quad z \in (-\infty, \infty), \tag{3.16}$$

where $\Phi(z) = \frac{1}{\frac{1}{K} + e^{-\sqrt{d}z}}$. *Then the wave speed of (3.8) with (3.4) is negative, which implies that the bistable traveling wave moves to the right direction, and thus the stable state* 0 *wins the competition.*

(2) *assume that* $K_1 > 2$ *is a solution of (3.2) (i.e.,* $p > \frac{1}{2}de^2$*) and*

$$K_1 \frac{\frac{K_1 e^{K_1}}{2a\sqrt{d}\Phi^2(z)} \int_{\Phi(z-a)}^{\Phi(z+a)} \frac{x}{K_1 - x} e^{-x} dx - 1}{K_1 - \Phi(z)} > 2, \;\; z \in (-\infty, \infty), \tag{3.17}$$

where $\Phi(z) = \frac{1}{\frac{1}{K_1} + e^{-\sqrt{d}z}}$*. Then the wave speed of (3.8) with (3.4) is positive, which implies that the bistable traveling wave moves to the left direction, and thus the stable state* K *wins the competition.*

Proof. (1). First we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} b(\Phi(z-y)) f(y) dy \\
&= \frac{p}{2a} \int_{-a}^{a} \Phi^2(z-y) e^{-\Phi(z-y)} dy = \frac{p}{2a} \int_{z-a}^{z+a} \Phi^2(s) e^{-\Phi(s)} ds \\
&= \frac{pK}{2a\sqrt{d}} \int_{\Phi(z-a)}^{\Phi(z+a)} \frac{\Phi(s)}{K - \Phi(s)} e^{-\Phi(s)} d\Phi(s) = \frac{pK}{2a\sqrt{d}} \int_{\Phi(z-a)}^{\Phi(z+a)} \frac{x}{K-x} e^{-x} dx
\end{aligned} \tag{3.18}$$

Using the equilibrium equation $d = pKe^{-K}$ on (3.18), we obtain

$$\begin{aligned}
\Psi(z) &= -2d + \frac{\int_{-\infty}^{\infty} b(\Phi(z-y)) f(y) dy - \frac{d\Phi^2}{K}}{\frac{\Phi^2}{K}(1 - \frac{\Phi}{K})} \\
&= -2d + dK \frac{\frac{Ke^K}{2a\sqrt{d}\Phi^2(z)} \int_{\Phi(z-a)}^{\Phi(z+a)} \frac{x}{K-x} e^{-x} dx - 1}{K - \Phi(z)}.
\end{aligned} \tag{3.19}$$

By Corollary 2.1, the first part is proved.

(2) . The second part of the proof is similar and omitted. ∎

4 Conclusion

In this work, the speed sign (i.e, propagation direction) of the bistable traveling wave to the nonlocal model (1.1) is studied for the first time. We rigorously prove that the bistable wave speed is bounded by the spreading speeds of two mono-stable waves, and that the sign of the bistable wave can be further determined by the speed of its upper or lower solutions. By constructing subtle upper or lower solutions, we obtain explicit conditions for determining the speed direction, which shows that the system parameters and the birth function jointly affects the propagation direction of the bistable traveling wave solution. As seen in the Application section, the obtained results are significant in the understanding of the competition of stable states in the biological environment. The idea can be extended to treat models with other significant kind of kernel functions such as (4)-(6) in [3].

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