Only (2.2.12) is satisfied by $u \equiv 0$ (of the linear conditions) and hence is homogeneous. It is not necessary that a boundary condition be $u(0,t) = 0$ for $u \equiv 0$ to satisfy it.

EXERCISES 2.2

2.2.1. Show that any linear combination of linear operators is a linear operator.

2.2.2. (a) Show that $L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator.

(b) Show that usually $L(u) = \frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$ is not a linear operator.

2.2.3. Show that $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(u,x,t)$ is linear if $Q = \alpha(x,t)u + \beta(x,t)$ and, in addition, homogeneous if $\beta(x,t) = 0$.

2.2.4. In this exercise we derive superposition principles for nonhomogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i, what is a particular solution for $f_1 + f_2$?

2.2.5 If L is a linear operator, show that $L(\sum_{n=1}^{M} c_n u_n) = \sum_{n=1}^{M} c_n L(u_n)$. Use this result to show that the principle of superposition may be extended to any finite number of homogeneous solutions.

2.3 Heat Equation with Zero Temperatures at Finite Ends

2.3.1 Introduction

Partial differential equation (2.1.1) is linear but it is homogeneous only if there are no sources, $Q(x,t) = 0$. The boundary conditions (2.1.3) are also linear, and they too are homogeneous only if $T_1(t) = 0$ and $T_2(t) = 0$. We thus first propose to study

$$\text{PDE:} \quad \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \quad \begin{array}{l} 0 < x < L \\ t > 0 \end{array} \tag{2.3.1}$$

$$\text{BC:} \quad \boxed{\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}} \tag{2.3.2}$$

EXERCISES 2.3

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

$*$(a) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$

(b) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} - v_0\dfrac{\partial u}{\partial x}$

$*$(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(d) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$

$*$(e) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^4 u}{\partial x^4}$

$*$(f) $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$

2.3.2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

*(b) $\phi(0) = 0$ and $\phi(1) = 0$

(c) $\dfrac{d\phi}{dx}(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$ (If necessary, see Sec. 2.4.1.)

*(d) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$

(e) $\dfrac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

*(f) $\phi(a) = 0$ and $\phi(b) = 0$ (You may assume that $\lambda > 0$.)

(g) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) + \phi(L) = 0$ (If necessary, see Sec. 5.8.)

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$

(b) $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

$*$(c) $u(x,0) = 2\cos\frac{3\pi x}{L}$

(d) $u(x,0) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$

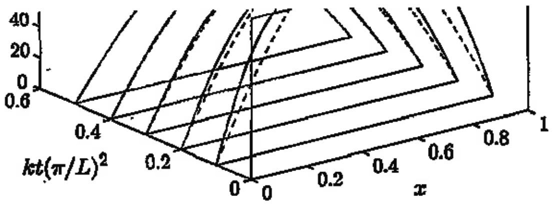

Figure 2.3.5 Time dependence of temperature (using the nfinite series) compared to the first term. Note the first erm is a good approximation if the time is not too small.

mary

ize the method of separation of variables as it appears for the one

$$\text{PDE:} \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$\text{BC:} \quad u(0,t) = 0, \quad u(L,t) = 0$$

$$\text{IC:} \quad u(x,0) = f(x).$$

e that you have a linear and homogeneous PDE with linear and ıous BC.

ily ignore the nonzero IC.

variables (determine differential equations implied by the assumption t solutions) and introduce a separation constant.

e separation constants as the eigenvalues of a boundary value prob-

er differential equations. Record all product solutions of the PDE e by this method.

e principle of superposition (for a linear combination of all product

to satisfy the initial condition.

e coefficients using the orthogonality of the eigenfunctions.

uld be *understood*, not memorized. It is important to note that

iple of superposition applies to solutions of the PDE (do not add up of various different ordinary differential equations).

ply the initial condition $u(x,0) = f(x)$ until *after* the principle of ion.

* (a) $\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

(b) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$

* (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$

* (e) $\frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4}$

* (f) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

2.3.2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

*(b) $\phi(0) = 0$ and $\phi(1) = 0$

(c) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$ (If necessary, see Sec. 2.4.1.)

*(d) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(e) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

*(f) $\phi(a) = 0$ and $\phi(b) = 0$ (You may assume that $\lambda > 0$.)

(g) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + \phi(L) = 0$ (If necessary, see Sec. 5.8.)

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$

(b) $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

* (c) $u(x,0) = 2\cos\frac{3\pi x}{L}$

(d) $u(x,0) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$

[Your answer in part (c) may involve certain integrals that do not need to be evaluated.]

2.3.4. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = 0, u(L,t) = 0$, and $u(x,0) = f(x)$.

*(a) What is the total heat energy in the rod as a function of time?

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

*(c) What relationship should exist between parts (a) and (b)?

2.3.5. Evaluate (be careful if $n = m$)

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \qquad \text{for } n > 0, m > 0.$$

Use the trigonometric identity

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right].$$

*2.3.6. Evaluate

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \qquad \text{for } n \geq 0, m \geq 0.$$

Use the trigonometric identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right].$$

(Be careful if $a - b = 0$ or $a + b = 0$.)

2.3.7. Consider the following boundary value problem (if necessary, see Sec. 2.4.1):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint:* The answer is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt}\cos\frac{n\pi x}{L}.\Bigg]$$

What is λ_n?

(c) Show that the initial condition, $u(x,0) = f(x)$, is satisfi

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

(d) Using Exercise 2.3.6, solve for A_0 and $A_n (n \geq 1)$.

(e) What happens to the temperature distribution as $t \to$ c
it approaches the steady-state temperature distribution

*2.3.8. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat lo
lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise
insulated lateral sides with a heat sink proportional to the
Suppose that the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

(a) What are the possible equilibrium temperature distribut

(b) Solve the time-dependent problem [$u(x,0) = f(x)$] if α
the temperature for large time ($t \to \infty$) and compare to

*2.3.9. Redo Exercise 2.3.8 if $\alpha < 0$. [Be especially careful if $-\alpha/k =$

2.3.10. For two- and three-dimensional vectors, the fundamental pr
products, $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta$, implies that

$$|\boldsymbol{A} \cdot \boldsymbol{B}| \leq |\boldsymbol{A}||\boldsymbol{B}|.$$

In this exercise we generalize this to n-dimensional vectors a
in which case (2.3.44) is known as Schwarz's inequality. [
Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that $|\boldsymbol{A} - \gamma\boldsymbol{B}|^2 > 0$ implies (2.3.44), where $\gamma = \boldsymbol{A}$

(b) Express the inequality using both

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

*(c) Generalize (2.3.44) to functions. [*Hint:* Let $\boldsymbol{A} \cdot \boldsymbol{B}$ mea
$\int_0^L A(x)B(x)\,dx$.]

2.3.11. Solve Laplace's equation inside a rectangle:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions

$$\begin{aligned} u(0,y) &= g(y) & u(x,0) &= 0 \\ u(L,y) &= 0 & u(x,H) &= 0. \end{aligned}$$

(*Hint:* If necessary, see Sec. 2.5.1.)

$$(x)\left\{\begin{matrix} L \\ \sin\frac{m\pi x}{L}\end{matrix}\right\} dx = \sum_{n=0} a_n \int_{-L} \cos\frac{n\pi x}{L}\left\{\begin{matrix} L \\ \sin\frac{m\pi x}{L}\end{matrix}\right\} dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\left\{\begin{matrix}\cos\frac{m\pi x}{L} \\ \sin\frac{m\pi x}{L}\end{matrix}\right\} dx.$$

e (2.4.40–2.4.42), we find that

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = a_m\int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,dx$$
$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = b_m\int_{-L}^{L}\sin^2\frac{m\pi x}{L}\,dx.$$

the coefficients in a manner that we are now familiar with yields

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ (m\geq 1)\quad a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx. \end{aligned} \tag{2.4.43}$$

n to the problem is (2.4.38), where the coefficients are given by (2.4.43).

ummary of Boundary Value Problems

roblems, including the ones we have just discussed, the specific simple efficient differential equation,

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

undamental part of the boundary value problem. We collect in the table e the relevant formulas for the eigenvalues and eigenfunctions for the ndary conditions already discussed. You will find it helpful to understand ts because of their enormous applicability throughout this text. It is o note that, in these cases, whenever $\lambda = 0$ is an eigenvalue, a constant function (corresponding to $n = 0$ in $\cos n\pi x/L$).

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0, 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} a_n\cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ $a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

EXERCISES 2.4

*2.4.1. Solve the heat equation $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, subject to

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t > 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0.$$

(a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ (b) $u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

(c) $u(x,0) = -2\sin\frac{\pi x}{L}$ (d) $u(x,0) = -3\cos\frac{8\pi x}{L}$

*2.4.2. Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{aligned} &\frac{\partial u}{\partial x}(0,t) = 0 \\ &u(L,t) = 0 \\ &u(x,0) = f(x). \end{aligned}$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

*2.4.3. Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

subject to

$$\phi(0) = \phi(2\pi) \quad \text{and} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

2.4.4. Explicitly show that there are no negative eigenvalues for

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{subject to} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0.$$

2.4.5. This problem presents an alternative derivation of the heat equation for a thin wire. The equation for a circular wire of finite thickness is the two-dimensional heat equation (in polar coordinates). Show that this reduces to (2.4.25) if the temperature does not depend on r and if the wire is very thin.

2.4.6. Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:

(a) Directly from the equilibrium problem (see Sec. 1.4)

(b) By computing the limit as $t \to \infty$ of the time-dependent problem

2.4.7. Solve Laplace's equation inside a circle of radius a,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

subject to the boundary condition

$$u(a,\theta) = f(\theta).$$

(*Hint:* If necessary, see Sec. 2.5.2.)

2.5 Laplace's Equation: Solutions and Qualitative Properties

2.5.1 Laplace's Equation Inside a Rectangle

In order to obtain more practice, we consider a different kind of proble analyzed by the method of separation of variables. We consider stea conduction in a two-dimensional region. To be specific, consider th temperature inside a rectangle ($0 \le x \le L$, $0 \le y \le H$) when the ten prescribed function of position (independent of time) on the boundary. rium temperature $u(x, y)$ satisfies Laplace's equation with the follow conditions:

PDE: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

BC1: $$u(0,y) = g_1(y)$$

BC2: $$u(L,y) = g_2(y)$$

BC3: $$u(x,0) = f_1(x)$$

BC4: $$u(x,H) = f_2(x),$$

where $f_1(x), f_2(x), g_1(y)$, and $g_2(y)$ are given functions of x and y Here the partial differential equation is linear and homogeneous, but conditions, although linear, are not homogeneous. We will not be the method of separation of variables to this problem in its present when we separate variables the boundary value problem (determinin tion constant) must have homogeneous boundary conditions. In th the boundary conditions are nonhomogeneous. We can get around th noting that the original problem is nonhomogeneous due to the fou neous boundary conditions. The idea behind the principle of superp used sometimes for nonhomogeneous problems (see Exercise 2.2.4). problem into four problems each having one nonhomogeneous condit

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),$$

where each $u_i(x,y)$ satisfies Laplace's equation with one nonhomogen condition and the related three homogeneous boundary conditions a

lposedness and uniqueness. The maximum principle is a very ıt tool for further analysis of partial differential equations, especially in ing qualitative properties (see, e.g., Protter and Weinberger [1967]). We a problem is **wellposed** if there exists a unique solution that depends usly on the nonhomogeneous data (i.e., the solution varies a small amount a are slightly changed). This is an important concept for physical problems. ution changed dramatically with only a small change in the data, then any measurement would have to be exact in order for the solution to be reliable. ely, most standard problems in partial differential equations are wellposed. ıple, the maximum principle can be used to prove that Laplace's equation with u specified as $u = f(x)$ on the boundary is wellposed. ose that we vary the boundary data a small amount such that

$$\nabla^2 v = 0 \quad \text{with} \quad v = g(x)$$

undary, where $g(x)$ is nearly the same as $f(x)$ everywhere on the boundary. ider the difference between these two solutions, $w = u - v$. Due to the

$$\nabla^2 w = 0 \quad \text{with} \quad w = f(x) - g(x)$$

oundary. The maximum (and minimum) principles for Laplace's equation at the maximum and minimum occur on the boundary. Thus, at any point

$$\min(f(x) - g(x)) \le w \le \max(f(x) - g(x)). \tag{2.5.60}$$

:) is nearly the same as $f(x)$ everywhere, w is small, and thus the solution ly the same as u; the solution of Laplace's equation slightly varies if the data are slightly altered.

an also prove that the solution of Laplace's equation is unique. We prove ontradiction. Suppose that there are two solutions, u and v as previously, sfy the same boundary condition [i.e., let $(f(x) = g(x))$]. If we again the difference ($w = u - v$), then the maximum and minimum principles e (2.5.60)] that inside the region

$$0 \le w \le 0.$$

ude that $w = 0$ everywhere inside, and thus $u = v$ proving that if a solution must be unique. These properties (uniqueness and continuous dependence ata) show that Laplace's equation with u specified on the boundary is a l problem.

ability condition. If on the boundary the heat flow $-K_0 \nabla u \cdot \hat{n}$ is instead of the temperature, Laplace's equation may have no solutions

$$0 = \iint \nabla^2 u \, dx \, dy = \iint \nabla \cdot (\nabla u) \, dx \, dy.$$

Using the (two-dimensional) divergence theorem, we conclude that (see Exercise 1.5.8)

$$0 = \oint \nabla u \cdot \hat{n} \, ds. \tag{2.5.61}$$

Since $\nabla u \cdot \hat{n}$ is proportional to the heat flow through the boundary, (2.5.61) implies that the *net* heat flow through the boundary must be zero in order for a steady state to exist. This is clear physically, because otherwise there would be a change (in time) of the thermal energy inside, violating the steady-state assumption. Equation (2.5.61) is called the **solvability condition** or **compatibility condition** for Laplace's equation.

EXERCISES 2.5

2.5.1. Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions:

*(a) $\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = f(x)$

(b) $\frac{\partial u}{\partial x}(0,y) = g(y)$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = 0$

*(c) $\frac{\partial u}{\partial x}(0,y) = 0$, $u(L,y) = g(y)$, $u(x,0) = 0$, $u(x,H) = 0$

(d) $u(0,y) = g(y)$, $u(L,y) = 0$, $\frac{\partial u}{\partial y}(x,0) = 0$, $u(x,H) = 0$

*(e) $u(0,y) = 0$, $u(L,y) = 0$, $u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0$, $u(x,H) = f(x)$

(f) $u(0,y) = f(y)$, $u(L,y) = 0$, $\frac{\partial u}{\partial y}(x,0) = 0$, $\frac{\partial u}{\partial y}(x,H) = 0$

(g) $\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$, $\frac{\partial u}{\partial y}(x,H) = 0$

2.5.2. Consider $u(x,y)$ satisfying Laplace's equation inside a rectangle ($0 < x < L$, $0 < y < H$) subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0 \quad \frac{\partial u}{\partial y}(x,0) = 0$$

$$\frac{\partial u}{\partial x}(L,y) = 0 \quad \frac{\partial u}{\partial y}(x,H) = f(x).$$

*(a) *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a).