EXERCISES 9.2

9.2.1. Consider

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\\ u(x,0) &= g(x).\end{aligned}$$

In all cases obtain formulas similar to (9.2.20) by introducing a Green's function.

(a) Use Green's formula instead of term-by-term spatial differentiation if

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

(b) Modify part (a) if

$$u(0,t) = A(t) \qquad \text{and} \qquad u(L,t) = B(t).$$

Do not reduce to a problem with homogeneous boundary conditions.

(c) Solve using any method if

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

*(d) Use Green's formula instead of term-by-term differentiation if

$$\frac{\partial u}{\partial x}(0,t) = A(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = B(t).$$

9.2.2. Solve by the method of eigenfunction expansion

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q(x,t)$$

subject to $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,0) = g(x)$, if $c\rho$ and K_0 are functions of x. Assume that the eigenfunctions are known. Obtain a formula similar to (9.2.20) by introducing a Green's function.

*9.2.3. Solve by the method of eigenfunction expansion

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} + Q(x,t)\\ u(0,t) = 0 \qquad & u(x,0) = f(x)\\ u(L,t) = 0 \qquad & \frac{\partial u}{\partial t}(x,0) = g(x).\end{aligned}$$

Define functions (in the simplest possible way) such that a relationship similar to (9.2.20) exists. It must be somewhat different due to the two initial conditions. (*Hint:* See Exercise 8.5.1.)

9.2.4. Modify Exercise 9.2.3 (using Green's formula if necessary) if instead

(a) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$

(b) $u(0,t) = A(t)$ and $u(L,t) = 0$

(c) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$

9.3 Green's Functions for Boundary Value F lems for Ordinary Differential Equations

9.3.1 One-Dimensional Steady-State Heat Equation

Introduction. Investigating the Green's functions for the time-depende equation is not an easy task. Instead, we first investigate a simpler problen of the techniques discussed will be valid for more difficult problems.

We will investigate the steady-state heat equation with homogeneous ary conditions, arising in situations in which the source term $Q(x,t) =$ independent of time:

$$0 = k\frac{d^2u}{dx^2} + Q(x).$$

We prefer the form

$$\boxed{\frac{d^2u}{dx^2} = f(x),}$$

in which case $f(x) = -Q(x)/k$. The boundary conditions we consider are

$$\boxed{u(0) = 0 \qquad \text{and} \qquad u(L) = 0.}$$

We will solve this problem in many different ways in order to suggest meth other harder problems.

Limit of time-dependent problem. One way (not the most nor easiest) to solve (9.3.1) is to analyze our solution (9.2.20) of the time-dep problem, obtained in the preceding section, in the special case of a steady s

$$\begin{aligned}u(x,t) = &\int_0^L g(x_0)G(x,t;x_0,0)\,dx_0\\ &+\int_0^L -kf(x_0)\left(\int_0^t G(x,t;x_0,t_0)\,dt_0\right)dx_0,\end{aligned}$$

where

$$G(x,t;x_0,t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2(t-t_0)}.$$

As $t \to \infty$, $G(x,t;x_0,0) \to 0$ such that the effect of the initial condition $u($ $g(x)$ vanishes at $t \to \infty$. However, even though $G(x,t;x_0,t_0) \to 0$ as $t \to$ steady source is still important as $t \to \infty$ since

$$\int_0^t e^{-k(n\pi/L)^2(t-t_0)}\,dt_0 = \left.\frac{e^{-k(n\pi/L)^2(t-t_0)}}{k(n\pi/L)^2}\right|_{t_0=0}^{t} = \frac{1-e^{-k(n\pi/L)^2 t}}{k(n\pi/L)^2}.$$

	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$	Gaussian (Sec. 10.3.3)
	$e^{-\beta\omega^2}$	
	$\frac{\partial F}{\partial t}$	Derivatives (Sec. 10.4.2)
	$-i\omega F(\omega)$	
	$(-i\omega)^2 F(\omega)$	
$d\overline{x}$	$F(\omega)G(\omega)$	Convolution (Sec. 10.4.3)
	$\frac{1}{2\pi} e^{i\omega x_0}$	Dirac delta function (Exercise 10.3.18)
	$e^{i\omega\beta} F(\omega)$	Shifting theorem (Exercise 10.3.5)
	$-i\frac{dF}{d\omega}$	Multiplication by x (Exercise 10.3.8)
	$e^{-\lvert\omega\rvert\alpha}$	Exercise 10.3.7
	$\frac{1}{\pi}\frac{\sin a\omega}{\omega}$	Exercise 10.3.6

Table 10.4.1: Fourier Transform

:onjugate. [For real functions $g(x)$ is the reflection of $f(x)$ ·al, their Fourier transforms are related:

$$\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} f(-s)e^{-i\omega s}\, ds$$
$$\int_{-\infty}^{\infty} g^*(x)e^{-i\omega x}\, dx = G^*(\omega), \qquad (10.4.27)$$

'hus, (10.4.25) becomes **Parseval's identity**.

$$\int_{-\infty}^{\infty} g(x)g^*(x)\, dx = \int_{-\infty}^{\infty} G(\omega)G^*(\omega)\, d\omega, \qquad (10.4.28)$$

$x)|^2$ and $G(\omega)G^*(\omega) = |G(\omega)|^2$. We showed a similar rela-ized Fourier series (see Sec. 5.10). This result, (10.4.28), is rpretation. Often energy per unit distance is proportional to $\int_{-\infty}^{\infty} |g(x)|^2\, dx$ represents the total energy. From (10.4.28),

tude squared is the spectral energy density (the amount of energy per unit wave number).

10.4.4 Summary of Properties of the Fourier Transform

Tables of Fourier transforms exist and can be very helpful. The results we have obtained are summarized in Table 10.4.1.

We list below some important and readily available tables of Fourier transforms. Beware of various different notations.

F. Oberhettinger, *Tabellen zur Fourier Transformation*, Springer-Verlag, New York, 1957.

R.V. Churchill, *Operational Mathematics*, 3rd ed., McGraw-Hill, New York, 1972.

G.A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications*, Van Nostrand, Princeton, NJ, 1948.

EXERCISES 10.4

10.4.1. Using Green's formula, show that

$$\mathcal{F}\left[\frac{d^2 f}{dx^2}\right] = -\omega^2 F(\omega) + \frac{e^{i\omega x}}{2\pi}\left(\frac{df}{dx} - i\omega f\right)\Big|_{-\infty}^{\infty}.$$

10.4.2. For the heat equation, $u(x,t)$ is given by (10.4.1). Show that $u \to 0$ as $x \to \infty$ even though $\phi(x) = e^{-i\omega x}$ does not decay as $x \to \infty$. (*Hint*: Integrate by parts.)

10.4.3. *(a) Solve the diffusion equation with **convection**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} \qquad -\infty < x < \infty$$
$$u(x,0) = f(x).$$

[*Hint*: Use the convolution theorem and the shift theorem (see Exercise 10.4.5).]

(b) Consider the initial condition to be $\delta(x)$. Sketch the corresponding $u(x,t)$ for various values of $t > 0$. Comment on the significance of the convection term $c\, \partial u/\partial x$.

✓ 10.4.4. (a) Solve by fourier transform :

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u \qquad -\infty < x < \infty$$
$$u(x,0) = f(x).$$

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10.4.9. Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{matrix} y > 0 \\ -\infty < x < \infty \end{matrix}$$

subject to

$$u(x, 0) = f(x).$$

(*Hint*: If necessary, see Sec. 10.6.3.)

10.4.10. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty \\ u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= 0. \end{aligned}$$

(*Hint*: If necessary, see Sec. 10.6.1.)

10.4.11. Derive an expression for the Fourier transform of the product $f(x)g(x)$.

10.5 Fourier Sine and Cosine Transforms: The Heat Equation on Semi-Infinite Intervals

10.5.1 Introduction

The Fourier series has been introduced to solve partial differential equations on the finite interval $-L < x < L$ with periodic boundary conditions. For problems defined on the interval $0 < x < L$, special cases of Fourier series, the sine and cosine series, were analyzed in order to satisfy the appropriate boundary conditions.

On an infinite interval, $-\infty < x < \infty$, instead we use the Fourier transform. In this section we show how to solve partial differential equations on a semi-infinite interval, $0 < x < \infty$. We will introduce special cases of the Fourier transform, known as the sine and cosine transforms. The modifications of the Fourier transform will be similar to the ideas we used for series on finite intervals.

10.5.2 Heat Equation on a Semi-Infinite Interval I

We will motivate the introduction of Fourier sine and cosine transforms by considering a simple physical problem. If the temperature is fixed at 0° at $x = 0$, then