MATH 2050 **Assignment 7** **Fall 2018**

Due: Nov. 8

[7] 1. Find the inverse of A and then solve the system is $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 8 \\ 5 \\ -7 \end{bmatrix}$.

Solution: We have $[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right],$$

so $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}$ and $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \\ -2 \end{bmatrix}$.

[5] 2. Determine if A is invertible and if so find A^{-1}:

$A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{bmatrix}$.

Solution:

$[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 1 & -1 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & -6 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{array}\right]$.

There is no inverse.

[3] 3. Which of the following matrices are elementary matrices, explain why.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution: Yes, this matrix is obtained from I_2 by applying $R_1 \Leftrightarrow R_2$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Solution: Yes, this matrix is obtained from I_3 by applying $-2R_1 + R_3 \Rightarrow R_3$.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Solution: No, this matrix cannot obtained from I_3 by a single operation, we require 2 operations.

[8] 4. Let $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$. Solve each of the following matrix equations:

(a) $AX + B = C$

Solution: We have $AX = C - B$ and so $X = A^{-1}(C - B)$. Thus, $X = \frac{1}{10-9}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -2 & -4 \\ -8 & -1 \end{bmatrix} = \begin{bmatrix} 20 & -5 \\ -34 & 7 \end{bmatrix}$.

(b) $XA + C = X$

Solution: We have $XA - X = -C$. Thus, $X(A - I) = -C$ which implies that $X = -C(A - I)^{-1}$. Thus, $X = -\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}^{-1} = \frac{-1}{5}\begin{bmatrix} -10 & 20 \\ 15 & -30 \end{bmatrix}$

[6] 5. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

(a) $A = \begin{bmatrix} -9 & -1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$.

Solution: Note that if we replace the first row of A by $R_2 + R_1$, we obtain B. The matrix E is obtained by applying the operation $R_2 + R_1 \to R_1$ to I_2:

$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{bmatrix}$

Solution: Note that if we switch rows two and three of A, we obtain B. The matrix E is obtained by applying the operation $R_2 \leftrightarrow R_3$ to I_3:

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix}$

Solution: Note that if we replace the third row of A by $2R_2 + R_3$, we obtain B. The matrix E is obtained by applying the operation $2R_2 + R_3 \to R_3$ to I_3:
$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

[8] 6. Let $A = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$.

(a) Express A as a product of elementary matrices.

(b) Express A^{-1} as a product of elementary matrices.

Solution: We have $\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = E_1 A \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -1 \end{bmatrix} = E_2 E_1 A$

$\to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = E_3 E_2 E_1 A \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = E_4 E_3 E_2 E_1 A$, where

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

Thus $A = (E_4 E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$.

and $A^{-1} = E_4 E_3 E_2 E_1 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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