MATH 2050 **Assignment 6** **Fall 2018**

Due: Thursday, November 1

[5] 1. For what value of c does

$$\begin{aligned} x+y+2z &= 2 \\ -x+y+z &= c \\ 4x\quad+2z &= 2 \end{aligned}$$

have a solution? Is it unique?

Solution: Writing the system as an augmented matrix, we have

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 2 \\ -1 & 1 & 1 & c \\ 4 & 0 & 2 & 2 \end{array}\right] \xrightarrow[R3 \leftarrow R3-4R1]{R2 \leftarrow R2+R1} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & c+2 \\ 0 & -4 & -6 & -6 \end{array}\right] \xrightarrow{R3 \leftarrow R3+2R2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & c+2 \\ 0 & 0 & 0 & -6+2(c+2) \end{array}\right]$$

Thus, the system has a solution only when $-6+2(c+2)=0$ or $c=1$, so that the last equation is consistent. The solution will not be unique, however, since there is a free variable in the row echelon form.

[20] 2. Write all solutions of the following linear systems in vector form.

(a)

$$\begin{aligned} x_1+2x_2-x_3+3x_4 &= 4 \\ x_2\qquad+2x_4 &= 1 \\ x_1+x_2\qquad-x_4 &= 3 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & -1 & 3 \end{array}\right] \xrightarrow{R3 \leftarrow R3-R1} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & -4 & -1 \end{array}\right] \xrightarrow{R3 \leftarrow R3+R2} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

Thus, we see that $x_4=t$ is a free variable, giving $x_3=0+2t$, $x_2=1-2t$, and $x_1=4-2x_2+x_3-3x_4=2+3t$. Writing this in vector form, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

(b)

$$\begin{aligned} x+2y+4z &= 3 \\ x+2y+6z &= 5 \\ x+3y+5z &= 4 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 1 & 2 & 6 & 5 \\ 1 & 3 & 5 & 4 \end{array}\right] \xrightarrow[R3 \leftarrow R3 - R1]{R2 \leftarrow R2 - R1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

This gives $z = 1$, $y = 0$, and $x = 3 - 2y - 4z = -1$, so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(c)

$$\begin{aligned} x + 5y - 2z &= -2 \\ 3x + 15y - 6z &= -6 \\ -x - 5y + 2z &= 2 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & -2 \\ 3 & 15 & -6 & -6 \\ -1 & -5 & 2 & 2 \end{array}\right] \xrightarrow[R3 \leftarrow R3 + R1]{R2 \leftarrow R2 - 3R1} \left[\begin{array}{ccc|c} 1 & 5 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, both y and z are free variables. Taking $y = s$ and $z = t$, we have $x = -2 - 5s + 2t$, giving

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

(d)

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 \\ 0 & -3 & 6 & -3 & 0 \\ 0 & 0 & 4 & -8 & 4 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & -3 & 6 & -3 & 0 & 0 \\ 0 & 0 & 4 & -8 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R2 \leftarrow R2 + R1} \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 \\ 0 & -3 & 6 & -3 & 0 & 0 \\ 0 & 0 & 4 & -8 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R3 \leftarrow R3 + R2} \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 4 & -8 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R4 \leftarrow R4 - R3} \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & -5 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R5 \leftarrow 5R5} \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & -5 & 4 & 0 \\ 0 & 0 & 0 & -5 & 5 & 5 \end{array}\right]$$

$$\xrightarrow{R5 \leftarrow R5 - R4} \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Solving, this gives $x_5 = 5$, $x_4 = 4$, $x_3 = 3$, $x_2 = 2$, and $x_1 = 1$, or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

[5] 3. Find conditions on a, b, and c (if any) such that the system

$$\begin{aligned} x \quad\; + \; z &= -1 \\ 2x - \; y \quad\;\; &= 2 \\ y + 2z &= -4 \\ ax + by + cz &= 3 \end{aligned}$$

has (i) no solution, (ii) a unique solution, and (iii) infinitely many solutions.

Solution: Working from the augmented matrix of the system, we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ a & b & c & 3 \end{array}\right] \xrightarrow[R4 \leftarrow R4 - aR1]{R2 \leftarrow R2 - 2R1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & -2 & 4 \\ 0 & 1 & 2 & -4 \\ 0 & b & c-a & 3+a \end{array}\right]$$

$$\xrightarrow[\substack{R3 \leftarrow R3 + R2 \\ R4 \leftarrow R4 + bR2}]{R2 \leftarrow -R2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (c-a)-2b & (3+a)+4b \end{array}\right] \xrightarrow{R3 \leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & (c-a)-2b & (3+a)+4b \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This is row echelon form.

(i) If $c - a - 2b = 0$ and $3 + a + 4b \neq 0$, the third equation says $0 = 3 + a + 4b$, which is a contradiction. There is no solution in this case.

(ii) If $c - a - 2b \neq 0$, there is a unique solution: $z = \frac{3+a+4b}{c-a-2b}$, $y = -4 - 2z$, $x = -1 - z$.

(iii) If $c - a + 2b = 0$ and $3 + a + 4b = 0$, then the row echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$,

so $z = t$ is a free variable, and there are infinitely many solutions.

[10] 4. Find all solutions of

$$\begin{aligned} x + 3y \quad\;\; &= 0 \\ x + 4y + \;\, 5z &= 0 \\ 2y + 10z &= 0 \end{aligned}$$

Does

$$\begin{aligned} x + 3y \quad\;\; &= \pi \\ x + 4y + \;\, 5z &= \sqrt{17/19} \\ 2y + 10z &= e^{-\sqrt{2}} \end{aligned}$$

have a unique solution?

Solution: First we put the system matrix in row echelon form

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 4 & 5 \\ 0 & 2 & 10 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 - R1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \end{bmatrix} \xrightarrow{R3 \leftarrow R3 - 2R2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this has only 2 pivot columns, we introduce $z = t$ as a free variable, giving $y = -5t$, and $x = 15t$. Thus, solutions are of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 15 \\ -5 \\ 1 \end{bmatrix}$$

For the non-homogeneous system, we know that if x, y, and z solve the given system, then so do $x + 15t$, $y - 5t$, $z + t$ for any value of t. Thus, it does not have a unique solution.

Indeed, this system has NO solution when we use Gaussian-elimination to solve it.

[5] 5. Show that the vectors $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 6 \end{bmatrix}$ are linearly independent.

Solution: The vectors are linearly independent if and only if the only solution of the homogeneous system, $A\vec{x} = \vec{0}$, with these vectors as columns of the matrix, A, is the trivial solution. So, we put A into row echelon form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & -1 \\ -1 & 0 & 1 \\ -2 & 4 & 6 \end{bmatrix} \xrightarrow[R4 \leftarrow R4 + 2R1]{\substack{R2 \leftarrow R2 - 3R1 \\ R3 \leftarrow R3 + R1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -4 \\ 0 & 2 & 2 \\ 0 & 8 & 8 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -4 \\ 0 & 8 & 8 \end{bmatrix} \xrightarrow{R4 \leftarrow R4 - 4R2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the row echelon form of the matrix has 3 pivot columns, there are no free variables and, thus, the unique solution of $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. This means that the vectors that make up the columns of A are linearly independent.

[5] 6. For what values of x (if any) are the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ -1 \\ x - 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 3x^2 - 3 \end{bmatrix}$ linearly independent?

Solution: The vectors are linearly independent if and only if the only solution of the homogeneous system, $A\vec{x} = \vec{0}$, with these vectors as columns of the matrix, A, is the trivial solution. So, we put A into row echelon form:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \\ 3 & x-3 & 3x^2-3 \end{bmatrix} \xrightarrow[R3 \leftarrow R3-3R1]{R2 \leftarrow R2-2R1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & x & 3x^2-6 \end{bmatrix} \xrightarrow{R3 \leftarrow R3-xR2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3x^2+3x-6 \end{bmatrix}$$

This system will have 3 pivot columns only when $3x^2 + 3x - 6 \neq 0$ and, consequently, the vectors will be linearly independent when $3x^2 + 3x - 6 \neq 0$.

Solving $3x^2 + 3x - 6 = 0$ we have $x = -2$ and $x = 1$. Thus, the vectors are linearly independent for all values of x *except* for $x = -2$ and $x = 1$.

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[50]