Stability of Traveling Waves to the Lotka-Volterra Competition Model

Ahmad Alhasanat* and Chunhua Ou†

Abstract

In this paper, the stability of traveling wave solutions to the Lotka-Volterra diffusive model is investigated. First, we convert the model into a cooperative system by a special transformation. The local and the global stability of the traveling wavefronts are studied in a weighted functional space. For the global stability, comparison principle together with the squeezing technique are applied to derive the main results.

Keywords and Phrases: Lotka-Volterra, traveling waves, stability
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1 Introduction

We are concerned here with the diffusive Lotka-Volterra competition model

$$\begin{cases} \phi_t = d_1\phi_{xx} + r_1\phi(1 - b_1\phi - a_1\psi), \\ \psi_t = d_2\psi_{xx} + r_2\psi(1 - a_2\phi - b_2\psi), \end{cases} \tag{1.1}$$

with the initial data

$$\phi(x,0) = \phi_0(x) \geq 0, \qquad \psi(x,0) = \psi_0(x) \geq 0, \quad \forall x \in \mathbb{R}.$$

Here $\phi(x,t)$ and $\psi(x,t)$ are the population densities at time t and location x; d_1 and d_2 are the diffusive coefficients; r_1 and r_2 are the net birth rates; a_1 and a_2 are the competition coefficients; $1/b_1$ and $1/b_2$ are the carrying capacities for each species. For derivation and biological interpretation of this model, we refer readers to [28, 29].

Using the following transformations

$$\sqrt{r_1/d_1}x \to x, \qquad r_1 t \to t,$$

$$b_1\phi(x,t) = \tilde{\phi}(x,t), \qquad b_2\psi(x,t) = \tilde{\psi}(x,t),$$

*Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, A1C 5S7, and Department of Mathematics and Statistics, Al-Hussein Bin Talal University, Ma'an, Jordan. Email: A.Alhasanat@mun.ca.

†Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, A1C 5S7. Email: ou@mun.ca. This work is partially supported by the NSERC discovery grant.

$$d=\frac{d_2}{d_1},\qquad r=\frac{r_2}{r_1},\qquad \frac{a_1}{b_2}\to a_1,\qquad \frac{a_2}{b_1}\to a_2,$$

the non-dimensional form of the system becomes

$$\begin{cases}\tilde{\phi}_t=\tilde{\phi}_{xx}+\tilde{\phi}(1-\tilde{\phi}-a_1\tilde{\psi}),\\ \tilde{\psi}_t=d\tilde{\psi}_{xx}+r\tilde{\psi}(1-a_2\tilde{\phi}-\tilde{\psi}).\end{cases}\tag{1.2}$$

By letting $u=\tilde{\phi},\ v=1-\tilde{\psi}$, this model can be further written as a cooperative system

$$\begin{cases}u_t=u_{xx}+u(1-a_1-u+a_1v),\\ v_t=dv_{xx}+r(1-v)(a_2u-v),\end{cases}\tag{1.3}$$

with

$$u(x,0)=u_0(x)=\tilde{\phi}(x,0),\qquad v(x,0)=v_0(x)=1-\tilde{\psi}(x,0),\ \ \forall x\in\mathbb{R}.$$

For our study, we will assume that $u_0(x)$ and $v_0(x)$ are non-negative. The existence and uniqueness of the solution of the above problem can be easily verified by a classical argument of Picard's iteration. Throughout this paper, we assume that the condition

$$0<a_1<1<a_2\tag{\textbf{C1}}$$

is satisfied. Under this condition, equilibria to system (1.3) in the region $\{(u,v)|0\le u\le 1, 0\le v\le 1\}$ are only $(0,0),(0,1)$, and $(1,1)$. In the absence of diffusion in the system (1.3), it can be shown that $(0,0)$ is unstable and $(1,1)$ is stable. For the system, we are particularly interested in the traveling wave solution, connecting $(1,1)$ and $(0,0)$, in the form

$$(u,v)(x,t)=(\overline{U},\overline{V})(z),$$

where $z=x-ct$ is the wave variable, $c\ge 0$ is the wave speed, and $(\overline{U},\overline{V})$ is called the wavefront and satisfies

$$\begin{cases}0=\overline{U}_{zz}+c\overline{U}_z+\overline{U}(1-a_1-\overline{U}+a_1\overline{V}),\\ 0=d\overline{V}_{zz}+c\overline{V}_z+r(1-\overline{V})(a_2\overline{U}-\overline{V}),\end{cases}\tag{1.4}$$

subject to

$$(\overline{U},\overline{V})(-\infty)=(1,1),\quad (\overline{U},\overline{V})(\infty)=(0,0).\tag{1.5}$$

This is equivalent to studying traveling waves for the original competition system (1.2) that connect the boundary equilibria $(0,1)$ and $(1,0)$.

The existence of traveling waves to the above problem is well-studied in literature. It is known that there exists $c^*\ge 0$ so that the problem (1.4)-(1.5) has a monotone solution $(\overline{U},\overline{V})(z)$ for $c\ge c^*$ and no wavefront exists for $c<c^*$, see [16, 20, 21, 36]. c^* is called the minimal wave speed for this system and satisfies $c^*\ge 2\sqrt{1-a_1}$. When $c^*=2\sqrt{1-a_1}$, we say that the minimal wave speed is *linearly determined*, see the details in [20].

We know that $(\overline{U},\overline{V})(x-ct)$ is a special pattern that only satisfies the first two equations in (1.3). For the stability of this pattern, we want to know if the solution of (1.3) tends to $(\overline{U},\overline{V})(x-ct)$ for given initial data $u_0(x)$ and $v_0(x)$. To this end, we use the (z,t)-coordinate and

$$(u,v)(x,t)=(U,V)(z,t),$$

to transform the uv-model (1.3) into the partial differential model

$$\begin{cases} U_t = U_{zz} + cU_z + U(1 - a_1 - U + a_1 V), \\ V_t = dV_{zz} + cV_z + r(1 - V)(a_2 U - V), \end{cases} \tag{1.6}$$

subject to

$$U(z, 0) = u_0(z), \qquad V(z, 0) = v_0(z), \quad \forall z \in \mathbb{R}.$$

It is easy to see that $(\overline{U}, \overline{V})(z)$ is the steady-state to the above new system.

We should mention that dynamics for (1.2) is very rich. There are always three non-negative equilibria $(0, 0), (1, 0)$, and $(0, 1)$. In the case when $a_1 < 1, a_2 < 1$, or the case when $a_1 > 1, a_2 > 1$, there exists a unique positive co-existence equilibrium

$$(\tilde{\phi}^*, \tilde{\psi}^*) = \left(\frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2} \right).$$

Based on the phase plane analysis to the ordinary differential system of (1.2) without diffusion terms, the nonlinearity of the model (1.2) when $a_1 < 1$ and $a_2 < 1$ is called the persistence case (or co-existence). Likewise, the nonlinearity is called the mono-stable case when $a_1 < 1$ and $a_2 > 1$ are satisfied, or the bistable case when $a_1 > 1$ and $a_2 > 1$. Traveling waves to (1.2) have been investigated considerably. For the bistable case, please see [5, 8] for the existence of traveling waves connecting $(1, 0)$ and $(0, 1)$, and [15] for the uniqueness and parameter dependence of wave speeds. For the mono-stable case, we refer to [14,16] for the existence of traveling waves, and [1,11] for the selection of the minimal speed. For the persistence (co-existence), the existence of traveling wave connecting $(0, 0)$ and $(\tilde{\phi}^*, \tilde{\psi}^*)$ has been studied in [32, 35]. When time delays are incorporated into (1.2) in the persistence case, Li et al. [19] and Gourley and Ruan [9] have proved the existence of traveling waves.

The stability of traveling waves to a scalar partial differential equation has been well-studied, e.g., [6,7,13,17,22,24,27,30,31,34,37], the monograph [3,36] and the survey paper [38]. Indeed, the extension of this study to a general system is not trivial. As we know, when time delays are directly incorporated in the competition terms in (1.2), the system becomes non-monotone and the comparison principle cannot work. Alternatively, in [23, 26], the authors studied the stability of traveling waves for the so-called cooperative delayed reaction diffusion system by changing the signs of a_1 and a_2. To be exact, with putting delay$= 0$, they studied the cooperative system

$$\begin{cases} \phi_t = d_1 \phi_{xx} + r_1 \phi(1 - \hat{b}_1 \phi + \hat{a}_1 \psi), \\ \psi_t = d_2 \psi_{xx} + r_2 \psi(1 + \hat{a}_2 \phi - \hat{b}_2 \psi), \end{cases}$$

where $d_i, r_i, \hat{a}_i$, and $\hat{b}_i$ are all positive. This corresponds to the persistence case in our model (1.2). Under the condition $\hat{b}_1 \hat{b}_2 - \hat{a}_1 \hat{a}_2 > 0$, a positive equilibrium

$$(\phi_+, \psi_+) = \left(\frac{\hat{a}_1 + \hat{b}_2}{\hat{b}_1 \hat{b}_2 - \hat{a}_1 \hat{a}_2}, \frac{\hat{b}_1 + \hat{a}_2}{\hat{b}_1 \hat{b}_2 - \hat{a}_1 \hat{a}_2} \right)$$

exists. They proved that the traveling wave fronts, connecting $(0, 0)$ and (ϕ_+, ψ_+), are exponentially stable in some weighted L^∞ spaces, and obtained the decay rates by the weighted energy estimate.

Despite the success in the study of the stability of traveling waves to the classical model (1.2) in the bistable and persistence cases, the stability of traveling wave in the mono-stable remains still unsolved.

The purpose of this paper is to systematically study the local and the global stability of the steady-state $(\overline{U},\overline{V})(z)$. Using the method of spectrum analysis in [12], we give the local stability. For the global stability, we construct an upper and a lower solutions to the system (1.6), and prove their convergence to the traveling wave $(\overline{U},\overline{V})(z)$. In view of comparison together with the squeezing technique, we arrive at new results on the global stability of the traveling waves. We remark that our method is different from that in [23, 26] where weighted energy method was applied.

The rest of the paper is organized as follows. Local analysis of the wave profile near the unstable point is studied in Section 2. In Section 3, we study the local stability of the steady-state by applying the standard linearization. The resulted spectrum problem is studied by the method in [12]. A suitable weighted functional space is chosen to proceed the analysis. In Section 4, besides the weighted functional space, the upper-lower solution method together with the squeezing technique are applied to derive the global stability results. Conclusions are presented in Section 5.

2 The local analysis of the wave profile near the equilibrium $(0,0)$

In this section, we study the behavior of the traveling wave $(\overline{U},\overline{V})(z)$ locally near the equilibrium $(0,0)$. Assume that the solution has exponential decay as $z \to \infty$. Indeed this claim can be easily verified by the maximum principal coupled with a comparison near the neighborhood of infinity. Therefore, we set

$$(\overline{U},\overline{V})(z) \sim (\zeta_1 e^{-\mu z}, \zeta_2 e^{-\mu z}) \quad \text{as} \quad z \to \infty,$$

for positive constants ζ_1, ζ_2, and μ. By substituting this into (1.4) and linearizing the equations we have

$$A(\mu)\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.1}$$

where $A(\mu)$ is given by

$$A(\mu) = \begin{pmatrix} \mu^2 - c\mu + 1 - a_1 & 0 \\ ra_2 & d\mu^2 - c\mu - r \end{pmatrix}. \tag{2.2}$$

The system of algebraic equations (2.1) has a non-trivial solution if and only if $\det(A) = 0$. This implies $\mu = \mu_{1,2,3} > 0$, where

$$\mu_1(c) = \frac{c - \sqrt{c^2 - 4(1-a_1)}}{2}, \quad \mu_2(c) = \frac{c + \sqrt{c^2 - 4(1-a_1)}}{2}, \tag{2.3}$$

and

$$\mu_3(c) = \frac{c + \sqrt{c^2 + 4dr}}{2d}. \tag{2.4}$$

Indeed, a condition so that μ_1 and μ_2 are reals is

$$c \geq 2\sqrt{1-a_1} := c_0.$$

For $c > c_0$, obviously $\mu_1 < \mu_2$. When $0 \leq d < 1$, we have also $\mu_2 < \mu_3$ for all $c > c_0$, i.e., $e^{-\mu_1 z}$ dominates both of $e^{-\mu_2 z}$ and $e^{-\mu_3 z}$. In this case, the eigenvector of $A(\mu)$ corresponding to μ_i, for $i = 1, 2$, is the strongly positive vector $(\zeta_1(\mu_i) \;\; \zeta_2(\mu_i))^T$, where

$$\zeta_1(\mu_i) = -(d\mu_i^2 - c\mu_i - r) \quad \text{and} \quad \zeta_2(\mu_i) = ra_2. \tag{2.5}$$

It follows that

$$\begin{pmatrix} \overline{U}(z) \\ \overline{V}(z) \end{pmatrix} = C_1 \begin{pmatrix} \zeta_1(\mu_1) \\ \zeta_2(\mu_1) \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} \zeta_1(\mu_2) \\ \zeta_2(\mu_2) \end{pmatrix} e^{-\mu_2 z}, \quad \text{as } z \to \infty, \tag{2.6}$$

for $C_1 > 0$ or $C_1 = 0, C_2 > 0$. For the case when

$$1 < d < 2 + \frac{r}{1 - a_1} := \hat{d},$$

the same behavior in (2.6) is still true if $c^* < c \le \hat{c}$, where

$$\hat{c} = \sqrt{\frac{r + 1 - a_1}{d - 1}} + (1 - a_1)\sqrt{\frac{d - 1}{r + 1 - a_1}}.$$

If $c > \hat{c}$, then $\mu_1 < \mu_3 < \mu_2$ and we have

$$\begin{pmatrix} \overline{U}(z) \\ \overline{V}(z) \end{pmatrix} = C_1 \begin{pmatrix} \zeta_1(\mu_1) \\ \zeta_2(\mu_1) \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} -\zeta_1(\mu_2) \\ -\zeta_2(\mu_2) \end{pmatrix} e^{-\mu_2 z} + C_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\mu_3 z}, \quad \text{as } z \to \infty, \tag{2.7}$$

for $C_1 > 0$ or $C_1 = 0, C_{2,3} > 0$. Here, $(0 \;\; 1)^T$ is the eigenvector of $A(\mu)$ corresponding to μ_3, and note that $\zeta_1(\mu_2) < 0$ in this case. On the other hand, when

$$d > \hat{d},$$

$(\overline{U}, \overline{V})(z)$ behaves like (2.7) if $c > \hat{c}$. For the case when $c^* < c < \hat{c}$, we have $\mu_3 < \mu_1 < \mu_2$. Hence,

$$\begin{pmatrix} \overline{U}(z) \\ \overline{V}(z) \end{pmatrix} = C_1 \begin{pmatrix} -\zeta_1(\mu_1) \\ -\zeta_2(\mu_1) \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} -\zeta_1(\mu_2) \\ -\zeta_2(\mu_2) \end{pmatrix} e^{-\mu_2 z} + C_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\mu_3 z}, \quad \text{as } z \to \infty, \tag{2.8}$$

for $C_{1,3} > 0$, or $C_1 = 0,\ C_{2,3} > 0$. We summarize the above behaviors in the following table

Condition on d	Condition on c	The asymptotic behavior
$0 \le d < 1$	$c > c^*$	(2.6)
$1 < d < \hat{d}$	$c^* < c < \hat{c}$	(2.6)
$1 < d$	$c > \hat{c}$	(2.7)
$d > \hat{d}$	$c^* < c < \hat{c}$	(2.8)

Table 1: The asymptotic behavior of the wave profile $(\overline{U}, \overline{V})$ near infinity.

Kan-on in [16] derived the asymptotic behaviors of $(\overline{U}, \overline{V})(z)$ near infinity when $c \ge c^*$. After deriving the behavior of $\overline{U}(z)$, he used it into the V-equation to find the behavior of $\overline{V}(z)$ when $\mu_1 \le \mu_2 \le \mu_3$ and when $\mu_3 \le \mu_1 \le \mu_2$. Our result here agrees with that in [16] when $c > c^*$. We further study the case when $\mu_1 < \mu_3 < \mu_2$.

Finally, we have the asymptotic behavior for the solution $\overline{U}(z)$ when the wave speed is greater than the minimal speed c^*.

Theorem 2.1. *For $c > c^*$, the wavefront $\overline{U}$ has the following behavior*

$$\overline{U}(z) \sim C_1 e^{-\mu_1 z}, \text{ as } z \to \infty$$

for some $C_1 > 0$.

Proof. On the contrary, assume that for some $c_1 > c^*$, the wavefront $\overline{U}$ has the following behavior

$$\overline{U}(z) \sim C_2 e^{-\mu_2 z}, \text{as } z \to \infty \tag{2.9}$$

for some $C_2 > 0$. By this assumption, it follows that $(\overline{U},\overline{V})(x - c_1 t)$ is a solution to the following partial differential equation

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = dv_{xx} + r(1 - v)(a_2 u - v), \end{cases} \tag{2.10}$$

with the initial conditions

$$u(x, 0) = \overline{U}(x) \quad \text{and} \quad v(x, 0) = \overline{V}(x).$$

We know that there exists a monotonic traveling wavefront to the system (2.10) for any $c \geq c^*$. In particular, assume $(U, V)(x - ct)$ is a solution for some $c \in (c^*, c_1)$ with the initial condition

$$u(x, 0) = U(x) \quad \text{and} \quad v(x, 0) = V(x).$$

By a simple computation of the asymptotic behavior of this solution to (1.4)-(1.5) near $\pm\infty$, we can always obtain (by shifting if necessary) $\overline{U}(x) \leq U(x)$ for all $x \in (-\infty, \infty)$. From the second equation of (1.4), we have $\overline{V}(x) \leq V(x)$ for all $x \in (-\infty, \infty)$. For (2.10), by comparison, we get

$$\begin{aligned} \overline{U}(x - c_1 t) &\leq U(x - ct), \\ \overline{V}(x - c_1 t) &\leq V(x - ct), \end{aligned} \tag{2.11}$$

for all $(x, t) \in (\mathbb{R}, \mathbb{R}^+)$. On the other hand, fix $\xi = x - c_1 t$. Then $\overline{U}(\xi) > 0$ is fixed, and we have

$$U(x - ct) = U(\xi + (c_1 - c)t) \sim U(+\infty) = 0 \text{ as } t \to \infty.$$

By (2.11), this implies that $\overline{U}(\xi) \leq 0$, which is a contradiction. The proof is complete. ∎

3 The local stability

To study the local stability, as usual, we add a small perturbation to the traveling wave and study the behavior of this perturbation for large time period. If this perturbation decays, then we say that the traveling wave is locally stable. For $\delta \ll 1$ and a parameter λ, let

$$\begin{aligned} U(z, t) &= \overline{U}(z) + \delta\phi_1(z)e^{\lambda t}, \\ V(z, t) &= \overline{V}(z) + \delta\phi_2(z)e^{\lambda t}, \end{aligned}$$

where ϕ_1 and ϕ_2 are two real functions. Substitute these formulas into (1.6) and linearize the system about $(\overline{U},\overline{V})$ to get the following spectrum problem

$$\lambda\Phi = \mathcal{L}\Phi := D\Phi'' + c\Phi' + J(z)\Phi, \tag{3.1}$$

where $\Phi = (\phi_1 \quad \phi_2)^T$, D and $J(z)$ are 2×2 matrices given by

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad J(z) = \begin{pmatrix} 1 - a_1 - 2\overline{U} + a_1\overline{V} & a_1\overline{U} \\ ra_2(1 - \overline{V}) & r(-1 - a_2\overline{U} + 2\overline{V}) \end{pmatrix}. \tag{3.2}$$

For Φ in a suitable space, we shall find sign of the maximal real part to the spectrum (λ) of the operator $\mathcal{L}$ to determine the local stability of the traveling wave solution. To proceed, we introduce a weighted functional space L^p_w,

$$L^p_w = \{f(z) \ : w(z)f(z) \in L^p(\mathbb{R}), p \geq 1\}$$

with the norm

$$\|f(z)\|_{L^p_w} = \left(\int_{-\infty}^{\infty} w(z)|f(z)|^p dz\right)^{\frac{1}{p}},$$

where

$$w(z) = (1/w_1(z), \quad 1/w_2(z)) \tag{3.3}$$

is the weight function with

$$w_1(z) = \begin{cases} e^{-\alpha(z-z_0)} & , z > z_0 \\ 1 & , z \leq z_0 \end{cases}, \quad w_2(z) = \begin{cases} e^{-\beta(z-z_0)} & , z > z_0 \\ 1 & , z \leq z_0 \end{cases}, \tag{3.4}$$

for some positive constants α, β and z_0 to be chosen. Here, $L^p(\mathbb{R})$, for $p \geq 1$, is the well-known Lebesgue space of integrable functions defined on $\mathbb{R}$. Then we consider the operator $\mathcal{L}$ on this new space and find its spectrum. To do this, we write $\Phi(z)$ in the form

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} w_1\psi_1 \\ w_2\psi_2 \end{pmatrix}, \tag{3.5}$$

for L^p-functions ψ_1 and ψ_2. Substituting (3.5) into (3.1) gives a new spectrum problem in the weighted space L^p_w,

$$\lambda\Psi = \mathcal{L}_w\Psi := D\Psi'' + M(z)\Psi' + N(z)\Psi,$$

where $\Psi = (\psi_1 \quad \psi_2)^T$, $M(z)$ and $N(z)$ are 2×2 matrices defined by

$$M(z) = \begin{pmatrix} c + 2\frac{w_1'}{w_1} & 0 \\ 0 & c + 2d\frac{w_2'}{w_2} \end{pmatrix} \tag{3.6}$$

and

$$N(z) = \begin{pmatrix} \frac{w_1''}{w_1} + c\frac{w_1'}{w_1} & 0 \\ 0 & d\frac{w_2''}{w_2} + c\frac{w_2'}{w_2} \end{pmatrix} + Y(z),$$

with the ik-element of the matrix $Y(z)$, y_{ik}, being given in terms of the ik-element of the matrix $J(z)$ as $y_{ik} = \frac{w_k}{w_i} j_{ik}$, that is,

$$N(z) = \begin{pmatrix} \frac{w_1''}{w_1} + c\frac{w_1'}{w_1} + 1 - a_1 - 2\overline{U} + a_1\overline{V} & a_1\overline{U}\frac{w_2}{w_1} \\ ra_2(1-\overline{V})\frac{w_1}{w_2} & d\frac{w_2''}{w_2} + c\frac{w_2'}{w_2} + r(-1 - a_2\overline{U} + 2\overline{V}) \end{pmatrix}. \tag{3.7}$$

The details to find the essential spectrum of the operator $\mathcal{L}_w$ can be finalized by using Theorem A.2 in [12] and are given below. After we choose the weight function so that the essential spectrum is on the left-half complex plane, we can determine the sign of the maximal real part of the point spectrum in the weighted space as well.

First of all, to apply the method in [12], we need to choose α and β so that the matrix functions $M(z)$ and $N(z)$ are bounded, i.e., the limits

$$\lim_{z\to\infty} \overline{U}(z)\frac{w_2(z)}{w_1(z)} = A_1 \quad \text{and} \quad \lim_{z\to\infty} (1-\overline{V}(z))\frac{w_1(z)}{w_2(z)} = A_2,$$

for some constants A_1 and A_2, are satisfied. We choose

$$\alpha - \mu_1 < \beta \leq \alpha, \tag{3.8}$$

where μ_1 is defined in (2.3). This makes, by using Theorem 2.1, $A_1 = 0$ and

$$A_2 = \begin{cases} 0 & \text{when } \beta < \alpha, \\ 1 & \text{when } \beta = \alpha. \end{cases}$$

Now, we define

$$S_\pm := \{\lambda \mid \det(-\tau^2 D + i\tau M_\pm + N_\pm - \lambda I) = 0, -\infty < \tau < \infty\},$$

where $M_\pm$ and $N_\pm$ are the limits of $M(z)$ and $N(z)$ as $z \to \pm\infty$, respectively. Then the essential spectrum of the operator $\mathcal{L}_w$ is contained in the union of regions inside or on the curves S_+ and S_-, see [12, pp. 140]. By letting $z \to +\infty$, M_+ and N_+ are given as (taking condition (3.8) into account)

$$M_+ = \begin{pmatrix} c-2\alpha & 0 \\ 0 & c-2d\beta \end{pmatrix} \quad \text{and} \quad N_+ = \begin{pmatrix} \alpha^2 - c\alpha + 1 - a_1 & 0 \\ A_2 & d\beta^2 - c\beta - r \end{pmatrix}.$$

The equation $\det(-\tau^2 D + i\tau M_+ + N_+ - \lambda I) = 0$ has two solutions $\lambda = \lambda_{1,2}$, where

$$\begin{aligned} \lambda_1 &= -\tau^2 + i\tau(c-2\alpha) + \alpha^2 - c\alpha + 1 - a_1, \\ \lambda_2 &= -\tau^2 d + i\tau(c - 2d\beta) + d\beta^2 - c\beta - r. \end{aligned}$$

This means that S_+ is the union of two parabolas in the complex plane which are symmetric about the real axis, namely

$$S_{+,1} = \{\lambda_1 \mid -\infty < \tau < \infty\} \quad \text{and} \quad S_{+,2} = \{\lambda_2 \mid -\infty < \tau < \infty\}.$$

The most right points of these curves are $\alpha^2 - c\alpha + 1 - a_1$ and $d\beta^2 - c\beta - r$, respectively, which are negative if

$$\alpha \in (\mu_1, \mu_2) \quad \text{and} \quad \beta \in (0, \mu_3), \tag{3.9}$$

where μ_1, μ_2, and μ_3 are defined in (2.3)-(2.4). Hence, when the above condition satisfies, $S_+ = S_{+,1} \cup S_{+,2}$ is on the left-half complex plane.

Similarly, we find S_- by solving the equation $\det(-\tau^2 D + i\tau M_- + N_- - \lambda I) = 0$, with

$$M_- = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad N_- = \begin{pmatrix} -1 & a_1 \\ 0 & r(1-a_2) \end{pmatrix}.$$

This gives two solutions $\lambda = \lambda_{3,4}$, where

$$\begin{aligned} \lambda_3 &= -\tau^2 + i\tau c - 1, \\ \lambda_4 &= -\tau^2 d + i\tau c + r(1-a_2). \end{aligned}$$

From (**C1**), $S_- = \{\lambda_3 \mid -\infty < \tau < \infty\} \cup \{\lambda_4 \mid -\infty < \tau < \infty\}$ is on the left-half complex plane.

The above analysis shows that the essential spectrum of $\mathcal{L}_w$ is on the left-half complex plane as long as conditions (3.8) and (3.9) are satisfied. In fact, there are many choices of α and β satisfying these conditions depending on μ_1, μ_2, and μ_3. We choose them by the following algorithm:

Algorithm 1. Two mechanisms are valid to choose α and β so that all conditions in (3.8) and (3.9) hold:

(1) If $\mu_1 < \mu_3$, then we choose $\beta = \alpha$ for any $\alpha \in (\mu_1, \min\{\mu_2, \mu_3\})$;

(2) If $\mu_1 \geq \mu_3$, then we choose $\epsilon < \beta < \mu_3$ and $\alpha = \mu_1 + \epsilon$ for small $\epsilon > 0$. In particular, we can choose $\beta = 2\epsilon$ and $\alpha = \mu_1 + \epsilon$, for $\epsilon < \min\{\mu_2 - \mu_1, \mu_3/2\}$.

Finally, in order to get a local stability result, we need to check the sign of the principal eigenvalue in the point spectrum for (3.1)-(3.2). Consider the associated linear partial differential system

$$u_t = Du_{zz} + cu_z + J(z)u, \tag{3.10}$$

where $u(z,t) = (u_1(z,t), u_2(z,t))$. The eigenpair (λ, Φ) of (3.1) implies a solution $e^{\lambda t}\Phi$ to the above system. Let $Q_t = u(t, z, \phi)$ denote the solution semiflow of (3.10) for any given initial data ϕ in L^p. It is easy to see Q_t is compact and strongly positive. By the well-known Krein-Rutman theorem (see e.g., [10]), Q_t has a simple principal eigenvalue $\lambda_{\max}$ with a strongly positive eigenvector, and all other eigenvalues $e^{\lambda t}$ must satisfy

$$|e^{\lambda t}| < e^{\lambda_{\max} t}.$$

For any $c > c^*$, we have from Theorem 2.1 that $\overline{U}(z) \sim C_1 e^{-\mu_1 z}$, $C_1 > 0$, as $z \to \infty$. $\lambda = 0$ is an eigenvalue to the operator $\mathcal{L}$ defined in (3.1) with the one-sign (strongly positive) eigenvector $(-\overline{U}', -\overline{V}')(z)$. By the choice of the weighted functional space L^p_w, the one-sign eigenvector $(\overline{U}', \overline{V}')(z)$ is not inside. Hence, the real parts of point spectrum of the operator $\mathcal{L}_w$ in L^p_w are all negative. We can also explain this in a simple analysis. Assume to the contrary that (λ, Φ) is an eigenpair of the eigenvalue problem (3.1)-(3.2) with $\lambda > 0$ and $\Phi \in L^p_w$. Obviously, the one-sign function $\bar{\Phi} = (-\overline{U}', -\overline{V}')(z)$ satisfies (3.10). For Φ in the L^p_w-space, we have essentially (or except for a set of zero measure) $\bar{\Phi}(z) > \Phi(z)$ as $z \to \infty$. On the other hand, when $z \to -\infty$, we can apply the method of asymptotic analysis and assume that the eigenfunction of (3.1) behaves like $ke^{\mu z}$ for some positive values k and μ. By substituting it into the eigenvalue problem and using the behavior of $J(z)$, we obtain that μ is increasing with respect to λ. This implies that $\bar{\Phi}(z) > \Phi(z)$ as $z \to -\infty$. Hence, by choosing $\bar{k}$ sufficient large, we can have $\bar{k}\bar{\Phi} \geq |\Phi|$. By comparison, from the partial differential system (3.10), we obtain $\bar{k}\bar{\Phi}(z) \geq |\Phi| e^{\lambda t}$, which contradicts $\lambda > 0$. This implies that for $\Phi \in L^p_w$, the real parts of all eigenvalues λ of (3.1) should be non-positive.

Now we are in a position to state the local stability result.

Theorem 3.1. *For any $c > c^*$, the wavefront $(\overline{U}, \overline{V})(z)$ is locally stable in the weighted functional space L^p_w with the weight function $w(z)$ defined in (3.3)-(3.4), where α and β in the formula of $w(z)$ are chosen by Algorithm 1.*

4 The global stability

We study here the global stability of the steady-state $(\overline{U}, \overline{V})(z)$ in a special choice of the weighted functional space $L^p_w(\mathbb{R})$. Let $p = \infty$ and define the norm $\|f\|_{L^\infty_w} = \text{ess } \sup_{z \in \mathbb{R}} |w(z)f(z)|$, for some weight function $w(z)$. Assume $\mu_1 < \mu_3$. By Algorithm 1, we choose $\alpha = \beta \in (\mu_1, \min\{\mu_2, \mu_3\})$. Specifically, let $\alpha = \beta = \mu_1 + \epsilon$, for small positive number ϵ. Also, we assume that the functions $\overline{U}(z)$ and $\overline{V}(z)$ satisfy the condition

$$\frac{\overline{V}(z)}{\overline{U}(z)} \leq \min\{a_2, 1/a_1\}, \quad \forall z \in (-\infty, +\infty). \tag{\textbf{C2}}$$

Theorem 4.1. *Suppose $c > c^*$, $\mu_1 < \mu_3$, and conditions **(C1)**-**(C2)** hold true. If the initial data $U(z,0) = U_0(z)$ and $V(z,0) = V_0(z)$ satisfy*

$$\begin{aligned} &(0,0) \leq (U_0, V_0)(z) \leq (1,1), \quad \forall z \in \mathbb{R}, \\ &\lim_{z\to-\infty} \inf (U_0, V_0)(z) > (0,0), \end{aligned}$$

and

$$\left[U_0(z) - \overline{U}(z)\right] \in L^\infty_w(\mathbb{R}), \quad \left[V_0(z) - \overline{V}(z)\right] \in L^\infty_w(\mathbb{R}).$$

Then the solution $(U,V)(z,t)$ to (1.6) exists globally with

$$(0,0) \leq (U,V)(z,t) \leq (1,1), \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+,$$

and converges to the steady-state $(\overline{U},\overline{V})(z)$ exponentially in the sense of

$$\begin{aligned} &\sup_{z\in\mathbb{R}} \left|U(z,t) - \overline{U}(z)\right| \leq k e^{-\eta t}, \quad t > 0, \\ &\sup_{z\in\mathbb{R}} \left|V(z,t) - \overline{V}(z)\right| \leq k e^{-\eta t}, \quad t > 0, \end{aligned}$$

for positive constants k and η.

To prove Theorem 4.1, we will find an upper and a lower solution to the partial differential equations system (1.6). For $z \in \mathbb{R}$, define

$$\begin{aligned} U_0^+(z) &= \max\left\{U_0(z), \overline{U}(z)\right\}, \quad V_0^+(z) = \max\left\{V_0(z), \overline{V}(z)\right\}, \\ U_0^-(z) &= \min\left\{U_0(z), \overline{U}(z)\right\}, \quad V_0^-(z) = \min\left\{V_0(z), \overline{V}(z)\right\}. \end{aligned}$$

It is easy to see that the following inequalities are true

$$\begin{aligned} &(0,0) \leq (U_0^-, V_0^-)(z) \leq (U_0, V_0)(z) \leq (U_0^+, V_0^+)(z) \leq (1,1), \\ &(0,0) \leq (U_0^-, V_0^-)(z) \leq (\overline{U},\overline{V})(z) \leq (U_0^+, V_0^+)(z) \leq (1,1). \end{aligned} \tag{4.1}$$

Denote $(U^+, V^+)(z,t)$ and $(U^-, V^-)(z,t)$ as the solutions to the system (1.6) with the initial data $(U_0^+, V_0^+)(z)$ and $(U_0^-, V_0^-)(z)$, respectively, that is,

$$\begin{cases} U_t^\pm = U_{zz}^\pm + cU_z^\pm + U^\pm(1 - a_1 - U^\pm + a_1 V^\pm), \\ V_t^\pm = dV_{zz}^\pm + cV_z^\pm + r(1 - V^\pm)(a_2 U^\pm - V^\pm), \\ (U^\pm, V^\pm)(z,0) = (U_0^\pm, V_0^\pm)(z). \end{cases} \tag{4.2}$$

By the comparison principle, one gets

$$\begin{aligned} &(0,0) \leq (U^-, V^-)(z,t) \leq (U,V)(z,t) \leq (U^+, V^+)(z,t) \leq (1,1), \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+, \\ &(0,0) \leq (U^-, V^-)(z,t) \leq (\overline{U},\overline{V})(z) \leq (U^+, V^+)(z,t) \leq (1,1), \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+. \end{aligned} \tag{4.3}$$

In the following lemmas we shall prove the convergence of $(U^+, V^+)(z,t)$ and $(U^-, V^-)(z,t)$ to the wavefront $(\overline{U},\overline{V})(z)$. Then we apply the squeezing theorem to obtain the result in Theorem 4.1.

Lemma 4.1. *Under the conditions in Theorem 4.1, $(U^+,V^+)(z,t)$ converges to $(\overline{U},\overline{V})(z)$.*

Proof. For $(z,t)\in\mathbb{R}\times\mathbb{R}^+$, define

$$P(z,t)=U^+(z,t)-\overline{U}(z) \text{ and } Q(z,t)=V^+(z,t)-\overline{V}(z).$$

These functions, P and Q, satisfy the initial value conditions

$$P(z,0)=U_0^+(z)-\overline{U}(z) \text{ and } Q(z,0)=V_0^+(z)-\overline{V}(z).$$

By (4.1) and (4.3), for all $z\in\mathbb{R}$ and $t\geq 0$, we have

$$(0,0)\leq (P,Q)(z,t)\leq (1,1).$$

By (1.4) and (4.2) and using condition (**C2**), we can verify that P and Q satisfy

$$\begin{aligned} P_t &\leq P_{zz}+cP_z+(1-a_1)P+(P+\overline{U})(-P+a_1Q),\\ Q_t &\leq Q_{zz}+cQ_z+r(a_2P-Q)+r(Q+\overline{V})(-a_2P+Q). \end{aligned} \tag{4.4}$$

To study the stability in the weighted functional space L_w^∞, with $w(z)$ defined in (3.3), we first let

$$\begin{pmatrix} P\\ Q\end{pmatrix}(z,t)=e^{-\alpha(z-z_0)}\begin{pmatrix} \overline{P}\\ \overline{Q}\end{pmatrix}(z,t), \quad \text{for all } (z,t)\in\mathbb{R}\times\mathbb{R}^+,$$

where $\overline{P}$ and $\overline{Q}$ are functions in $L^\infty(\mathbb{R})$ and z_0 is the same number used in the weight function $w(z)$. This gives

$$\begin{aligned} \begin{pmatrix} \overline{P}\\ \overline{Q}\end{pmatrix}_t &\leq D\begin{pmatrix} \overline{P}\\ \overline{Q}\end{pmatrix}_{zz}+M\begin{pmatrix} \overline{P}\\ \overline{Q}\end{pmatrix}_z+A(\alpha)\begin{pmatrix} \overline{P}\\ \overline{Q}\end{pmatrix}+\begin{pmatrix} (\overline{U}+e^{-\alpha(z-z_0)}\overline{P})(-\overline{P}+a_1\overline{Q})\\ r(\overline{V}+e^{-\alpha(z-z_0)}\overline{Q})(-a_2\overline{P}+\overline{Q})\end{pmatrix}\\ &:=\begin{pmatrix} \mathcal{L}_1(\overline{P},\overline{Q})\\ \mathcal{L}_2(\overline{P},\overline{Q})\end{pmatrix}, \end{aligned} \tag{4.5}$$

where $A(\alpha)$ is the same matrix defined in (2.2) and $M=diag(c-2\alpha,c-2d\alpha)$.

Define $\overline{P}_1(z,t)$ and $\overline{Q}_1(z,t)$ as

$$\overline{P}_1(z,t)=k_1\zeta_1e^{-\eta_1t} \text{ and } \overline{Q}_1(z,t)=k_1\zeta_2e^{-\eta_1t}, \quad \forall (z,t)\in\mathbb{R}\times\mathbb{R}^+,$$

for some constants $k_1,\eta_1>0$ to be chosen and $(\zeta_1,\zeta_2)=(\zeta_1(\alpha),\zeta_2(\alpha))$ is the eigenvector of the matrix $A(\alpha)$ associated to the eigenvalue $\alpha^2-c\alpha+1-a_1$. Simple computations give

$$\begin{aligned} \zeta_1(\alpha)&=(\alpha^2-c\alpha+1-a_1)-(d\alpha^2-c\alpha-r)\\ &=(\mu_1^2+\epsilon)(1-d)+1-a_1+r,\\ \zeta_2(\alpha)&=ra_2, \end{aligned}$$

which are positive for small ϵ and $\mu_1<\mu_3$. Since the initial values $\overline{P}(z,0)$ and $\overline{Q}(z,0)$ are in the space L_w^∞, we can choose $k_1\geq \max_{z\in\mathbb{R}}\{\overline{P}(z,0)/\zeta_1,\overline{Q}(z,0)/\zeta_2\}$. Direct computations and using condition (**C2**) show that both of $\mathcal{L}_1(\overline{P}_1,\overline{Q}_1)$ and $\mathcal{L}_2(\overline{P}_1,\overline{Q}_1)$ are negative. This allows to choose a positive value to η_1 so that the inequality

$$\begin{pmatrix} \overline{P}_1 \\ \overline{Q}_1 \end{pmatrix}_t = -\eta_1 k_1 \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} e^{-\eta_1 t} \geq \begin{pmatrix} \mathcal{L}_1(\overline{P}_1,\overline{Q}_1) \\ \mathcal{L}_2(\overline{P}_1,\overline{Q}_1) \end{pmatrix}. \tag{4.6}$$

holds. Hence, since $(\overline{P}_1,\overline{Q}_1)(0,z) \geq (\overline{P},\overline{Q})(0,z)$ and by comparison on unbounded domain, see e.g. [2, Proposition 2.1],

$$(P,Q)(z,t) = (\overline{P},\overline{Q})e^{-\alpha(z-z_0)} \leq k_1(\zeta_1,\zeta_2)e^{-\alpha(z-z_0)-\eta_1 t}, \quad \forall (z,t) \in \mathbb{R}\times\mathbb{R}^+.$$

In particular, this is true when $z \in [z_0,\infty)$, for any fixed z_0.

Now, we introduce the weight function $w(z)$ defined in (3.3)-(3.4) with $\alpha = \beta = \mu_1 + \epsilon$. By the above analysis, we need to prove the convergence of $(P,Q)(z,t)$ to $(0,0)$ for $z \in (-\infty, z_0]$. Note that the full system of $(P,Q)(z,t)$ can be expressed as

$$\begin{pmatrix} P \\ Q \end{pmatrix}_t = D\begin{pmatrix} P \\ Q \end{pmatrix}_{zz} + c\begin{pmatrix} P \\ Q \end{pmatrix}_z + J(z)\begin{pmatrix} P \\ Q \end{pmatrix} + \begin{pmatrix} (-P+a_1Q)P \\ r(-a_2P+Q)Q \end{pmatrix}. \tag{4.7}$$

Here, $J(z)$ is the same 2×2 matrix defined in (3.2). Let z_0 be chosen so that

$$J(z) \leq \begin{pmatrix} -1+\epsilon_1 & a_1+\epsilon_1 \\ \epsilon_1 & r(1-a_2)+\epsilon_1 \end{pmatrix} := J_{\epsilon_1},$$

for some given small $\epsilon_1 > 0$, when $z \leq z_0$. This is equivalent to require that $(\overline{U},\overline{V})(z)$ is close to $(1,1)$ for all $z \leq z_0$. Define $(\widehat{P},\widehat{Q})(t)$ as the solution of the autonomous system

$$\begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix}_t = J_{\epsilon_1}\begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix} + \begin{pmatrix} (-\widehat{P}+a_1\widehat{Q})\widehat{P} \\ r(-a_2\widehat{P}+\widehat{Q})\widehat{Q} \end{pmatrix}, \tag{4.8}$$

with the initial data

$$\widehat{P}(0) \geq \overline{P}(z,0), \quad \widehat{Q}(0) \geq \overline{Q}(z,t), \ \forall z \in \mathbb{R}.$$

Then $(\widehat{P},\widehat{Q})$ is an upper solution to the system (4.7).

Now we need to prove the convergence of $(\widehat{P},\widehat{Q})(t)$ to $(0,0)$ as $t\to\infty$. The Jacobian matrix $J(0,0) = J_{\epsilon_1}$ of system (4.8) at the fixed point $(0,0)$ has two eigenvalues, $\hat{\lambda}_2 < \hat{\lambda}_1 < 0$. By the phase plane analysis, there exists $0 < \delta \leq 1$ so that the flow in the $\widehat{P}\widehat{Q}-$space converges to origin for any initial data $(\widehat{P},\widehat{Q})(0)$ in the box $[0,1]\times[0,\delta]$. Hence, we conclude that

$$(\widehat{P},\widehat{Q}) = \hat{k}_1(\widehat{C}_1,\widehat{C}_2)e^{\hat{\lambda}_1 t} \ \text{ as } \ t\to\infty,$$

for positive constant $\hat{k}_1$ and $(\widehat{C}_1 \ \ \widehat{C}_2)^T$ is the eigenvector of J_{ϵ_1} corresponding to $\hat{\lambda}_1$. For the maximal possible choice of the constant δ so that we have the convergence result inside the box $[0,1]\times[0,\delta]$, see Remark 4.1 below.

We can choose $\hat{k}_1$ large and $\bar{\lambda}_1 = \min\{\eta_1, -\hat{\lambda}_1\}$ so that, at the boundary $z = z_0$, we have

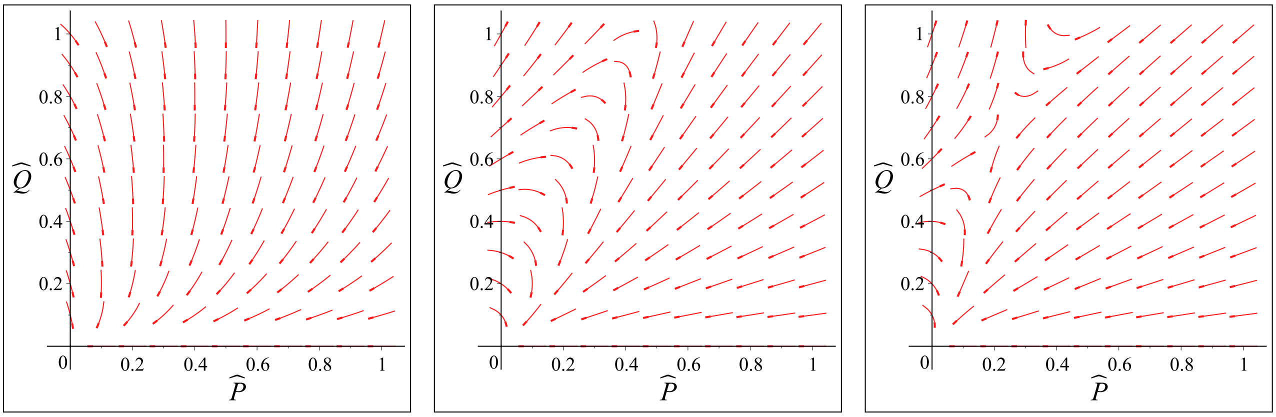
$$(P,Q)(z_0,t) \leq k_1(\zeta_1,\zeta_2)e^{-\eta_1 t} \leq \hat{k}_1(\zeta_1,\zeta_2)e^{-\bar{\lambda}_1 t}.$$

Hence, by comparison on the domain $(-\infty,z_0]\times[0,\infty)$, see e.g. [33, Lemma 3.2],

$$(P,Q)(z,t) \leq \hat{k}_1(\zeta_1,\zeta_2)e^{-\bar{\lambda}_1 t}, \quad \forall (z,t) \in (-\infty,z_0]\times\mathbb{R}^+.$$

This completes the proof. ∎

Remark 4.1. The maximal possible value of the constant δ, which could be 1, depends on the location of the fourth fixed point to the system (4.8) near or inside the box $[0,1] \times [0,1]$. See Figure 1 for all possible different cases. In (a), the positive fixed point is far away from the box $[0,1] \times [0,1]$ and does not effect the flow. This happens when $a_2 > 2$. Hence we set $\delta = 1$. The second figure (b) shows the effect of the positive fixed point on the flow, which still outside the box. The maximal choice of δ for this case exists in the interval $(a_2 - 1 - \epsilon_1/r, 1)$. The number $a_2 - 1 - \epsilon_1/r$ is the positive $\widehat{Q}$-intercept of the nullcline $\widehat{Q}_t = 0$. A fixed point exists inside the box $[0,1] \times [0,1]$ in (c), where δ becomes close to the value $a_2 - 1 - \epsilon_1/r$.



(a) $a_1 = 0.5$ and $a_2 = 2.4$. We choose $\delta = 1$

(b) $a_1 = 0.5$ and $a_2 = 1.4$. The maximal possible value of δ is in $(0.3984, 1)$.

(c) $a_1 = 0.3$ and $a_2 = 1.4$. The maximal possible value of δ becomes close to 0.3984.

Figure 1: The phase portrait of system (4.8) when $\epsilon_1 = 0.003$ and $r = 1.875$.

Lemma 4.2. *Under the conditions in Theorem 4.1, $(U^-, V^-)(z,t)$ converges to $(\overline{U}, \overline{V})(z)$.*

Proof. For $(z,t) \in \mathbb{R} \times \mathbb{R}^+$, define

$$R(z,t) = \overline{U}(z) - U^-(z,t) \quad \text{and} \quad S(z,t) = \overline{V}(z) - V^-(z,t).$$

These functions, R and S, satisfy the initial value conditions

$$R(z,0) = \overline{U}(z) - U_0^-(z) \quad \text{and} \quad S(z,0) = \overline{V}(z) - V_0^-(z).$$

From (4.1) and (4.3), for all $z \in \mathbb{R}$ and $t \geq 0$, we have

$$(0,0) \leq (R,S)(z,t) \leq (1,1).$$

From (1.4) and (4.2), R and S satisfy the system

$$\begin{pmatrix} R \\ S \end{pmatrix}_t = D \begin{pmatrix} R \\ S \end{pmatrix}_{zz} + c \begin{pmatrix} R \\ S \end{pmatrix}_z + J(z) \begin{pmatrix} R \\ S \end{pmatrix} - \begin{pmatrix} (-R + a_1 S)R \\ r(-a_2 R + S)S \end{pmatrix}, \tag{4.9}$$

with $J(z)$ defined in (3.2). By condition (**C2**), we have

$$\begin{aligned} R_t &\leq R_{zz} + cR_z + (1-a_1)R + (R - \overline{U})(R - a_1 S), \\ S_t &\leq dS_{zz} + cS_z + r(a_2 R - S) + r(S - \overline{V})(a_2 R - S). \end{aligned} \tag{4.10}$$

Similar to the previous analysis in the proof of Lemma 4.1, and making a use of the facts $R < \overline{U}$ and $S < \overline{V}$, we can prove that there exist $\eta_2 > 0$ and

$$k_2 \geq e^{\alpha(z-z_0)} \max_{z\in\mathbb{R}}\{R(z,0)/\zeta_1, S(z,0)/\zeta_2\}$$

so that

$$(R,S)(z,t) \leq k_2(\zeta_1,\zeta_2)e^{-\eta_2 t}, \quad \forall (z,t) \in \mathbb{R}\times\mathbb{R}^+.$$

For the choice of z_0 in proof of Lemma 4.1, we study the stability in the weighted space L^∞_w. To this end, define $(\widehat{R},\widehat{S})(t)$ as the solution of the system

$$\begin{pmatrix} \widehat{R} \\ \widehat{S} \end{pmatrix}_t = J_{\epsilon_1}\begin{pmatrix} \widehat{R} \\ \widehat{S} \end{pmatrix} - w_1\begin{pmatrix} (-\widehat{R}+a_1\widehat{S})\widehat{R} \\ r(-a_2\widehat{R}+\widehat{S})\widehat{S} \end{pmatrix}, \tag{4.11}$$

with the initial data

$$\widehat{R}(0) \geq R(z,0), \quad \widehat{S}(0) \geq S(z,0), \ \forall z \in \mathbb{R}. \tag{4.12}$$

It is easy to see that $(\widehat{R},\widehat{S})$ is an upper solution to the system (4.9). The phase plane analysis shows that $(\widehat{R},\widehat{S})(t)$ converges to origin for any initial data in the region $[0,1]\times[0,1]$ except the point $(1,1)$. Similar to the previous lemma,

$$(R,S)(z,t) \leq \hat{k}_2(\zeta_1,\zeta_2)e^{-\bar{\lambda}_2 t}, \quad \forall (z,t) \in (-\infty,z_0]\times\mathbb{R}^+.$$

for some positive constants $\hat{k}_2$ and $\bar{\lambda}_2$. This completes the proof. ∎

Now, we are ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. From (4.3), for all $(z,t) \in \mathbb{R}\times\mathbb{R}^+$, we have

$$|R(z,t)| \leq |U(z,t) - \overline{U}(z)| \leq |P(z,t)|,$$
$$|S(z,t)| \leq |V(z,t) - \overline{V}(z)| \leq |Q(z,t)|.$$

By lemmas 4.1-4.2 and the squeezing theorem, it follows that there exist $k > 0$ and $\eta > 0$ so that

$$|U(z,t) - \overline{U}(z)| \leq ke^{-\eta t},$$
$$|V(z,t) - \overline{V}(z)| \leq ke^{-\eta t},$$

for all $(z,t) \in \mathbb{R}\times\mathbb{R}^+$. This proves the desired result. ∎

Condition (**C2**) is used in the previous analysis to construct the upper solutions in the proof of Lemmas 4.1-4.2. It implies that, at $c = c_0$ and $z \to +\infty$,

$$\frac{\zeta_2(\mu_1)}{\zeta_1(\mu_1)} \leq \min\{a_2, 1/a_1\},$$

and it can be guaranteed by

$$\begin{cases} d \leq 2, \\ (a_1a_2 - 1)r \leq (2-d)(1-a_1). \end{cases}$$

This condition arose in the linear speed selection studies, see [18]. To see that the condition (**C2**) can be realized for all $z \in \mathbb{R}$, we prove the following claim.

Claim 1. *$d = 0$ and $a_1 a_2 \leq 1$ imply (**C2**).*

Proof. In the case when $d = 0$, the $\overline{V}-$equation can be written in the form

$$\begin{cases} \overline{V}' = \dfrac{r}{c}(1-\overline{V})(\overline{V}-a_2\overline{U}), \\ \overline{V}(-\infty) = 1, \overline{V}(+\infty) = 0. \end{cases} \tag{4.13}$$

Since $a_1 a_2 \leq 1$, we need to prove $\overline{V}(z) \leq a_2 \overline{U}(z)$ for all $z \in \mathbb{R}$. Assume, for contrary, this is not true for some $\bar{z} \in \mathbb{R}$. By (4.13), $\overline{V}$ is increasing at the neighborhood of $\bar{z}$. Since $\overline{U}(z)$ is a decreasing function, we have $\overline{V}(\bar{z}+\delta) > \overline{V}(\bar{z}) > a_2\overline{U}(\bar{z}) > a_2\overline{U}(\bar{z}+\delta)$, for some $\delta > 0$. Similarly, we can show that $\overline{V}(z)$ is increasing for all $z \geq \bar{z}$, which contradicts the fact $\overline{V}(+\infty) = 0$. This implies that condition (**C2**) holds true. ∎

5 Conclusions

The local and the global stability of traveling waves to the two-species Lotka-Volterra competition model (1.3) under the condition (**C1**) are investigated. Using the linearization and the essential spectrum analysis in [12], we find that the traveling wavefront is stable in some weighted functional space, see Theorem 3.1. Many choices of the exponential weight functions are valid, see Algorithm 1.

Under some further condition, (**C2**), we apply the upper-lower solution method to obtain a global stability result. Indeed, we prove that both the upper and the lower solutions tend to the wavefront. Our main results are presented in Theorem 4.1.

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We have used Maple codes to do the figure in the paper. They are available from the corresponding author upon request. No other data were used in this study.

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