Minimal-speed selection of traveling waves to the Lotka-Volterra competition model

Ahmad Alhasanat* and Chunhua Ou†

Abstract

In this paper the minimal-speed determinacy of traveling wave fronts of a two-species competition model of diffusive Lotka-Volterra type is investigated. First, a cooperative system is obtained from the classical Lotka-Volterra competition model. Then, we apply the upper-lower solution technique on the cooperative system to study the traveling waves as well as its minimal-speed selection mechanisms: linear or nonlinear. New types of upper and lower solutions are established. Previous results for the linear speed selection are extended, and novel results on both linear and nonlinear selections are derived.

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1 Introduction

The Lotka-Volterra competition model with diffusion

$$\begin{cases} \phi_t = d_1\phi_{xx} + r_1\phi(1 - b_1\phi - a_1\psi), \\ \psi_t = d_2\psi_{xx} + r_2\psi(1 - a_2\phi - b_2\psi), \end{cases}$$

has been established to describe the growth of competitive two-species population, see [20]. Here, $\phi(x,t)$ and $\psi(x,t)$ are the population densities at time t and location x; d_1 and d_2 are the diffusion coefficients; r_1 and r_2 are the net birth rates; a_1 and a_2 are competition coefficients; $1/b_1$ and $1/b_2$ are the carrying capacities. All these parameters are assumed to be non-negative. Based on the model, Okubo *et al.* [20] have successfully investigated the

*Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, A1C 5S7, and Department of Mathematics and Statistics, Al-Hussein Bin Talal University, Ma'an, Jordan Email: A.Alhasanat@mun.ca.

†Corresponding author. Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, A1C 5S7. Email: ou@mun.ca. This work is supported by the NSERC discovery grant of Canada # 204509.

interaction between the externally introduced gray squirrels and the indigenous red squirrels in Britain. The equivalent non-dimensional system to the above classical model can be given by

$$\begin{cases} \phi_t = \phi_{xx} + \phi(1 - \phi - a_1\psi), \\ \psi_t = d\psi_{xx} + r\psi(1 - a_2\phi - \psi), \end{cases}$$

via the scaling

$$\sqrt{r_1/d_1}\ x \to x, \qquad r_1 t \to t,$$

$$b_1\phi(x,t) \to \phi(x,t), \qquad b_2\psi(x,t) \to \psi(x,t),$$

$$d = \frac{d_2}{d_1}, \qquad r = \frac{r_2}{r_1}, \qquad \frac{a_1}{b_2} \to a_1, \qquad \frac{a_2}{b_1} \to a_2.$$

Furthermore, a cooperative system can be generated from the latter system by using $u = \phi$ and $v = 1 - \psi$. The system satisfied by the new functions, $u(x,t)$ and $v(x,t)$, reads

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = dv_{xx} + r(1 - v)(a_2 u - v), \end{cases} \tag{1.1}$$

with initial data

$$u(x,0) = \phi(x,0), \qquad v(x,0) = 1 - \psi(x,0), \quad \forall x \in \mathbb{R}.$$

Throughout this paper, we assume that the competition coefficients a_1 and a_2 satisfy the inequality

$$0 < a_1 < 1 < a_2. \tag{1.2}$$

When (1.2) is satisfied, the system (1.1) has three equilibria $e_0 := (0,0)$, $e_1 := (1,1)$, and $e_2 := (0,1)$ in the box $\{(u,v) | 0 \le u \le 1, 0 \le v \le 1\}$.

Introduce a new variable $z = x - ct$, for $c \ge 0$, and define

$$(u,v)(x,t) = (U,V)(z)$$

as a traveling wave solution to the system (1.1) connecting e_1 and e_0. Thus, it follows that $U(z)$ and $V(z)$ should satisfy

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) = 0, \\ dV'' + cV' + r(1 - V)(a_2 U - V) = 0, \end{cases} \tag{1.3}$$

subject to

$$(U,V)(-\infty) = e_1, \quad (U,V)(\infty) = e_0. \tag{1.4}$$

Here, prime denotes the derivative d/dz, (U,V) is called the wavefront, and c is the wave speed. It is known that there exists $c_{\min} \ge 0$ so that (1.3)-(1.4) has a non-negative monotone solution if and only if $c \ge c_{\min}$, see [13, 15, 16, 25]. Standard linearization near the equilibrium point e_0 shows

$$c_{\min} \ge c_0 = 2\sqrt{1 - a_1}.$$

The significance of the minimal speed is well-known. It is the spreading speed of species invasion onto an unstable state. The formula for the minimal wave speed is usually not easy to determine unless it is equal to c_0. However, for some choices of the parameters, this speed may be strictly greater than c_0. For convenience, we give the following definition of the linear or nonlinear determinacy of the minimal wave speed.

Definition 1. *If $c_{\min} = c_0$, then we say that the minimal wave speed is linearly selected; otherwise, if $c_{\min} > c_0$, we say that the minimal wave speed is nonlinearly selected.*

The speed selection mechanism has been studied widely in literature, e.g. [5, 8, 9, 10, 11, 12, 14, 15, 17, 19, 20, 22, 23, 24, 26, 28]. In [19, 20], the linearization of the system (1.3) gives a linear speed c_0 and it was conjectured to be the spreading speed. Later on, it was showed that this is not true for arbitrary choice of the parameters. For example, numerically, Hosono [10] demonstrated that either linear or nonlinear speed selection can be realized for a proper choice of the parameters. Analytically, there have been great progress regarding the spreading speed of the model. However, most of the rigorous results are concerned with the linear selection due to the difficulty of the study of the nonlinear selection, see [5, 10, 11, 14, 15]. Particularly, in [10], it has been shown that the linear speed selection for the system (1.3) is realized if

$$d = 0 \quad \text{and} \quad (a_1 a_2 - 1) r \leq 2(1 - a_1). \tag{1.5}$$

Weinberger *et al.* [28] studied the linear speed selection for the cooperative models of the form

$$\begin{aligned} &(\mathbf{u}_i)_t = d_i (\mathbf{u}_i)_{xx} + \ \mathbf{f}_i(\mathbf{u}), \ \ i = 1, 2, \ldots k, \\ &\mathbf{u}(x, 0) = \mathbf{u}_0(x), \end{aligned}$$

with $d_i \geq 0$ and $\mathbf{f} = (f_1, f_2, \ldots, f_k)$ being independent of x and t. They proved that the minimal wave speed is linearly selected when $\mathbf{f}$ satisfies certain conditions, see [28] for details. Lewis *et al.* [14] applied the results in [28] to study the speed selection mechanism for the system (1.3). They obtained that the linear speed selection is realized if

$$0 \leq d \leq 2, \quad \text{and} \quad (a_1 a_2 - 1) r \leq (2 - d)(1 - a_1). \tag{1.6}$$

Note that (1.6) includes (1.5). Huang [11], by applying the upper-lower solution method, proved that the linear speed selection for the system (1.3) is realized when the condition

$$\frac{(2 - d)(1 - a_1) + r}{r a_2} \geq \max\left\{a_1, \frac{d - 2}{2|d - 1|}\right\} \tag{1.7}$$

holds. When $d \leq 2$, this is equivalent to (1.6).

For a complete classification on the speed selection of this model, we should mention that, based on his numerical simulations, Hosono in [10] raised the following conjecture.

Hosono's conjecture. *If $a_1 a_2 \leq 1$, then $c_{\min} = c_0$ for all $r > 0$. If $a_1 a_2 > 1$, then there exits a positive number r_c such that $c_{\min} = c_0$ for $0 < r \leq r_c$, and $c_{\min} > c_0$ for $r > r_c$.*

Despite the great progress in the study of the problem of the speed selection, the confirmation or denial of this conjecture has been outstanding for two decades. Fortunately when $d = 0$, the speed selection problem has been completely solved in our recent paper [1]. It was found that this conjecture is not completely correct and a modified version was presented and proved. In particular, we extended the linear determinacy in the above conditions to

$$(a_1 a_2 - M)r \leq 2M(1 - a_1), \quad M := \max\{1, 2(1 - a_2)\}. \tag{1.8}$$

Huang in [11] strongly believed that the condition (1.6) could be necessary and sufficient for the linear speed selection, while the results in [1] are against this claim. Also, numerical computations when $d > 0$ in [22] show that, for different values of the parameters, the region of $c^* = c_0$ is larger than that determined by the conditions (1.6) and (1.7). For nonlinear selection, Huang and Han [12] proved that the minimal wave speed of the system (1.3) is nonlinearly selected when $d = r$ and $a_1 \in (1 - \bar{\epsilon}, 1)$ for small $\bar{\epsilon}$. Traveling waves for a similar model are also studied by the method of geometrical singular perturbation in [7].

The purpose of this paper is to further study the speed selection mechanism for the system (1.3) in the much more important case $d \neq 0$. Now the wave profile is a four-dimensional system which is naturally more complicated than the case $d = 0$. The method in [1] cannot be directly applied. Even though we still resort to the upper-lower solution method coupled with the comparison theorem, the construction of them becomes more technical. We will incorporate a novel transform by way of abstractly solving the second equation and reduce the system into a scalar non-local equation. By a proper construction of the upper and the lower solutions that are completely different from the classical contribution in [3], we successfully establish some new results on the linear or nonlinear speed selection. In particular, a condition that is weaker than (1.6) and includes (1.8) is provided for the linear speed selection. A new condition that includes the condition (1.7) for some cases is also provided. Also, another new nonlinear selection result, which covers the result in [12], is obtained. More precisely, our novel results are that the minimal speed of the system is nonlinearly selected if

$$\frac{(d+2)(1-a_1)+r}{ra_2} < 1 - 2(1-a_1),$$

and linearly selected if $a_1 < \frac{1}{3}$ and

$$\frac{(d-4)(1-a_1)}{4} < r < \frac{d(1-a_1)}{2a_2} \quad \text{or} \quad \frac{d(1-a_1)}{2a_2} \leq r < \frac{(d+4)(1-a_1)}{4(a_2-1)}.$$

The rest of the paper is organized as follows. In Section 2, we derive general conditions (inequalities) for the linear and nonlinear speed selections from a particular choice of upper and lower solutions. These inequalities are used to find some explicit conditions for the speed selection in Section 3. Conclusions are presented in Section 4. The last two sections are appendices, where we study the wave profile behavior locally near the equilibrium solution e_0 in Section 5, and the idea of the upper-lower solution method is provided for interested readers in Section 6.

2 The speed selection mechanism

We study here the speed selection mechanism of the problem (1.3)-(1.4) using the upper-lower solution method illustrated in Section 6. Following the idea in [1], we shall first prove the existence of solution $V(z)$ to the V-equation, for any given continuous function $U(z)$ satisfying $U(-\infty)=1$ and $U(\infty)=0$. The function $V(z)$ can be thought as an upper or a lower solution to the V-equation. Upper and lower solutions to the U-equation can be constructed after the substitution of the function $V(z)$ into the U-equation. To prove the existence of $V(z)$ for a given $U(z)$, we follow the idea in [2], see also the application in [4].

Lemma 2.1. *The second order ordinary differential equation*

$$dw''(t)-cw'(t)+rw(t)(\eta(t)-w(t))=0,$$

where $\eta(t)$ is the step function

$$\eta(t)=\begin{cases}\eta_+, & t\le t_0,\\ \eta_-, & t>t_0,\end{cases}$$

for some $t_0\in(-\infty,\infty)$ and constants η_-,η_+ satisfying $\eta_-<0<\eta_+$, has a monotone non-negative solution $w(t)$ satisfying $w(-\infty)=\eta_+$ and $w(\infty)=0$ if $c>-2\sqrt{dr\eta_+}$.

For the proof of this lemma, we refer interesting readers to [2], which is also applied in Section 5.2 in [4].

Based on the above lemma, we can prove the following result.

Lemma 2.2. *For $c\ge c_0=2\sqrt{1-a_1}$ and any given continuous non-increasing function $U(z)$, with $U(-\infty)=1$ and $U(\infty)=0$, there exists a non-increasing function $V(z)$ satisfying the equation*

$$\begin{cases}dV''+cV'+r(1-V)(a_2U-V)=0,\\ V(-\infty)=1,\quad V(\infty)=0.\end{cases}\tag{2.1}$$

Proof. Let $w(z)=1-V(z)$, then the equation of $w(z)$ reads

$$\begin{cases}dw''(z)+cw'(z)+rw(z)(1-a_2U(z)-w(z))=0,\\ w(-\infty)=0,\quad w(\infty)=1.\end{cases}$$

Rescale the variable z as $z=-t$. Define $w(z)=w_1(t)$ and $1-a_2U(z)=a(t)$. Then we have

$$\begin{cases}dw_1''(t)-cw_1'(t)+rw_1(t)(a(t)-w_1(t))=0,\\ w_1(-\infty)=1,\quad w_1(\infty)=0,\end{cases}\tag{2.2}$$

with $a(-\infty)=1$, $a(\infty)=1-a_2<0$. Clearly, proving the existence of non-increasing $w_1(t)$ with $w_1(-\infty)=1$ and $w_1(\infty)=0$ in (2.2) gives the result for $V(z)$ in (2.1).

The upper-lower solution method can be applied here to prove the existence of $w_1(t)$. We use Lemma 6.1 in the Appendix B to find an upper and a lower solutions in the sense of

Definition 2. It is easy to see that $\bar{w}(t) = 1$ is an upper solution to the system (2.2). To construct a lower solution, we choose $t_0 \in (-\infty, \infty)$ and small $\epsilon_4 > 0$, and define a step function

$$a_-(t) = \begin{cases} 1 - \epsilon_4, & t \leq t_0, \\ 2(1 - a_2), & t > t_0, \end{cases}$$

so that $a(t) \geq a_-(t)$, for all $t \in (-\infty, \infty)$. By Lemma 2.1, the equation

$$dw''(t) - cw'(t) + rw(t)(a_-(t) - w(t)) = 0$$

has a positive monotonic solution $\underline{w}(t)$ with $\underline{w}(-\infty) = 1 - \epsilon_4$ and $\underline{w}(\infty) = 0$ if $c \geq c_0$. This function is a lower solution to (2.2). Indeed,

$$\begin{aligned} d\underline{w}'' - c\underline{w}' + r\underline{w}(a(t) - \underline{w}) &= -r\underline{w}(a_-(t) - \underline{w}) + r\underline{w}(a(t) - \underline{w}) \\ &= r\underline{w}(a(t) - a_-(t)) \\ &\geq 0. \end{aligned}$$

This implies that, by Theorem 6.2, $w_1(t)$ exists and $\underline{w}(t) \leq w_1(t) \leq 1$. Note that $w_1(t) \not\equiv 1$. Otherwise if $w_1(t)$ is identical to 1, then $V(z) = 0$. By direct substitution, we have $a_2 U(z) = 0$, which is a contradiction. Moreover, $w_1(-\infty) = 1, w_1(\infty) = 0$, and $w_1'(t) \leq 0$. The proof is complete. ∎

Remark 2.3. *By the above lemmas, it is easy to establish that the linear speed selection is realized when $a_1 = 0$. In this case, system (1.3) reads*

$$\begin{cases} U'' + cU' + U(1 - U) = 0, \\ dV'' + cV' + r(1 - V)(a_2 U - V) = 0, \\ (U, V)(-\infty) = e_1, \quad (U, V)(\infty) = e_0. \end{cases}$$

The first equation is the well-known Fisher equation. It has a traveling wave solution for all $c \geq 2$. By Lemma 2.2, the solution $V(z)$ exists.

We can now construct a suitable upper and lower solutions to the U-equation. Linearizing the U equation (1.3) gives a characteristic equation

$$\Gamma_1(\mu) := \mu^2 - c\mu + 1 - a_1 = 0 \tag{2.3}$$

which gives two positive solutions

$$\mu_1 = \mu_1(c) = \frac{c - \sqrt{c^2 - 4(1 - a_1)}}{2} \text{ and } \mu_2 = \mu_2(c) = \frac{c + \sqrt{c^2 - 4(1 - a_1)}}{2}, \tag{2.4}$$

if $c \geq c_0$.

Let $c = c_0 + \epsilon_1$ and $\bar{k} = 1 + \bar{\epsilon}_1$, for sufficiently small non-negative numbers ϵ_1 and $\bar{\epsilon}_1$. Define a continuous monotonic function

$$\overline{U}(z) = \frac{\bar{k}}{1 + Ae^{\mu_1 z}}, \tag{2.5}$$

where A is a constant and $\mu_1 = \frac{1}{2}(c-\sqrt{c^2-4(1-a_1)})$ is defined in (2.4). Making use of Lemma 2.2, let $\overline{V}(z)$ be the solution of the V-equation when $U(z) = \overline{U}(z)$. Substituting $(\overline{U},\overline{V})(z)$ into the U-equation in (1.3), the left-hand side becomes

$$\overline{U}\left(1-\frac{\overline{U}}{\bar{k}}\right)\left\{\left(\mu_1^2-c\mu_1+1-a_1\right)+\frac{\overline{U}}{\bar{k}}\left(-2\mu_1^2+a_1\frac{\overline{V}-\overline{U}\left(\frac{a_1-1+\bar{k}}{a_1\bar{k}}\right)}{(1-\frac{\overline{U}}{\bar{k}})\frac{\overline{U}}{\bar{k}}}\right)\right\}. \tag{2.6}$$

Here, we have used $\overline{U}' = -\mu_1\overline{U}\left(1-\frac{\overline{U}}{\bar{k}}\right)$, and found $\overline{U}''$ by deriving this relation implicitly. Note that, when $c = c_0+\epsilon_1$, $\mu_1(c) = \sqrt{1-a_1}+\delta_1(\epsilon_1)$, for small $\delta_1(\epsilon_1)$. Since $\bar{\epsilon}_1$ is sufficiently small, we may only consider the limiting case $\bar{k}=1$ to conclude from (2.6) that $(\overline{U},\overline{V})(z)$ is an upper solution to the system (1.3) if

$$-2(1-a_1)+a_1J(z)<0, \quad \text{where} \quad J(z) = \frac{\overline{V}-\overline{U}}{(1-\overline{U})\overline{U}} \tag{2.7}$$

where $\overline{U}$ is defined in (2.5) with $\bar{k}=1$.

Remark 2.4. *If (2.7) holds for $\bar{k}=1$, then the value of (2.6) is negative when $\bar{k}=1+\bar{\epsilon}_1$, for sufficiently small $\bar{\epsilon}_1$. This fact can be easily confirmed.*

Remark 2.5. *Realization of the inequality in (2.7) requires that $J(z)$ is bounded above. For $c = c_0+\epsilon_1$ and $0 \le d < 2+\frac{r}{1-a_1}$, simple calculations show that $\lim_{z\to\pm\infty} J(z) < \infty$ and the required boundedness holds.*

In order to use Theorem 6.2 to find linear speed selection conditions, we need to construct a lower solution to (1.3) when the value of c is near c_0. Define a continuous function

$$\underline{U}(z) = \begin{cases} \xi_1 e^{-\mu_1 z}(1-Me^{-\epsilon_2 z}) & , z > z_1, \\ 0 & , z \le z_1, \end{cases}$$

where $0<\epsilon_2 \ll 1$, M is a positive constant to be determined, $z_1 = \frac{1}{\epsilon_2}\log M$, and ξ_1 is defined in Section 5.

Lemma 2.6. *When $c = c_0+\epsilon_1$, the pair of functions $(\underline{U}(z),\underline{V}(z))$ is a lower solution to the system (1.3), where $\underline{V}(z)$ is the solution of the V-equation with $U(z)=\underline{U}(z)$.*

Proof. We need to show that, for all $z \in (-\infty,\infty)$,

$$\begin{aligned} \underline{U}''+c\underline{U}'+\underline{U}(1-a_1-\underline{U}+a_1\underline{V}) &\ge 0, \\ d\underline{V}''+c\underline{V}'+r(1-\underline{V})(a_2\underline{U}-\underline{V}) &\ge 0. \end{aligned}$$

The first inequality, when $z \le z_1$, and the second inequality, for all z, are naturally satisfied. For the first inequality when $z > z_1$, we get

$$\begin{aligned} \underline{U}''+c\underline{U}'+\underline{U}(1-a_1-\underline{U}+a_1\underline{V}) = {} & \xi_1 e^{-\mu_1 z}\Gamma_1(\mu_1) - M\xi_1 e^{-(\mu_1+\epsilon_2)z}\Gamma_1(\mu_1+\epsilon_2) \\ & - \xi_1^2 e^{-2\mu_1 z}\left(1-Me^{-\epsilon_2 z}\right) + \xi_1^2 M e^{-(2\mu_1+\epsilon_2)z}\left(1-Me^{-\epsilon_2 z}\right) \\ & + a_1\xi_1\underline{V}e^{-\mu_1 z}\left(1-Me^{-\epsilon_2 z}\right). \end{aligned}$$

In view of the definition of $\Gamma(\mu_1)$ in (2.3), the first term equals to 0, and the second term is positive for sufficiently small ϵ_2. We choose M sufficiently large so that $z_1 > 0$ and the exponential function in the second term dominates that in the third term. The last two terms are positive. Hence, the proof is complete. ∎

Since the condition $\underline{U}'(z_1^-) \leq \underline{U}'(z_1^+)$ can be easily verified, we use $(\overline{U}, \overline{V})(z)$ and $(\underline{U}, \underline{V})(z)$ as upper and lower solutions in Theorem 6.2. This leads to the following linear speed selection result.

Theorem 2.7. *The linear speed selection of the system (1.3)-(1.4) is realized when (2.7) is satisfied.*

To study the nonlinear speed selection by the upper-lower solution method, we shall construct a lower solution $U(z)$ that behaves like $e^{-\mu_2 z}$, as $z \to \infty$, (with the faster decay rate). The lemma below provides a justification.

Lemma 2.8. *For $c_1 > c_0$, assume $(\underline{U}, \underline{V})(x - c_1 t) \geq 0$ is a lower solution to the partial differential system*

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = d v_{xx} + r(1 - v)(a_2 u - v). \end{cases} \tag{2.8}$$

Furthermore, suppose that $\underline{U}(z^*)$, $z^* = x - c_1 t$, is monotonic and satisfies $\limsup_{z^* \to -\infty} \underline{U}(z^*) < 1$, $\underline{U}(z^*) \sim e^{-\mu_2 z^*}$ as $z^* \to \infty$, where $\mu_2 = \frac{1}{2}(c + \sqrt{c^2 - 4(1 - a_1)})$ is defined in (2.4). Then no traveling wave solution exists for (1.3) with speed c in $[c_0, c_1)$.

Proof. To the contrary, assume there exists a monotone traveling wave solution $(U, V)(x - ct)$ to the system (2.8), with the initial conditions

$$u(x, 0) = U(x) \quad \text{and} \quad v(x, 0) = V(x),$$

for some c in $[c_0, c_1)$. Definitely (U, V) satisfies (1.3)-(1.4). By shifting if necessary, we can assume that, $\underline{U}(x) \leq U(x)$. Using the second equation, we can obtain $(\underline{U}, \underline{V})(x) \leq (U, V)(x)$. Since $(\underline{U}, \underline{V})(x - c_1 t)$ is a lower solution to the system (2.8) with the initial data $(\underline{U}, \underline{V})(x)$, and by comparison, we have

$$\underline{U}(x - c_1 t) \leq U(x - ct), \tag{2.9}$$

$$\underline{V}(x - c_1 t) \leq V(x - ct), \tag{2.10}$$

for all $(x, t) \in (-\infty, \infty) \times (0, \infty)$. Fix $z^* = x - c_1 t$ to have $\underline{U}(z^*) > 0$. Furthermore, we have

$$U(x - ct) = U(z^* + (c_1 - c)t) \sim U(\infty) = 0 \text{ as } t \to \infty.$$

By the above comparison in (2.9), we conclude that $\underline{U}(z^*) \leq 0$, which is a contradiction. This completes the proof. ∎

Now we are ready to define a lower solution to study the nonlinear speed selection. Define

$$\underline{U}_1(z) = \frac{\underline{k}}{1 + Be^{\mu_2 z}}, \tag{2.11}$$

where B is a positive constant and $0 < \underline{k} < 1$. Let $\underline{V}_1$ be the corresponding solution of the V-equation with $U(z) = \underline{U}_1(z)$. Substituting this pair of functions into the U-equation in (1.3), the left-hand side becomes

$$\underline{U}_1\left(1 - \frac{\underline{U}_1}{\underline{k}}\right)\left\{(\mu_2^2 - c\mu_2 + 1 - a_1) + \frac{\underline{U}_1}{\underline{k}}\left(-2\mu_2^2 + a_1\frac{\underline{V}_1 - \underline{U}_1\left(\frac{a_1 - 1 + \underline{k}}{a_1 \underline{k}}\right)}{(1 - \frac{\underline{U}_1}{\underline{k}})\frac{\underline{U}_1}{\underline{k}}}\right)\right\}.$$

From this, it is easy to see that the pair $(\underline{U}_1(z), \underline{V}_1(z))$ is a lower solution to the system (1.3) if

$$-2\mu_2^2 + a_1 J_1(z) > 0, \quad \text{where} \quad J_1(z) = \frac{\underline{V}_1 - \underline{U}_1\left(\frac{a_1 - 1 + \underline{k}}{a_1 \underline{k}}\right)}{\left(1 - \frac{\underline{U}_1}{\underline{k}}\right)\frac{\underline{U}_1}{\underline{k}}}. \tag{2.12}$$

As such, we have the following theorem.

Theorem 2.9. *The nonlinear speed selection of the system (1.3)-(1.4) is realized when (2.12) is satisfied.*

Remark 2.10. *A straight forward computation shows that* $\lim_{z\to-\infty} J_1(z) = \infty$, *for* $\underline{k} < 1$, *and* $\lim_{z\to\infty} J_1(z)$ *is finite.*

Remark 2.11. *The idea of this section can be extended by other novel choices of the upper-lower solutions of* U *so that they are geometrically the hetero-clinic connections of the following differential equations*

$$U' = -\mu U(1 - U^m), \; m > 0,$$

or

$$U' = -\mu U\left(1 - \frac{1}{1 - \ln U}\right),$$

or

$$U' = -\mu((1 - U)^\gamma - (1 - U)), \; 0 < \gamma < 1,$$

for some $\mu > 0$. *We will illustrate one example in next section.*

3 Explicit conditions on the speed selection

We shall use the idea of the previous section to derive some explicit conditions for linear or nonlinear speed selection from the construction of appropriate upper or lower solutions to the V-equation. This gives a number of new results.

In Theorems 3.1-3.6, we will use the upper solution $\overline{U}(z)$ defined in (2.5) with $\bar{k} = 1$.

Theorem 3.1. *If $0 \le d < 2$ and $a_1 a_2 \le 2(1-a_1)$, then the minimal wave speed is linearly selected.*

Proof. We re-define

$$\overline{V}(z) = \min\{1, a_2\overline{U}(z)\} = \begin{cases} 1, & z \le z_2, \\ a_2\overline{U}(z), & z > z_2, \end{cases}$$

where z_2 satisfies $a_2\overline{U}(z_2) = 1$. This function is an upper solution to the V-equation, when c is near c_0. Indeed, when $z \le z_2$, $d\overline{V}'' + c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = 0$, and when $z > z_2$, we have

$$d\overline{V}'' + c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = a_2\overline{U}(1-\overline{U})(d\mu_1^2 - c\mu_1 - 2d\mu_1^2\overline{U}).$$

By formula of $\mu_1(c)$ when $c = c_0 + \epsilon$ and since $d < 2$, the inequality

$$a_2\overline{U}(1-\overline{U})(d\mu_1^2 - c\mu_1 - 2d\mu_1^2\overline{U}) \le 0$$

is satisfied. $J(z)$ in (2.7) satisfies

$$J(z) = \begin{cases} \dfrac{1}{\overline{\overline{U}}} \le a_2, & \text{when } z \le z_2, \\ \dfrac{a_2 - 1}{1 - \overline{\overline{U}}} \le a_2, & \text{when } z > z_2, \end{cases}$$

which implies $-2(1-a_1) + a_1 J(z) < 0$ if $-2(1-a_1) + a_1 a_2 < 0$, i.e., the inequality in (2.7) is satisfied. By using Theorem 2.7, we know that the main result holds. When $-2(1-a_1)+a_1a_2 = 0$, a limiting argument can be used to prove the linear selection mechanism. ∎

Remark 3.2. *When $0 \le d < 2$, we can show that the function $(\xi_1, \xi_2)e^{-\mu_1 z}$, where ξ_1 and ξ_2 are defined in Section 5, is an upper solution to the system (1.3) when the condition (1.6) is satisfied. This recovers the result in [21].*

We combine the result in the above remark with Theorem 3.1 to have the following corollary.

Corollary 3.3. *When $0 \le d < 2$ and $a_1 a_2 \le \max\{1, 2(1-a_1)\}$, the minimal wave speed is linearly selected.*

Theorem 3.4. *The minimal wave speed is linearly selected if the following conditions*

$$\begin{cases} 0 \le d < 2, \\ a_1 \le 2/3, \ \ a_1 a_2 > 2(1-a_1), \\ r < \dfrac{2(2-d)(1-a_1)^2}{a_1 a_2 - 2(1-a_1)}, \end{cases}$$

are satisfied.

Proof. Here we choose $\overline{V}(z)$ as

$$\overline{V}(z) = \min\left\{1, \frac{2(1-a_1)}{a_1}\overline{U}\right\} = \begin{cases} 1, & z \le z_3, \\ \dfrac{2(1-a_1)}{a_1}\overline{U}(z), & z > z_3, \end{cases}$$

so that $2(1-a_1)\overline{U}(z_3) = a_1$. When $z \le z_3$, we have $d\overline{V}'' + c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = 0$, and when $z > z_3$, we have

$$\begin{aligned} d\overline{V}'' + c\overline{V}' &+ r(1-\overline{V})(a_2\overline{U}-\overline{V}) \\ &= \frac{2(1-a_1)}{a_1}\overline{U}(1-\overline{U})(d\mu_1^2 - c\mu_1 - 2d\mu_1^2\overline{U}) + r\left(1 - \frac{2(1-a_1)}{a_1}\overline{U}\right)\left(a_2\overline{U} - \frac{2(1-a_1)}{a_1}\overline{U}\right). \end{aligned}$$

Since $a_1 \le 2/3$, the inequality $1 - \frac{2(1-a_1)}{a_1}\overline{U} \le 1 - \overline{U}$ is true. Hence,

$$\begin{aligned} d\overline{V}'' + c\overline{V}' &+ r(1-\overline{V})(a_2\overline{U}-\overline{V}) \\ &\le \frac{2(1-a_1)}{a_1}\overline{U}(1-\overline{U})\left\{d\mu_1^2 - c\mu_1 - 2d\mu_1^2\overline{U} + r\left(\frac{a_1a_2}{2(1-a_1)} - 1\right)\right\} \\ &\le 0, \end{aligned}$$

when

$$0 \le d < 2 \quad \text{and} \quad r < \frac{2(2-d)(1-a_1)^2}{a_1a_2 - 2(1-a_1)}.$$

For the inequality (2.7), we have $J(z) \le \dfrac{2(1-a_1)}{a_1}$. Then $-2(1-a_1) + a_1J(z) < 0$. The result follows from Theorem 2.7. ∎

Again, from Remark 3.2 we have seen that, when $0 \le d < 2$, $a_1a_2 > 1$, and

$$r < \frac{(2-d)(1-a_1)}{a_1a_2 - 1},$$

the linear speed selection is realized. To connect this with the above theorem, define $M := \max\{1, 2(1-a_1)\}$. When $a_1 \le 1/2 < 2/3$, $M = 2(1-a_1)$. For this case, when $a_1a_2 > M$, we have proved that the minimal wave speed is linearly selected if

$$r < \frac{2(2-d)(1-a_1)^2}{a_1a_2 - 2(1-a_1)} = \frac{M(2-d)(1-a_1)}{a_1a_2 - M}.$$

This gives

Corollary 3.5. *The minimal wave speed is linearly selected when $0 \le d < 2$, $a_1a_2 > M$, and*

$$r < \frac{M(2-d)(1-a_1)}{a_1a_2 - M}.$$

Note that the above results, corollaries 3.3 and 3.5, extend (1.6).

Theorem 3.6. *For $2 < d < 2 + \frac{r}{1-a_1}$, the minimal wave speed is linearly selected when*

$$\frac{r-(d-2)(1-a_1)}{ra_2} > \max\left\{\frac{d-2}{2d}, \frac{a_1}{2(1-a_1)}\right\}. \tag{3.1}$$

Proof. We choose

$$\overline{V}(z) = \min\{1, a_2k\overline{U}\} = \begin{cases} 1, & z \le z_5, \\ a_2k\overline{U}(z), & z > z_5, \end{cases}$$

where

$$k = k(d, a_1, r) = \frac{r}{r-(d-2)(1-a_1)} + \eta_1 > 1$$

for a sufficiently small positive constant η_1 so that

$$\frac{1}{a_2k} > \max\left\{\frac{d-2}{2d}, \frac{a_1}{2(1-a_1)}\right\},$$

and z_5 satisfies $a_2k\overline{U}(z_5) = 1$.

When $z \le z_5$, we have $d\overline{V}'' + c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = 0$. When $z > z_5$, we have

$$\begin{aligned} d\overline{V}'' + c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) &= a_2\overline{U}\left\{k(1-\overline{U})\left(d\mu_1^2 - c\mu_1 - 2d\mu_1^2\overline{U}\right) + r(1-a_2k\overline{U})(1-k)\right\} \\ &:= a_2\overline{U}F(\overline{U}). \end{aligned}$$

When $z \ge z_5$, we obtain $0 \le \overline{U} \le 1/(a_2k)$. Note that $F''(\overline{U}) = 4d\mu_1^2k > 0$, which means that F is convex for $\overline{U}$ in $[0, 1/(a_2k)]$. To prove the inequality $F(\overline{U}) < 0$, we need to show that it is negative at the end points, 0 and $1/(a_2k)$. By direct substitution, when $c = c_0 + \epsilon_1,\ \epsilon_1 \ll 1$, we can find $F(0) = -\eta_1(r-(d-2)(1-a_1))$. In view of the condition (3.1), $F(0) < 0$. For the right-hand end point,

$$\begin{aligned} F\left(\frac{1}{a_2k}\right) &= k\left(1 - \frac{1}{a_2k}\right)\left(d\mu_1^2 - c\mu_1 - 2d\mu_1^2\frac{1}{a_2k}\right) \\ &< k\left(1 - \frac{d-2}{2d}\right)\left((d-2)(1-a_1) - 2d(1-a_1)\ \frac{d-2}{2d}\right) \\ &= 0. \end{aligned}$$

This implies that $d\overline{V}'' + c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) \le 0$, for all $z \in (-\infty, \infty)$.

To complete the proof, we need to prove the inequality (2.7). Since

$$J(z) = \frac{\overline{V} - \overline{U}}{(1-\overline{U})\overline{\overline{U}}} = \begin{cases} \dfrac{1}{\overline{\overline{U}}}, & z \le z_5, \\[2ex] \dfrac{a_2k - 1}{1 - \overline{U}}, & z > z_5, \end{cases}$$

$J(z) \le a_2k$. Then the inequality $-2(1-a_1) + a_1J(z) < 0$ follows from the condition. Hence, by Theorem 2.7, the result is proved. ∎

Next we want to establish a nonlinear speed selection condition by constructing a smooth lower solution to the V-equation. We use $\underline{U}_1(z)$ defined in (2.11) and re-define $\underline{V}_1(z)$, that is, we set

$$\underline{U}_1(z) = \frac{\underline{k}}{1+Be^{\mu_2 z}},\ \underline{V}_1(z) = \frac{\underline{U}_1(z)}{\underline{k}}. \tag{3.2}$$

Here B is a positive constant and $0<\underline{k}<1$. Then we have the following theorem.

Theorem 3.7. *The minimal speed of the system is nonlinearly selected if*

$$\frac{(d+2)(1-a_1)+r}{ra_2} < 1-2(1-a_1). \tag{3.3}$$

Proof. Under (3.3), we can take $\underline{k}$ such that

$$\frac{(d+2)(1-a_1)+r}{ra_2} < \underline{k} < 1-2(1-a_1). \tag{3.4}$$

After the substitution of (3.2), we need to verify the inequality (2.12) and

$$\underline{V}(1-\underline{V})(d\mu_2^2 - c\mu_2 - r + ra_2\underline{k} - 2d\mu_2^2) > 0. \tag{3.5}$$

Note that $\mu_2 \sim \sqrt{1-a_1}$ when $c = c_0+\epsilon_1$ for small ϵ_1. Then we can directly confirm that (3.2) is a lower solution to the wave profile system under (3.4). By Lemma 2.8, the proof is complete. ∎

Remark 3.8. *The above theorem implies that the minimal speed of the system is nonlinearly selected if a_1 is sufficiently close to 1. This extends the result in [12] without the requirement $d=r$.*

We proceed to establish a new linear selection result by constructing an upper solution inspired by Remark 2.11. We set $\overline{V} = \overline{U}^{\frac{1}{2}}$ and $\overline{U}$ is a solution to the following differential equation

$$\overline{U}' = -\mu_1\overline{U}(1-\overline{U}^{\frac{1}{2}}),\ \overline{U}(-\infty)=1, \overline{U}(\infty)=0. \tag{3.6}$$

When substituting them into (1.3) and making use of the facts

$$\overline{U}'' = \mu_1^2\overline{U}\left(1-\frac{3}{2}\overline{U}^{\frac{1}{2}}\right)\left(1-\overline{U}^{\frac{1}{2}}\right),$$

$$\overline{V}' = -\frac{1}{2}\mu_1\overline{V}(1-\overline{V}),$$

and

$$\overline{V}'' = \frac{1}{4}\mu_1^2\overline{V}(1-\overline{V})(1-2\overline{V}),$$

we can verify that $(\overline{U},\overline{V})$ is an upper solution to the system (1.3) if

$$\mu_1^2 - c\mu_1 + (1-a_1) + \left(-\frac{3}{2}\mu_1^2+1\right)\overline{U}^{\frac{1}{2}} \le 0,$$

and

$$\left(\frac{1}{4}d\mu_1^2 - \frac{1}{2}c\mu_1 - r\right) + \left(-\frac{1}{2}d\mu_1^2 + ra_2\right)\overline{V} \leq 0.$$

This, by Theorem 6.2, will lead to a linear speed selection result. Indeed, when $c = c_0 + \epsilon_1$, for small $\epsilon_1 > 0$, we have $\mu_1 \sim \sqrt{1 - a_1}$. Hence, the first inequality is true if $a_1 < 1/3$. For the second one, if

$$\frac{(d-4)(1-a_1)}{4} < r < \frac{d(1-a_1)}{2a_2}, \tag{3.7}$$

then both terms inside the brackets are negative and the result is valid. On the other hand, if

$$r \geq \frac{d(1-a_1)}{2a_2},$$

then we have

$$\frac{1}{4}d\mu_1^2 - \frac{1}{2}c\mu_1 - r + \left(-\frac{1}{2}d\mu_1^2 + ra_2\right)\overline{V} \leq -\frac{1}{4}d\mu_1^2 - \frac{1}{2}c\mu_1 + r(a_2 - 1),$$

which is negative when

$$r < \frac{(d+4)(1-a_1)}{4(a_2-1)}.$$

Thus, the above derivation is summarized in the following linear selection result.

Theorem 3.9. *The minimal wave speed of the system (1.3)-(1.4) is linearly selected if $a_1 < \frac{1}{3}$ and one of the following conditions is satisfied:*

$$\frac{(d-4)(1-a_1)}{4} < r < \frac{d(1-a_1)}{2a_2},$$

or

$$\frac{d(1-a_1)}{2a_2} \leq r < \frac{(d+4)(1-a_1)}{4(a_2-1)}.$$

Remark 3.10. *In the above linear and nonlinear results, by taking the same values of a_2 and r as that of [22] and plotting the regions for each condition in a_1d-plane, we can see that our results agree with Figure 3 in [22].*

Remark 3.11. *Assume that the wave speed is linearly selected when $a_2 = a_\beta$, for some $1 < a_\beta < \infty$. It can be easily proved that the corresponding solution $(U_\beta, V_\beta)(z)$ is an upper solution to the system for all $a_2 \leq a_\beta$. By using the upper-lower solution technique with $(U_\beta, V_\beta)(z)$ and the lower solution defined in Lemma 2.6, we conclude that the minimal wave speed is linearly selected for all $a_2 \leq a_\beta$. A similar discussion of this can also be found in [14] where the authors claimed that the spreading speed c^* is non-decreasing in a_2 (due to the fact that the spreading speed is the minimal speed of the system, we arrive at the same conclusion). Hence, either the minimal wave speed does not change for all $a_2 \geq 1$, or there exists $a_c \in (1, \infty)$ so that the linear speed selection is realized when $a \leq a_c$ and the nonlinear speed selection is realized when $a_2 > a_c$. The conditions of the above theorems give some estimations of a_c. Unfortunately, we could not verify a similar property with respect to r. This means that existence of r_c in Hosono's Conjecture is still unknown for $d > 0$, except for the case $d = 0$ in [1] where the conjecture is completely solved.*

4 Conclusions

The speed selection mechanism (linear and nonlinear) for traveling waves to a two-species Lotka-Volterra competition model (1.1) is investigated when $0 \le a_1 < 1 < a_2$ via the upper-lower solution method.

We give new conditions to the linear speed selection that extend the previous conditions. Indeed, the linear determinacy (1.6) in [14] has been extended to the condition

$$\begin{cases} 0 \le d < 2, \\ (a_1 a_2 - M) r \le M(2-d)(1-a_1), \ M = \max\{1, 2(1-a_1)\}, \end{cases}$$

see Theorems 3.1-3.4. This shows that condition (1.6) is sufficient but not necessary for the linear speed selection. The linear speed selection condition in [11], condition (1.7), is different from (1.6) only if $d > 2$. It also can not be satisfied if $d \ge 2 + \frac{r}{1-a_1}$. This means that the condition (1.7) contributes if and only if

$$2 < d < 2 + \frac{r}{1-a_1}.$$

The new result for this case can be seen from Theorem 3.6. A nonlinear selection result that extends the previous result in [12] is given in Theorem 3.7, and a novel linear selection result is given in Theorem 3.9.

5 Appendix A: Local analysis of the wave profile near e_0

In this section, we analyze the wave profile of the nonlinear system (1.3) locally near the equilibrium point e_0. By linearizing the system (1.3) around e_0, we get a constant-coefficient system

$$\begin{cases} U'' + cU' + U(1-a_1) = 0, \\ dV'' + cV' + r(a_2 U - V) = 0, \end{cases} \tag{5.1}$$

Let $(U, V)(z) = (\xi_1 e^{-\mu z}, \xi_2 e^{-\mu z})$, for some positive constants ξ_1, ξ_2, and μ. Substitute it into (5.1) to get

$$A(\mu) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.2}$$

where $A(\mu)$ is a 2×2 matrix given by

$$A(\mu) = \begin{pmatrix} \mu^2 - c\mu + (1-a_1) & 0 \\ ra_2 & d\mu^2 - c\mu - r \end{pmatrix} := \begin{pmatrix} \Gamma_1(\mu) & 0 \\ ra_2 & \Gamma_2(\mu) \end{pmatrix}.$$

The system of algebraic equations (5.2) has a non-trivial solution if and only if

$$\Gamma_1(\mu)\Gamma_2(\mu) = 0. \tag{5.3}$$

$\Gamma_1(\mu) = 0$ implies that μ equals one of the following values

$$\mu_1(c) = \frac{c - \sqrt{c^2 - 4(1-a_1)}}{2} \quad \text{and} \quad \mu_2(c) = \frac{c + \sqrt{c^2 - 4(1-a_1)}}{2}, \tag{5.4}$$

and $\Gamma_2(\mu) = 0$ implies that

$$\mu = \mu_3(c) = \frac{c + \sqrt{c^2 + 4dr}}{2d} > 0. \tag{5.5}$$

These solutions of equation (5.3) can also be written as

$$c = c_1(\mu) = \mu + \frac{1-a_1}{\mu} \quad \text{or} \quad c = c_2(\mu) = d\mu - \frac{r}{\mu}.$$

Clearly, for positive solution (U, V), we need $c \geq c_0 = 2\sqrt{1-a_1}$. Hence, $0 < \mu_1 \leq \mu_2$ is satisfied. Based on the relation among these values and μ_3, we shall divide the analysis into three cases. To this end, first, we define a constant $\hat{c}$, related to r, d, and a_1, as the intersection point of $c_1(\mu)$ and $c_2(\mu)$ at $\hat{\mu} = \sqrt{(r+1-a_1)/(d-1)}$. Direct computations show that $\hat{c}$ is well-defined if and only if $d > 1$ and, in such a case, it satisfies

$$\hat{c} = \sqrt{\frac{r+1-a_1}{d-1}} + (1-a_1)\sqrt{\frac{d-1}{r+1-a_1}}.$$

Now we consider the following three cases:

Case 1. $0 \leq d \leq 1$, or $1 < d < 2 + \frac{r}{1-a_1}$ with $c_0 < c < \hat{c}$. In this case, the inequality $\mu_1 < \mu_2 < \mu_3$ is satisfied, see Fig. 1 (a),(b) and Fig. 2 (a). Since $\Gamma_1(\mu_i) = 0$ and $\Gamma_2(\mu_i) < 0$, for $i = 1, 2$, the eigenvector of the matrix $A(\mu)$ corresponding to μ_i, $i = 1, 2$, is the strongly positive vector

$$\xi(\mu_i) = (-\Gamma_2(\mu_i) \quad ra_2)^T, \quad i = 1, 2.$$

Hence, as $z \to \infty$, the decaying positive solution (U, V) should satisfy

$$\begin{pmatrix} U(z) \\ V(z) \end{pmatrix} \sim C_1 \begin{pmatrix} -\Gamma_2(\mu_1) \\ ra_2 \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} -\Gamma_2(\mu_2) \\ ra_2 \end{pmatrix} e^{-\mu_2 z},$$

for constant $C_1 > 0$, or $C_1 = 0, C_2 > 0$.

Case 2. $d > 1$ and $c > \hat{c}$. This condition implies the inequality $\mu_1 < \mu_3 < \mu_2$, see Fig. 1 (b), (c). The eigenvector of $A(\mu)$ corresponding to μ_3 is $\xi = (0 \quad 1)^T$. We also have $\Gamma_2(\mu_1) < 0$ and $\Gamma_2(\mu_2) > 0$, see Fig. 2 (b). Consequently, as $z \to \infty$,

$$\begin{pmatrix} U(z) \\ V(z) \end{pmatrix} \sim C_1 \begin{pmatrix} -\Gamma_2(\mu_1) \\ ra_2 \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} \Gamma_2(\mu_2) \\ -ra_2 \end{pmatrix} e^{-\mu_2 z} + C_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\mu_3 z},$$

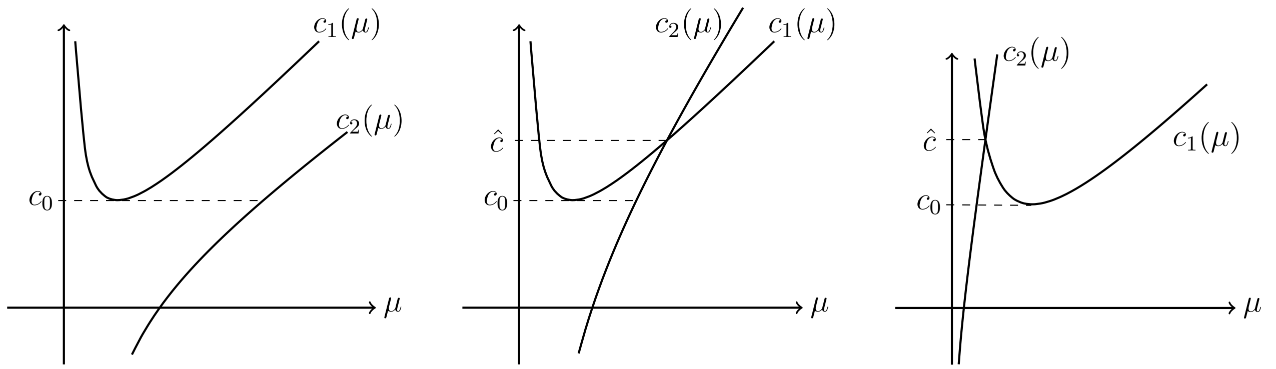
for constant $C_1 > 0$, or $C_1 = 0, C_2, C_3 > 0$.

Case 3. $d > 2 + \frac{r}{1-a_1}$ and $c_0 < c < \hat{c}$. In this case, we have $\mu_3 < \mu_1 < \mu_2$ and $\Gamma_2(\mu_i) > 0$, $i = 1, 2$, see Fig. 1 (c) and Fig. 2 (c). The solution, as $z \to \infty$, satisfies

$$\begin{pmatrix} U(z) \\ V(z) \end{pmatrix} \sim C_1 \begin{pmatrix} \Gamma_2(\mu_1) \\ -ra_2 \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} \Gamma_2(\mu_2) \\ -ra_2 \end{pmatrix} e^{-\mu_2 z} + C_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\mu_3 z},$$

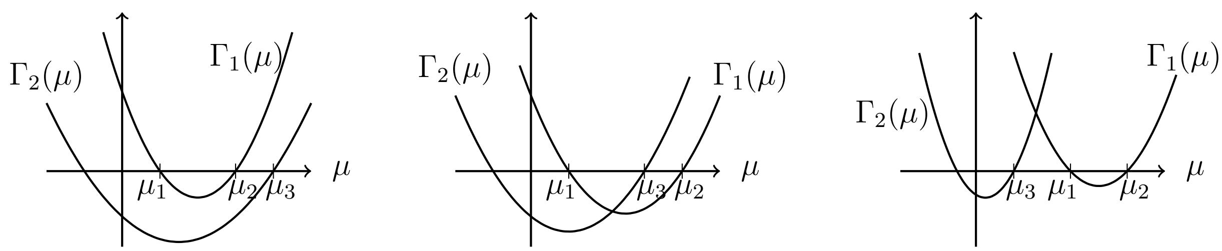
for constant $C_1, C_3 > 0$ or $C_1 = 0, C_2, C_3 > 0$.

A similar result can be found in Appendix of [6].



(a) $0 \leq d < 1$ (b) $1 \leq d \leq 2 + \frac{r}{1-a_1}$ (c) $d > 2 + \frac{r}{1-a_1}$

Figure 1: Graph of $c_1(\mu)$ and $c_2(\mu)$ for different values of d.



(a) Case 1. (b) Case 2. (c) Case 3.

Figure 2: Graph of $\Gamma_1(\mu)$ and $\Gamma_2(\mu)$ for cases 1, 2, and 3.

6 Appendix B: Upper-lower solution method

The method of upper-lower solution, originated in [3, 27], is used to prove the existence of monotone traveling wave solution to the partial differential equations. As a byproduct, the wave speed is likely to be determined. Here, we introduce this method and show how it works on system (1.3).

Let α be large enough so that

$$\alpha U + U(1 - a_1 - U + V) := F(U, V)$$

and

$$\alpha V + r(1 - V)(a_2 U - V) := G(U, V)$$

are monotone in U and V, respectively. System (1.3) can be written as

$$\begin{cases} U'' + cU' - \alpha U = -F(U, V), \\ dV'' + cV' - \alpha V = -G(U, V). \end{cases} \tag{6.1}$$

Define constants $\lambda_1^\pm$ and $\lambda_2^\pm$, for $d \neq 0$, as

$$\begin{aligned} \lambda_1^- &= \frac{-c-\sqrt{c^2+4\alpha}}{2} < 0, \quad \lambda_1^+ = \frac{-c+\sqrt{c^2+4\alpha}}{2} > 0, \\ \lambda_2^- &= \frac{-c-\sqrt{c^2+4\alpha d}}{2d} < 0, \quad \lambda_2^+ = \frac{-c+\sqrt{c^2+4\alpha d}}{2d} > 0. \end{aligned} \tag{6.2}$$

By the variation-of-parameters method, the integral form of (6.1) is given by

$$\begin{aligned} U(z) &:= T_1(U,V)(z), \\ V(z) &:= T_2(U,V)(z). \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} T_1(U,V)(z) &:= \frac{1}{\lambda_1^+ - \lambda_1^-} \left\{ \int_{-\infty}^{z} e^{\lambda_1^-(z-s)} F(U,V)(s)ds + \int_z^\infty e^{\lambda_1^+(z-s)} F(U,V)(s)ds \right\} \\ T_2(U,V)(z) &:= \frac{1}{d(\lambda_2^+ - \lambda_2^-)} \left\{ \int_{-\infty}^{z} e^{\lambda_2^-(z-s)} G(U,V)(s)ds + \int_z^\infty e^{\lambda_2^+(z-s)} G(U,V)(s)ds \right\}. \end{aligned}$$

Definition 2. *A pair of continuous functions $(U(z), V(z))$ is an upper (a lower) solution to the integral system (6.3) if*

$$\begin{cases} U(z) \geq (\leq) T_1(U,V)(z), \\ V(z) \geq (\leq) T_2(U,V)(z). \end{cases}$$

In the lemma below, we shall find inequalities in terms of the differential equations themselves that give the inequalities in Definition 2. These new inequalities shall be used in our analysis.

Lemma 6.1. *A continuous function $(U,V)(z)$ which is differentiable on $(-\infty,\infty)$ except at finite number of points $z_i, i = 1, \cdots, n$, and satisfies*

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) \leq \ 0, \\ dV'' + cV' + r(1-V)(a_2 U - V) \leq \ 0 \end{cases}$$

for $z \neq z_i$, and $(U',V')(z_i^-) \geq (U',V')(z_i^+)$, for all z_i, is an upper solution to the integral equations system (6.3). This is true for the lower solution by reversing the inequalities.

Proof. When the above inequalities hold, we have

$$\begin{aligned} T_1(U,V)(z) &= \frac{1}{\lambda_1^+ - \lambda_1^-} \left\{ \int_{-\infty}^{z} e^{\lambda_1^-(z-s)} F(U,V)(s)ds + \int_z^\infty e^{\lambda_1^+(z-s)} F(U,V)(s)ds \right\} \\ &\leq \frac{-1}{\lambda_1^+ - \lambda_1^-} \left\{ \int_{-\infty}^{z} e^{\lambda_1^-(z-s)} (U'' + cU' - \alpha U)(s)ds + \int_z^\infty e^{\lambda_1^+(z-s)} (U'' + cU' - \alpha U)(s)ds \right\}. \end{aligned}$$

By calculations similar to that of [18, proof of Lemma 2.5], we can show that

$$T_1(U,V)(z) \leq U(z).$$

The same is true for $T_2(U,V) \leq V(z)$. This implies that $(U,V)(z)$ is an upper solution to the system (6.3). The proof for the lower solution is the same and omitted. ∎

Now we assume the following hypothesis and state a result that shows how the existence of an upper and of a lower solution gives the existence of the actual solution and its estimation.

Hypothesis 1. *There exists a monotone non-increasing upper solution* $(\overline{U},\overline{V})(z)$ *and a non-zero lower solution* $(\underline{U},\underline{V})(z)$ *to the integral system (6.3) satisfying the properties:*

(1) $(\underline{U},\underline{V})(z) \leq (\overline{U},\overline{V})(z),$ *for all* $z \in \mathbb{R}$*,*

(2) $(\overline{U},\overline{V})(+\infty) = e_0, \quad (\overline{U},\overline{V})(-\infty) = (\bar{k}_1,\bar{k}_2),$

(3) $(\underline{U},\underline{V})(+\infty) = e_0, \quad (\underline{U},\underline{V})(-\infty) = (\underline{k}_1,\underline{k}_2),$

for $e_0 \leq (\underline{k}_1,\underline{k}_2) \leq e_1$ *and* $(\bar{k}_1,\bar{k}_2) \geq e_1 = (1,1)$ *so that no equilibrium solution to (1.3) exists in the set* $\{(U,V)|e_1 < (U,V) \leq (\bar{k}_1,\bar{k}_2)\}$*.*

When the above Hypothesis holds true, we can define

$$\begin{cases} (U_0,V_0) = (\overline{U},\overline{V}), \\ U_{n+1} = T_1(U_n,V_n), \quad n = 0,1,2,\ldots, \\ V_{n+1} = T_2(U_n,V_n), \quad n = 0,1,2,\ldots, \end{cases} \tag{6.4}$$

By results in [3, 27], we have the following result.

Theorem 6.2. *If Hypothesis 1 holds, then the iteration (6.4) converges to a non-increasing function* $(U,V)(z)$*. This function is a solution to the system (1.3) satisfying* $(U,V)(-\infty) = e_1$*,* $(U,V)(\infty) = e_0$*, and* $(\underline{U},\underline{V})(z) \leq (U,V)(z) \leq (\overline{U},\overline{V})(z),\ z \in (-\infty,\infty)$*.*

Remark 6.3. *When* $d = 0$*, we replace* $T_2(U,V)$ *by (see [1])*

$$T_3(U,V) = \frac{1}{c}\int_z^{\infty} e^{\frac{\alpha}{c}(z-s)} G(U,V)(s)ds.$$

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