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Propagation Speed of the Bistable Traveling Wave to the Lotka–Volterra Competition System in a Periodic Habitat

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Abstract

We study propagation direction of the traveling wave for the diffusive Lotka–Volterra competition system with bistable nonlinearity in a periodic habitat. By directly proving the strong stability of two semitrivial equilibria, we establish a new and sharper result on the existence of traveling wave. Using the method of upper and lower solutions, we provide two comparison theorems concerning the direction of traveling wave propagation. Several explicit sufficient conditions on the determination of the speed sign are established. In addition, an interval estimation of the bistable-wave speed reveals the relations among the bistable speed and the spreading speeds of two monostable subsystems. Biologically, our idea and insight provide an effective approach to find or control the direction of wave propagation for a system in heterogeneous environments.

Keywords Reaction diffusion equations · Bistable Lotka–Volterra competition system · Traveling wave · Propagation direction

Mathematics Subject Classification Primary 35K57 · 35B20 · 92D25.

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1 Introduction

This paper is devoted to the propagation direction of traveling wave solution for the following bistable Lotka–Volterra competition system

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= d_1(x)\frac{\partial^2 u_1}{\partial x^2} + u_1(b_1(x) - a_{11}(x)u_1 - a_{12}(x)u_2), \\ \frac{\partial u_2}{\partial t} &= d_2(x)\frac{\partial^2 u_2}{\partial x^2} + u_2(b_2(x) - a_{21}(x)u_1 - a_{22}(x)u_2), \quad t > 0,\ x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

in an L-periodic habitat for some positive number L, where $u_i(t, x)$, $i = 1, 2$, represent the population densities of two competing species at time t and location x. As usual, the coefficients $d_i(x)$, $b_i(x)$ stand for diffusion, resource concentration, respectively; $a_{11}(x)$ and $a_{22}(x)$ are the intraspecific competing coefficients; and $a_{12}(x)$ and $a_{21}(x)$ are the interspecific competing coefficients. Furthermore, we assume the following conditions: (i) $d_i(x)$, $b_i(x)$ and $a_{ij}(x) > 0$, $1 \le i, j \le 2$ are L-periodic Hölder continuous functions in $C^{\nu}(\mathbb{R})$ with $\nu \in (0, 1)$; (ii) $d_i(x) \ge d_0$ for some positive number d_0, such that the operators $d_i(x)\frac{\partial^2 u_i}{\partial x^2}$ are uniformly elliptic. For more details, we refer the readers to Yu and Zhao (2017).

Recently, the study of Lotka–Volterra competition systems in heterogeneous environments has drawn considerable attention (see, e.g., Lou 2006; Berestycki et al. 2005a, b; Xin 1991a, b, 1992, 1993; Ducrot 2016). For space-periodic monostable system, the existence, global stability and uniqueness (up to translation) of pulsating fronts of system (1.1) with advection terms can be found in Du et al. (2020). Particularly, the existence was derived, based on the abstract results in Fang and Zhao (2015). For time-periodic monostable system, the existence, uniqueness and asymptotic stability of traveling waves have been proved in Zhao and Ruan (2011). For a specific time-periodic bistable system, the existence and global stability of traveling waves have been established in Zhang and Zhao (2013). For time-space-periodic monotone system with monostable structure, the existence of traveling waves and spreading speeds have been studied in Fang et al. (2017). Additionally, in the case of bounded domain, the uniqueness and complete dynamics of the system (1.1) with $d_1(x) = d_2(x) := m(x)$ and other coefficients to be constants have been considered in Lam and Ni (2012). The global dynamics of such systems, influenced by the combination of diffusion, spatial variation and competition ability, can be seen in a series of investigations in He and Ni (2016a, b, 2017).

In the current paper, we study propagation direction of the traveling wave for the bistable system (1.1). To understand bistable nonlinearity, we will make a few assumptions. Denote $\lambda(d, g, c)$ as the principal eigenvalue of the following problem

$$\begin{aligned} &\lambda\phi(x) = d(x)\phi''(x) + g(x)\phi'(x) + c(x)\phi(x), \\ &\phi(x) = \phi(x + L), \quad x \in \mathbb{R}, \end{aligned}$$

where $\phi(x)$ is a continuous and nonnegative L-periodic function. Under the following assumption

(H1) $\lambda(d_i(x), 0, b_i(x)) > 0, \quad i = 1, 2,$

the trivial periodic steady state (0, 0) is unstable and it can be seen from [Yu and Zhao (2017), Proposition 2.1] or [Zhao (2003), Theorem 2.3.4] that system (1.1) has two semitrivial nonnegative periodic steady states $(p(x), 0)$ and $(0, q(x))$.

To normalize these steady states, we make a transformation

$$v_1(t,x) = \frac{u_1(t,x)}{p(x)}, \quad v_2(t,x) = 1 - \frac{u_2(t,x)}{q(x)}, \tag{1.2}$$

which transforms the competitive system (1.1) into the following cooperative system

$$\begin{aligned} \frac{\partial v_1}{\partial t} &= d_1(x)\frac{\partial^2 v_1}{\partial x^2} \\ &\quad + 2d_1(x)\frac{p'(x)}{p(x)}\frac{\partial v_1}{\partial x} + v_1[a_{11}(x)p(x)(1-v_1) - a_{12}(x)q(x)(1-v_2)], \\ \frac{\partial v_2}{\partial t} &= d_2(x)\frac{\partial^2 v_2}{\partial x^2} \\ &\quad + 2d_2(x)\frac{q'(x)}{q(x)}\frac{\partial v_2}{\partial x} + (1-v_2)[a_{21}(x)p(x)v_1 - a_{22}(x)q(x)v_2]. \end{aligned} \tag{1.3}$$

As a consequence of (1.2), two equilibria $(p(x), 0)$ and $(0, q(x))$ become $\beta := (1, 1)$ and $\mathbf{o} := (0, 0)$, respectively. In addition, the original equilibrium (0, 0) becomes $\alpha_1 := (0, 1)$.

We further assume that

(H2) $\lambda\big(d_1(x), 2d_1(x)\frac{p'(x)}{p(x)}, \Lambda_1(x)\big) < 0$ and $\lambda\big(d_2(x), 2d_2(x)\frac{q'(x)}{q(x)}, -\Lambda_2(x)\big) < 0,$

where $\Lambda_1(x) = a_{11}(x)p(x) - a_{12}(x)q(x)$, $\Lambda_2(x) = a_{21}(x)p(x) - a_{22}(x)q(x)$, so that $\mathbf{o}$ and β are two locally linearly stable steady states to the system (1.3). Meanwhile, there exists at least another unstable coexistence steady state for system (1.3), see [Furter and López-Gómez (1995), Theorem 2.4]. We denote this unstable state(s) by $\alpha_2(x)$ in the later section.

Since we focus on standard bistable structure (for the interpretation of which, see, e.g., Zhang and Zhao 2013), we need an additional assumption.

(H3) System (1.3) has no stable steady state in $\{(v_1, v_2) | 0 < v_1 < 1, 0 < v_2 < 1\}$.

The above assumption means that system (1.3) has and only has two stable steady states $\mathbf{o}$ and β, and any other coexistence steady states α between $\mathbf{o}$ and β are unstable and satisfy $\mathbf{o} \ll \alpha \ll \beta$ due to the strong maximum principle.

For a traveling wave solution connecting $\mathbf{o}$ and β to (1.3), we are referred to as a special solution which is given by

$$\begin{aligned} &(v_1, v_2)(t,x) = (U, V)(x, z), \quad z = x + ct, \\ &(U, V)(x, z) = (U, V)(x + L, z), \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{1.4}$$

subjected to

$$(U, V)(x, -\infty) = \mathbf{o}, \quad (U, V)(x, +\infty) = \beta, \tag{1.5}$$

where c reads as the wave speed. The limits in (1.5) are understood as

$$\lim_{z \to -\infty} |(U, V)(x, z)| = \mathbf{0}, \quad \lim_{z \to +\infty} |(U, V)(x, z) - \beta| = \mathbf{0},$$

uniformly for all x in $\mathbb{R}$.

Substituting (1.4) into (1.3) gives

$$\begin{aligned} & d_1(x)(U_{xx} + 2U_{xz} + U_{zz}) + 2d_1(x)\frac{p'(x)}{p(x)}(U_x + U_z) - cU_z \\ & \qquad + U[a_{11}(x)p(x)(1 - U) - a_{12}(x)q(x)(1 - V)] = 0, \\ & d_2(x)(V_{xx} + 2V_{xz} + V_{zz}) + 2d_2(x)\frac{q'(x)}{q(x)}(V_x + V_z) - cV_z \\ & \qquad + (1 - V)[a_{21}(x)p(x)U - a_{22}(x)q(x)V] = 0. \end{aligned} \tag{1.6}$$

From the biological point of view, the sign of the traveling wave speed for bistable competition models plays a significant role in predicting which species will eventually win the competition. It is quite challenging to establish explicit conditions on the determination of the sign of the bistable-wave speed. Due to the heterogeneous nature of the system, the study of wave propagation direction of (1.1) becomes even more challenging. To our knowledge, this paper would be the first study in this direction. For the existence of traveling wave, we directly prove the strong stability of two equilibria $\mathbf{o}$ and β. This removes the technique condition in Du et al. (2020) and provides a sharper condition for the existence of bistable wave. For the speed sign of the wave, technically, the higher phase space dimension coupled with heterogeneity of the environment causes much more difficulties than the case of a scalar equation with homogeneous environment. Taking the scalar constant-coefficient Hodgkin–Huxley equation as an example, the integration of the reaction term between the two stable equilibria gives the wave speed sign, which now is referred as Maxwell integral condition (Murray 1989). However, this method usually does not work for general systems including the Lotka–Volterra competition model. One of the obstacles is that between the two stable steady states, there might exist multiple unstable steady states in the higher-dimensional phase plane, not mentioning the fact that the method of phase plane analysis is almost impossible to handle the geometrical flow of such a heterogeneous system that the study of its dynamics becomes highly non-trivial. To overcome these difficulties, we first abstractly solve one of the equations in (1.6) such that the system can be reduced equivalently to a non-local scalar equation (see Lemma 3.7), which is of independent interest. Then, by virtue of the upper and lower solution method, we establish two comparison principles to help determine the sign of bistable-wave speed. Furthermore, to obtain explicit conditions, we develop and construct some novel upper and lower solutions to (1.6), based on the specific heterogeneous environments.

This provides a valid and effective approach to find or control the wave propagation direction, by setting or adjusting the parameters in the system.

The paper is organized as follows. The existence of a traveling wave solution of system (1.1) is proved in Sect. 2. A lemma that enables us to transform the system (1.1) into a scalar equation equivalently is provided in Sect. 3. Two crucial comparison principles are established there. By constructing several novel upper and lower solutions, explicit formulas to determine the speed sign are obtained in Sect. 4. An interval estimation of the bistable-wave speed is established in Sect. 5. Numerical results are shown in Sect. 6. Finally, a brief conclusion and discussion are included in Sect. 7.

2 The Existence of Traveling Wave Solution to System (1.3)

We first introduce some notations for the sake of convenience. We use $\mathcal{C}$ to represent the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ and use $\mathcal{C}^+$ to represent the positive cone. Additionally, $\mathcal{C}_\beta = \{\varphi \in \mathcal{C} : \mathbf{o} \le \varphi \le \beta\}$, where $\mathbf{o} = (0, 0)$. And the notations "$u \le v$," "$u < v$" and "$u \ll v$" for the partial orders are understood as the classic definitions, see, for example, Fang and Zhao (2015). Finally, we define $\Pi_\beta := \{\phi \in \mathcal{C} : \phi(x + L) = \phi(x), \mathbf{o} \le \phi(x) \le \beta, x \in \mathbb{R}\}$.

Definition 2.1 ([Fang and Zhao 2015, Definition 4.2]) A steady sate $\psi \in \Pi_\beta$ is called strongly stable from above for the map $Q : \Pi_\beta \to \Pi_\beta$ if there exists $\eta_0 > 0$ and a strongly positive element $\omega_0 \in \Pi_\beta$ so that

$$Q[\psi + \eta\omega_0] \ll \psi + \eta\omega_0, \quad \forall \eta \in (0, \eta_0].$$

Similarly, the definition of strong stability from below can be given by reversing the inequality and replacing "+" by "−."

In this paper, a semiflow Q_t is called spatially periodic with a positive period L if $Q_t \circ T_L = T_L \circ Q_t$ for all $t \in (0, +\infty)$, where T_L is the L-translation operator. To apply the abstract result in Fang and Zhao (2015), we give the following assumptions on Q_t:

(A1) (Continuity) $Q_t : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous in the compact-open topology.
(A2) (Monotonicity) Q_t is order preserving in the sense that $Q[\phi] \ge Q[\psi]$ whenever $\phi \ge \psi$ in $\mathcal{C}_\beta$.
(A3) (Compactness) $Q_t : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is compact in the compact-open topology.
(A4) (Bistability) $\mathbf{o}$ and $\beta \gg \mathbf{o}$ are strongly stable L-periodic steady states from above and below, respectively, for $Q_t : \Pi_\beta \to \Pi_\beta$, and the set of all intermediate L-periodic steady states is totally unordered in Π_β.
For a semiflow in a periodic habitat, we define the leftward spreading speed for Q_1 (Q_t with $t = 1$) as follows:

$$c_-^*(\alpha, \beta) := \sup\{c : \lim_{i\to-\infty, i\in\mathbb{Z}} a(c; iL + \theta) = \beta, \theta \in [0, L]\} \tag{2.1}$$

where

$$a(c;x) = \lim_{n\to\infty} a_n(c;x).$$

In our setting, for a given real number c, the sequence of functions $\{a_n\}_{n=0}^{\infty}$ is defined as

$$a_0(c;x) = \phi(x), \quad a_{n+1}(c;x) = R_c[a_n(c;\cdot)](x), \tag{2.2}$$

and

$$R_c[a_n](x) = \max\{\phi(x), T_c[Q_1[a_n]](x)\}, \tag{2.3}$$

where $\phi(x) = \alpha(x) + (\beta(x) - \alpha(x))\phi_0(x)$ with non-decreasing function $\phi_0(x)$ satisfying

$$\phi_0(x) = 0 \quad \text{for } \ x \le 0 \text{ and } \lim_{x\to\infty}(\phi(x) - \omega) = 0,$$

$0 < \omega < 1$, for a constant ω close to 1. Similarly, we can define the rightward spreading speed $c_+^*(\mathbf{o}, \alpha)$.

In our definition of the spreading speed, we avoid the incorporation of the homeomorphism operator F and the conjugate FQ_1F^{-1}, but the spreading speed is still well defined. For a detailed introduction of the operator F, the reader is referred to Lemma 2.2 in Fang et al. (2017), or [Fang and Zhao (2015), pp. 2267–2268].

Denoting E as the set of all fixed points of Q_t restricted on Π_β, we add a further assumption on Q_t:

(A5) (Counter-propagation) For each $\alpha \in E\backslash\{\mathbf{o}, \beta\}$, $c_-^*(\alpha, \beta) + c_+^*(\mathbf{o}, \alpha) > 0$, where $c_-^*(\alpha, \beta)$ is the leftward spreading speed of the operator Q_1 (Q_t with $t = 1$) in the phase space $\mathcal{C}_{\alpha,\beta} = \{\varphi \in \mathcal{C} : \alpha \le \varphi \le \beta\}$, and $c_+^*(\mathbf{o}, \alpha)$ is rightward spreading speed of the operator Q_1 in the phase space $\mathcal{C}_{\mathbf{o},\alpha} = \{\varphi \in \mathcal{C} : \mathbf{o} \le \varphi \le \alpha\}$.

Lemma 2.2 ([Fang and Zhao 2015, Theorem 4.1]) *For the map Q_t, assume that (A1)–(A5) are satisfied. Then, the spatially L-periodic semiflow $\{Q_t\}_{t\ge 0}$ has an L-periodic traveling wave $V(x, x+ct)$. Moreover, $V(x,z)$ is non-decreasing in z and connects $\mathbf{o}$ and $\beta(x)$.*

Hence, by applying Lemma 2.2 on system (1.3), we have the following result.

Theorem 2.3 *Under the assumptions (H1)–(H3), the spatially L-periodic system (1.3) possesses an L-periodic traveling wave $(U, V)(x, x+ct)$. Moreover, $U(x,z)$ and $V(x,z)$ are non-decreasing in z and connect $\mathbf{o}$ and β.*

Proof We define

$$\{Q_t := Q_t[(\phi, \psi)](x) = (v_1(t,x,\phi), v_2(t,x,\psi)), \quad \forall(\phi,\psi) \in \mathcal{C}, \ x \in \mathbb{R}, \ t \ge 0\}$$

as the solution semiflow associated with (1.3). Firstly, it is fairly straightforward to show that $\{Q_t\}$ satisfies the continuous condition (A1) and the monotone condition (A2). The compactness of $\{Q_t\}$ follows from the compactness of the equivalent integral form for the system (1.3).

To verify (A5), for the equilibrium $\alpha_1 = (0, 1)$, we can easily see that both of the two spreading speeds are positive. Indeed, this can be found when we return to the original system (1.1) and study the traveling waves connecting $(0, 0)$ and $(p(x), 0)$, or connecting $(0, 0)$ and $(0, q(x))$. Since we do not have advection terms in (1.1), each spreading speed is positive. For any other unstable equilibrium $\alpha(x)$ satisfying $\mathbf{o} \ll \alpha \ll \beta$, due to the fact that the linear system of Q_t along α is irreducible, we can follow the idea in [Fang and Zhao (2015), p. 2277] to derive that (A5) is true.

Next, we focus on the verification of the bistable condition (A4). Let η be a positive constant, and $\omega_0 := (v_{10}, v_{20})(x)$ in Π_β be a strongly positive vector which is to be determined. We first consider the linearization of (1.3) at $\mathbf{o}$, that is

$$
\begin{aligned}
&\frac{\partial \widetilde{v}_1}{\partial t} = d_1(x)\frac{\partial^2 \widetilde{v}_1}{\partial x^2} + 2d_1(x)\frac{p'(x)}{p(x)}\frac{\partial \widetilde{v}_1}{\partial x} + [a_{11}(x)p(x) - a_{12}(x)q(x)]\widetilde{v}_1,\\
&\frac{\partial \widetilde{v}_2}{\partial t} = d_2(x)\frac{\partial^2 \widetilde{v}_2}{\partial x^2} + 2d_2(x)\frac{q'(x)}{q(x)}\frac{\partial \widetilde{v}_2}{\partial x} + a_{21}(x)p(x)\widetilde{v}_1 - a_{22}(x)q(x)\widetilde{v}_2,\\
&\widetilde{v}_1(0, x) = \eta v_{10}(x) > 0, \quad \widetilde{v}_2(0, x) = \eta v_{20}(x) > 0.
\end{aligned}
\tag{2.4}
$$

To choose $\omega_0 = (v_{10}, v_{20})(x)$, we consider two eigenvalue value problems

$$
\begin{aligned}
&\lambda\phi = d_1(x)\phi'' + 2d_1(x)\frac{p'(x)}{p(x)}\phi' + [a_{11}(x)p(x) - a_{12}(x)q(x)]\phi,\\
&\phi(x + L) = \phi(x), \quad x \in \mathbb{R},
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
&\lambda\psi = d_2(x)\psi'' + 2d_2(x)\frac{q'(x)}{q(x)}\psi' - a_{22}(x)q(x)\psi,\\
&\psi(x + L) = \psi(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{2.6}
$$

By (H2), one can prove that there exists a principal eigenvalue $\lambda_1 < 0$ associated with a positive L-periodic eigenfunction $\phi_0(x)$ for (2.5). Similarly, one can verify that there also exists a principal eigenvalue $\lambda_2 < 0$ associated with a positive L-periodic eigenfunction $\psi_0(x)$ for (2.6). Let

$$
\lambda_3 = \min\left\{\frac{\lambda_1}{2}, \frac{\lambda_2}{2}\right\}.
$$

Since a constant multiple of an eigenfunction ψ_0 is still an eigenfunction, we can choose ψ_0 so that

$$
(-\lambda_2 + \lambda_3)\psi_0 \geq a_{21}(x)p(x)\phi_0. \tag{2.7}
$$

Now, we choose $v_{10} = \phi_0$, $v_{20} = \psi_0$ and we can show that

$$(e^{\lambda_3 t}\eta\phi_0(x), e^{\lambda_3 t}\eta\psi_0(x))$$

becomes an upper solution of (2.4). By comparison, we get

$$(\widetilde{v}_1(t,x), \widetilde{v}_2(t,x)) \le (e^{\lambda_3 t}\eta\phi_0(x), e^{\lambda_3 t}\eta\psi_0(x)) \ll (\eta\phi_0(x), \eta\psi_0(x))$$

for any fixed $t > 0$.

We have verified that the linear operator is strongly stable. Since the topological structure of the linear system near the equilibrium $\mathbf{o}$ is hyperbolic, one can select a suitable small positive number η so that the map $Q_t : \mathcal{C}_\beta \to \mathcal{C}_\beta$ can be well approximated by the linear Poincaré map. This together with a perturbation argument enables us to find a sufficiently small number $\eta_0 > 0$ to get $Q[\eta\omega_0] \ll \eta\omega_0$, $\forall \eta \in (0, \eta_0]$, which implies $\mathbf{o}$ is strongly stable from above. By a similar way, we can prove that $\beta = (1, 1)$ is strongly stable from below. As a result, the proof is complete. □

Remark 2.4 The existence is also considered in [Du et al. (2020), Theorem 2.12]. Precisely, they required the linearized system of (1.6) at $\mathbf{o}$ and β has principal eigenvalue with strongly positive eigenfunction, respectively. Therefore, [Du et al. (2020), Lemma 2.3] required a technique condition

$$\begin{aligned}
&\lambda\Big(d_2(x), 2d_2(x)\frac{q'(x)}{q(x)}, -a_{22}(x)q(x)\Big) < \lambda\Big(d_1(x), 2d_1(x)\frac{p'(x)}{p(x)}, \Lambda_1(x)\Big),\\
&\lambda\Big(d_1(x), 2d_1(x)\frac{p'(x)}{p(x)}, -a_{11}(x)p(x)\Big) < \lambda\Big(d_2(x), 2d_2(x)\frac{q'(x)}{q(x)}, -\Lambda_2(x)\Big).
\end{aligned}$$

We remove this condition by a direct proof of the strong stability.

3 The Sign of the Bistable-Wave Speed

In this section, we investigate the sign of the bistable-wave speed to (1.3) by comparing the wave solution to any chosen upper or lower solutions with given speed. For clarity, we begin with the definition of regular upper/lower solution.

Definition 3.1 A pair of continuous functions $(U, V)(x, z)$, which is twice continuously differentiable in x and z, is said to be a regular upper/lower solution of (1.6), if it satisfies

$$\begin{aligned}
&d_1(x)(U_{xx} + 2U_{xz} + U_{zz}) + 2d_1(x)\frac{p'(x)}{p(x)}(U_x + U_z) - cU_z\\
&\quad + U[a_{11}(x)p(x)(1-U) - a_{12}(x)q(x)(1-V)] \le 0 \text{ (resp. } \ge 0),\\
&d_2(x)(V_{xx} + 2V_{xz} + V_{zz}) + 2d_2(x)\frac{q'(x)}{q(x)}(V_x + V_z) - cV_z\\
&\quad + (1-V)[a_{21}(x)p(x)U - a_{22}(x)q(x)V] \le 0 \text{ (resp. } \ge 0).
\end{aligned}$$

Remark 3.2 The conception of upper/lower solution of (1.3) can be stated in a similar way.

In general, the construction of regular upper or lower solutions in the whole domain is very difficult to carry out. Hence, upper or lower solutions defined domain by domain are needed. In other words, we turn to establish piecewisely defined upper or lower solutions. This leads to the following definition concerning irregular upper or lower solutions.

Definition 3.3 (see, [Fife and Tang 1981, Definition 4]) A pair of continuous functions $(\bar{U}, \bar{V})$ is said to be an irregular upper solution of (1.6), if there exist regular upper solutions $(\bar{U}^1, \bar{V}^1), \ldots, (\bar{U}^k, \bar{V}^k)$ of (1.6) such that $(\bar{U}, \bar{V}) = \min\limits_{1\le i\le k} (\bar{U}^i, \bar{V}^i)$ componentwise. Similarly, $(\underline{U}, \underline{V})$ is called an irregular lower solution of (1.6) if there exist regular lower solutions $(\underline{U}^1, \underline{V}^1), \ldots, (\underline{U}^k, \underline{V}^k)$ of (1.6) such that $(\underline{U}, \underline{V}) = \max\limits_{1\le i\le k} (\underline{U}^i, \underline{V}^i)$ componentwise.

The following lemma is important to the derivation of our main results.

Lemma 3.4 ([Du et al. 2020, Lemma 3.3]) *There exist positive constants $\beta_0, \delta_0, \sigma_0$ such that for any $\xi^{\pm} \in \mathbb{R}$ and $\delta \in (0, \delta_0]$, the functions $(v_1^{\pm}, v_2^{\pm})(t, x)$ defined by*

$$
\begin{aligned}
v_1^{\pm}(t, x) = U(&x, x + ct + \xi^{\pm} \pm \sigma_0\delta(1 - e^{-\beta_0 t}))\\
&\pm\delta p_1(x, x + ct + \xi^{\pm} \pm \sigma_0\delta(1 - e^{-\beta_0 t}))e^{-\beta_0 t},\\
v_2^{\pm}(t, x) = V(&x, x + ct + \xi^{\pm} \pm \sigma_0\delta(1 - e^{-\beta_0 t}))\\
&\pm\delta p_2(x, x + ct + \xi^{\pm} \pm \sigma_0\delta(1 - e^{-\beta_0 t}))e^{-\beta_0 t},
\end{aligned}
$$

are a pair of upper/lower solutions (or super- and subsolutions) of (1.3) for any $(t, x) \in (0, +\infty) \times \mathbb{R}$. Here, (U, V) is the traveling wave solution of (1.3), and $p_i(x, \cdot), i = 1, 2$, are two bounded functions. For their expressions, see Du et al. (2020).

Now, we are ready to establish two sufficient conditions on the determinacy of the sign of bistable-wave speed. Since a regular upper (lower) solution is also an irregular one (with $k = 1$ in Definition 3.3), from now on, by an upper (lower) solution, we mean that it is either regular or irregular.

Theorem 3.5 *If $(\overline{U}, \overline{V})(x, z)$ is an upper solution, with speed $\overline{c} < 0$, which is nonnegative for all $(x, z) \in \mathbb{R}^2$, non-decreasing in z and satisfies*

$$
(\overline{U}, \overline{V})(x, -\infty) < (1, 1), \quad (\overline{U}, \overline{V})(x, +\infty) \ge (1, 1), \tag{3.1}
$$

then the speed of the bistable-wave solution of (1.3) satisfies

$$
c \le \overline{c} < 0. \tag{3.2}
$$

Proof Choose a pair of continuous and non-decreasing initial functions $(v_1, v_2)(0, x)$ satisfying

$$v_1(0,x) = v_2(0,x) = \begin{cases} 0, & x < -H, \\ 1-\kappa, & x > H, \end{cases}$$

where H is a positive number and $\kappa \in (0, 1)$ is assumed to be a small number. In view of (3.1), one can suppose that (by a shift if it is necessary) the initial value of the upper solution $(\overline{U}, \overline{V})(x, z)$ satisfies

$$\overline{U}(x,x) \geq v_1(0,x), \quad \overline{V}(x,x) \geq v_2(0,x), \quad \text{for } x \in \mathbb{R}.$$

Then, it follows from the comparison principle that

$$\overline{U}(x,z) \geq v_1(t,x), \quad \overline{V}(x,z) \geq v_2(t,x), \quad \text{for } t \geq 0. \tag{3.3}$$

On the other hand, the comparison principle, combined with Lemma 3.4, ensures

$$\begin{aligned} v_1(t,x) \ &\geq U(x, x+ct+\xi^- - \sigma_0\delta(1-e^{-\beta_0 t})) \\ &\quad -\delta p_1(x, x+ct+\xi^- - \sigma_0\delta(1-e^{-\beta_0 t}))e^{-\beta_0 t}. \end{aligned} \tag{3.4}$$

Again due to (3.1), we may suppose that $\overline{U}_1(x, \eta) < 1$ at the line $z = x + \overline{c}t = \eta$. To the contrary of our theorem, if $c > \overline{c}$, then we have from (3.3) and (3.4) that

$$\begin{aligned} \overline{U}(x,\eta) \ &\geq U(x, \eta+(c-\overline{c})t+\xi^- - \sigma_0\delta(1-e^{-\beta_0 t})) \\ &\quad -\delta p_1(x, x+ct+\xi^- - \sigma_0\delta(1-e^{-\beta_0 t}))e^{-\beta_0 t}. \end{aligned} \tag{3.5}$$

Let $t \to +\infty$, we obtain $\overline{U}(x, \eta) \geq 1$, which is a contradiction. Thus, the proof is complete. $\square$

Theorem 3.6 *If $(\underline{U}, \underline{V})(x, z)$ is a lower solution, with speed $\underline{c} > 0$, which is nonnegative for all $(x, z) \in \mathbb{R}^2$, non-decreasing in z and satisfies*

$$(\underline{U}, \underline{V})(x, -\infty) = (0, 0) < (\underline{U}, \underline{V})(x, +\infty) \leq (1, 1), \tag{3.6}$$

then the speed of the bistable-wave solution of (1.3) satisfies

$$c \geq \underline{c} > 0. \tag{3.7}$$

Proof The proof is similar to that of Theorem 3.5. It is worth mentioning that the application of lower solution in (3.5) should be replaced by the upper solution provided in Lemma 3.4. $\square$

Let $c^*_-(\alpha_1, \beta)$ and $c^*_+(\mathbf{o}, \alpha_1)$ be the left and right spreading speeds as defined in (A5). We have the following lemma.

Lemma 3.7 *Suppose that c is a given number satisfying*

$$-c_+^*(\mathbf{o},\alpha_1) < c < c_-^*(\alpha_1,\beta), \tag{3.8}$$

then we have the following conclusions.

(1) *For any given continuous function $U(x,z)$ which is non-decreasing in z and L-periodic in x and satisfies $U(x,-\infty)=0$, $U(x,+\infty)>\frac{a_{22}(x)q(x)}{a_{21}(x)p(x)}$, the equation*

$$\begin{aligned}
&d_2(x)(V_{xx}+2V_{xz}+V_{zz})+2d_2(x)\frac{q'(x)}{q(x)}(V_x+V_z)-cV_z\\
&\quad+(1-V)[a_{21}(x)p(x)U-a_{22}(x)q(x)V]=0,\\
&V(x,-\infty)=0,\ \ V(x,+\infty)=1,\ \ x\in\mathbb{R},\\
&V(x,z)=V(x+L,z),
\end{aligned} \tag{3.9}$$

has a continuous solution $V(x,z)$ which is non-decreasing in z and is L-periodic in x.

(2) *For any given continuous function $V(x,z)$ which is non-decreasing in z and L-periodic in x and satisfies $V(x,-\infty)=0$, $V(x,+\infty)=1$, the equation*

$$\begin{aligned}
&d_1(x)(U_{xx}+2U_{xz}+U_{zz})+2d_1(x)\frac{p'(x)}{p(x)}(U_x+U_z)-cU_z\\
&\quad+U[a_{11}(x)p(x)(1-U)-a_{12}(x)q(x)(1-V)]=0,\\
&U(x,-\infty)=0,\ \ U(x,+\infty)=1,\quad x\in\mathbb{R},\\
&U(x,z)=U(L+x,z),
\end{aligned} \tag{3.10}$$

has a continuous solution $U(x,z)$ which is non-decreasing in z and L-periodic in x.

Proof We shall apply upper and lower solution method to prove the results. For case (1), instead of working on (3.9), we make a transformation $W(x,z)=1-V(x,z)$. It is not difficult to get the equation for $W(x,z)$ as

$$\begin{aligned}
&d_2(x)(W_{xx}+2W_{xz}+W_{zz})+2d_2(x)\frac{q'(x)}{q(x)}(W_x+W_z)-cW_z\\
&\quad+a_{22}(x)q(x)W[r(x,z)-W]=0,\\
&W(x,-\infty)=1,\ \ W(x,+\infty)=0,\quad x\in\mathbb{R},\\
&W(x,z)=W(x+L,z),
\end{aligned} \tag{3.11}$$

where

$$r(x,z)=1-\frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x,z)$$

with

$$r(x,-\infty)=1,\ r(x,+\infty)=1-\frac{a_{21}(x)p(x)}{a_{22}(x)q(x)}U(x,+\infty)<0,\quad x\in\mathbb{R}.$$

It is clear that we can choose $W\equiv 1$ as an upper solution to (3.11). However, the construction of the lower solution is non-trivial. Define

$$f(x,\hat{W})=\begin{cases} a_{22}(x)q(x)\hat{W}(1-\epsilon-\hat{W}), & \hat{W}\geq 0,\\ a_{22}(x)q(x)\hat{W}(\epsilon+\hat{W}), & \hat{W}<0,\end{cases}$$

where $0<\epsilon\ll 1$, and consider the following partial differential equation

$$\begin{aligned} d_2(x)(\hat{W}_{xx}+2\hat{W}_{xz}+\hat{W}_{zz})+2d_2(x)\frac{q'(x)}{q(x)}(\hat{W}_x+\hat{W}_z)\\ -\hat{c}_\epsilon\hat{W}_z+f(x,\hat{W})=0. \end{aligned}\tag{3.12}$$

Obviously, (3.12) has three equilibria $-\epsilon, 0, 1-\epsilon$. Furthermore, there exists a non-increasing solution $\hat{W}$ satisfying Eq. (3.12) and

$$\hat{W}(x,-\infty)=1-\epsilon,\quad \hat{W}(x,+\infty)=-\epsilon,\tag{3.13}$$

for the details, see Fang and Zhao (2015). When $\hat{W}\geq 0$, letting $\epsilon\to 0$ in (3.12) gives

$$\begin{aligned} &d_2(x)(\tilde{W}_{xx}+2\tilde{W}_{xz}+\tilde{W}_{zz})+2d_2(x)\frac{q'(x)}{q(x)}(\tilde{W}_x+\tilde{W}_z)\\ &-\hat{c}\tilde{W}_z+a_{22}(x)q(x)\tilde{W}(1-\tilde{W})=0,\\ &\tilde{W}(x,-\infty)=1,\quad \tilde{W}(x,+\infty)=0, \end{aligned}$$

which has a nonnegative solution for $\hat{c}=\hat{c}_0$ where

$$\hat{c}_0=-\min_{\mu>0}\left\{\frac{\hat{\lambda}(\mu)}{\mu}\right\},$$

and $\hat{\lambda}(\mu)$ is the principal eigenvalue of the following eigenvalue problem

$$\begin{aligned} &\lambda(\mu)\phi(x)=d_2(x)\phi''(x)-2d_2(x)\left(\mu-\frac{q'(x)}{q(x)}\right)\phi'(x)\\ &\quad+\left(d_2(x)\mu^2-2\mu d_2(x)\frac{q'(x)}{q(x)}+a_{22}(x)q(x)\right)\phi(x),\\ &\phi(x+L)=\phi(x),\quad x\in\mathbb{R}. \end{aligned}$$

It can be seen from the continuity of $\hat{c}_\epsilon$ with respect to ϵ that $\hat{c}_\epsilon\to\hat{c}_0$ as $\epsilon\to 0$. Owing to the fact that any translation of $\hat{W}(x,z)$ in z is still a solution to (3.12), we can assume that $\hat{W}(x,z) \geq 0$ when $z \leq z_0$, and $\hat{W}(x,z) < 0$ when $z > z_0$ for any given z_0. Meanwhile, for such a number z_0, we may suppose that $r(x,z) \geq 1-\epsilon$ if $z \leq z_0$ since $r(x,-\infty) = 1$. Recall that $\hat{W}(x,z)$ is the solution to (3.12) satisfying (3.13) and define

$$\underline{W}(x,z) = \max\{0, \hat{W}(x,z)\} = \begin{cases} 0, & \text{for } z > z_0, \\ \hat{W}(x,z), & \text{for } z \leq z_0. \end{cases}$$

One can verify directly that $\underline{W}(x,z)$ is a lower solution to (3.11) if $c > \hat{c}_0 = -c_+^*(\mathbf{o}, \alpha_1)$. The proof is done.

Since the proof of part (2), where the right inequality of (3.8) is required, is similar to the one of part (1), we omit it here. □

To establish proper upper and lower solutions and apply Theorems 3.5 and 3.6, we consider the eigenvalue problem of (1.6). To be exact, near $\mathbf{o}$ and β, we denote $-\kappa(\lambda)$ and $-\kappa(\mu)$, respectively, as the principal eigenvalue of the operator

$$\begin{aligned} \mathcal{L}_\lambda \phi &= d_1(x)\phi'' + 2d_1(x)\left(\frac{p'(x)}{p(x)} + \lambda\right)\phi' \\ &\quad + \left[d_1(x)\lambda^2 + 2d_1(x)\frac{p'(x)}{p(x)}\lambda + \Lambda_1(x)\right]\phi, \quad \phi \in C_{\text{per}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_\mu \psi &= d_2(x)\psi'' + 2d_2(x)\left(\frac{q'(x)}{q(x)} - \mu\right)\psi' \\ &\quad + \left[d_2(x)\mu^2 - 2d_2(x)\frac{q'(x)}{q(x)}\mu - \Lambda_2(x)\right]\psi, \quad \psi \in C_{\text{per}}, \end{aligned}$$

where $C_{\text{per}} = \{\omega(x) \in C^2(\mathbb{R}) : \omega(\cdot + L) = \omega(\cdot), \omega(x) > 0, x \in \mathbb{R}\}$. The following results come from [Du et al. (2019), Lemma 2.3].

Lemma 3.8 *(i) The function $\kappa(\lambda)$ is analytic and concave in $\mathbb{R}$, and the set*

$$\{\lambda \in \mathbb{R} | \kappa(\lambda) + c\lambda = 0\}$$

is equal to $\{\lambda_1(c), \lambda_2(c)\}$ with $\lambda_2(c) < 0 < \lambda_1(c)$.

(ii) The function $\kappa(\mu)$ is analytic and concave in $\mathbb{R}$, and the set

$$\{\mu \in \mathbb{R} | \kappa(\mu) - c\mu = 0\}$$

is equal to $\{\mu_1(c), \mu_2(c)\}$ with $\mu_2(c) < 0 < \mu_1(c)$.

In the sequel, we will use the following notations for simplicity to denote

$$\begin{aligned} H_1(U,V) &:= d_1(x)(U_{xx}+2U_{xz}+U_{zz}) + 2d_1(x)\frac{p'(x)}{p(x)}(U_x+U_z) - cU_z \\ &\quad + U[a_{11}(x)p(x)(1-U) - a_{12}(x)q(x)(1-V)], \\ H_2(U,V) &:= d_2(x)(V_{xx}+2V_{xz}+V_{zz}) + 2d_2(x)\frac{q'(x)}{q(x)}(V_x+V_z) - cV_z \\ &\quad + (1-V)[a_{21}(x)p(x)U - a_{22}(x)q(x)V]. \end{aligned}$$

We proceed to deduce a series of results concerning the sign of bistable-wave speed by virtue of Theorems 3.5 and 3.6. For $0<\epsilon\ll 1$, define

$$\overline{U}(x,z) = \frac{1}{1+\frac{e^{-\lambda_1(-\epsilon)z}}{\phi(x)}}, \quad \overline{V}(x,z) \ \text{ is the function determined by Lemma 3.7}, \tag{3.14}$$

where $\phi(x)$ is the L-periodic eigenfunction of the operator $\mathcal{L}_\lambda\phi$ with respect to $\lambda_1(-\epsilon)$.

Corollary 3.9 *The speed c of the bistable traveling wave of* (1.6) *satisfies $c\le -\epsilon<0$ provided that*

$$\begin{aligned} &-2d_1(x)\left(\lambda_1(-\epsilon)+\frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)J_1(x,z)\le 0, \\ &\text{where } \ J_1(x,z) = \frac{\overline{V}-\overline{U}}{\overline{U}(1-\overline{U})}. \end{aligned} \tag{3.15}$$

Proof Inserting (3.14) into the left of the first equation in (1.6) gives

$$\begin{aligned} H_1(\overline{U},\overline{V}) &= \overline{U}(1-\overline{U})\left\{d_1(x)\left[\frac{\phi''}{\phi}+\left(2\lambda_1+2\frac{p'(x)}{p(x)}\right)\frac{\phi'}{\phi}+\lambda_1^2+2\lambda_1\frac{p'(x)}{p(x)}\right]+\epsilon\lambda_1\right. \\ &\quad \left. -2d_1(x)\overline{U}\left(\lambda_1+\frac{\phi'}{\phi}\right)^2\right\} + \overline{U}\big[a_{11}(x)p(x)(1-\overline{U}) - a_{12}(x)q(x)(1-\overline{V})\big] \\ &= \overline{U}(1-\overline{U})\left\{-\Lambda_1(x) - 2d_1(x)\overline{U}\left(\lambda_1+\frac{\phi'}{\phi}\right)^2\right\} \\ &\quad + \overline{U}\big[a_{11}(x)p(x)(1-\overline{U}) - a_{12}(x)q(x)(1-\overline{V})\big] \\ &= \overline{U}^2(1-\overline{U})\left[-2d_1(x)\left(\lambda_1+\frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)\frac{\overline{V}-\overline{U}}{\overline{U}(1-\overline{U})}\right]. \end{aligned}$$

It follows from (3.15) that $H_1(\overline{U},\overline{V})\le 0$. As a result, the pair of functions $(\overline{U},\overline{V})(x,z)$ defined in (3.14) turns to be an upper solution. By Theorem 3.5 with $\overline{c}=-\epsilon<0$, the proof is complete. □

Define

$$\overline{V}(x,z) = \frac{1}{1+\psi(x)e^{-\mu_1(-\epsilon)z}}, \quad \overline{U}(x,z) \quad \text{is the function determined by (2) of Lemma 3.7,} \tag{3.16}$$

where $\psi(x)$ is the L-periodic eigenfunction of the operator $\mathcal{L}_\mu\psi$ with respect to $\mu_1(-\epsilon)$.

Corollary 3.10 *The speed c of the bistable traveling wave of (1.6) satisfies $c \le -\epsilon < 0$ provided that*

$$2d_2(x)\left(\mu_1(-\epsilon) - \frac{\psi'}{\psi}\right)^2 + a_{21}(x)p(x)J_2(x,z) \le 0,$$
$$\textit{where } J_2(x,z) = \frac{\overline{U}-\overline{V}}{\overline{V}(1-\overline{V})}. \tag{3.17}$$

Proof Take $\bar{c} = -\epsilon$ and write $\mu_1(-\epsilon) = \mu_1$ for short. Then, plugging (3.16) into the right of the second equation in (1.6) yields

$$\begin{aligned}
H_2(\overline{U},\overline{V}) &= \overline{V}(1-\overline{V})\Bigg\{d_2(x)\bigg[-\frac{\psi''}{\psi} + 2\left(\frac{\psi'}{\psi}\right)^2 - \left(2\mu_1 + 2\frac{q'(x)}{q(x)}\right)\frac{\psi'}{\psi} + \mu_1^2 + 2\mu_1\frac{q'(x)}{q(x)}\bigg] \\
&\quad + \epsilon\mu_1 - 2d_2(x)\overline{V}\left(\mu_1 - \frac{\psi'}{\psi}\right)^2\Bigg\} + (1-\overline{V})\big[a_{21}(x)p(x)\overline{U} - a_{22}(x)q(x)\overline{V}\big] \\
&= \overline{V}(1-\overline{V})\left\{2d_2(x)\left(\mu_1 - \frac{\psi'}{\psi}\right)^2(1-\overline{V}) - \Lambda_2(x)\right\} \\
&\quad + (1-\overline{V})\big[a_{21}(x)p(x)\overline{U} - a_{22}(x)q(x)\overline{V}\big] \\
&= \overline{V}(1-\overline{V})^2\left[2d_2(x)\left(\mu_1 - \frac{\psi'}{\psi}\right)^2 + a_{21}(x)p(x)\frac{\overline{U}-\overline{V}}{\overline{V}(1-\overline{V})}\right].
\end{aligned}$$

It follows from (3.17) that $H_2(\overline{U},\overline{V}) \le 0$, which implies the pair of functions $(\overline{U},\overline{V})(x,z)$ defined in (3.16) forms an upper solution. By virtue of Theorem 3.5, the proof is complete. □

Corollaries 3.9 and 3.10 are concerning the negativity of the bistable traveling wave speed, which means the bistable wave spreads to the right. Next, we shall utilize Theorem 3.6 to study under what conditions the bistable traveling wave speed is positive.

For $0 < \epsilon \ll 1$ and $p \in (\max\limits_{x\in[0,L]}\left\{\frac{a_{22}(x)q(x)}{a_{21}(x)p(x)}\right\}, 1]$, define

$$\underline{U}(x,z) = \frac{p}{1+\frac{e^{-\lambda_1(\epsilon)z}}{\phi(x)}}, \quad \underline{V}(x,z) \quad \text{is the function determined by (1) of Lemma 3.7,} \tag{3.18}$$

where $\phi(x)$ is the L-periodic eigenfunction of the operator $\mathcal{L}_\lambda\phi$ with respect to $\lambda_1(\epsilon)$.

Corollary 3.11 *The speed c of the bistable traveling wave of (1.6) satisfies $c \geq \epsilon > 0$ provided that*

$$-2d_1(x)\left(\lambda_1(\epsilon)+\frac{\phi'}{\phi}\right)^2 + J_3(x,z) \geq 0,$$
$$\text{where } J_3(x,z) = \frac{a_{12}(x)q(x)\underline{V} - \big(a_{11}(x)p(x) - \frac{\Lambda_1(x)}{p}\big)\underline{U}}{\big(1-\frac{\underline{U}}{p}\big)\frac{\underline{U}}{p}}. \tag{3.19}$$

Proof By letting $\underline{c} = \epsilon$ and substituting (3.18) into the left of the first equation in (1.6), we obtain

$$\begin{aligned} H_1(\underline{U},\underline{V}) &= \underline{U}(1-\frac{\underline{U}}{p})\Big\{d_1(x)\Big[\frac{\phi''}{\phi} + \Big(2\lambda_1 + 2\frac{p'(x)}{p(x)}\Big)\frac{\phi'}{\phi} + \lambda_1^2 + 2\lambda_1\frac{p'(x)}{p(x)}\Big] - \epsilon\lambda_1 \\ &\quad - 2d_1(x)\frac{\underline{U}}{p}\Big(\lambda_1 + \frac{\phi'}{\phi}\Big)^2\Big\} + \underline{U}\big[a_{11}(x)p(x)(1-\underline{U}) - a_{12}(x)q(x)(1-\underline{V})\big] \\ &= \underline{U}(1-\frac{\underline{U}}{p})\Big\{-\Lambda_1(x) - 2d_1(x)\frac{\underline{U}}{p}\Big(\lambda_1+\frac{\phi'}{\phi}\Big)^2\Big\} \\ &\quad + \underline{U}\big[a_{11}(x)p(x)(1-\underline{U}) - a_{12}(x)q(x)(1-\underline{V})\big] \\ &= \frac{\underline{U}^2}{p}(1-\frac{\underline{U}}{p})\Big[-2d_1(x)\Big(\lambda_1+\frac{\phi'}{\phi}\Big)^2 + \frac{a_{12}(x)q(x)\underline{V} - \big(a_{11}(x)p(x) - \frac{\Lambda_1(x)}{p}\big)\underline{U}}{\big(1-\frac{\underline{U}}{p}\big)\frac{\underline{U}}{p}}\Big], \end{aligned}$$

where we have written $\lambda_1(\epsilon) = \lambda_1$ for short. Hence, we can conclude that the pair of functions $(\underline{U},\underline{V})(x,z)$ defined in (3.18) turns to be a lower solution if (3.19) is valid. By Theorem 3.6, the proof is thus complete. □

For $0 < \epsilon \ll 1$ and $0 < k \leq 1$, we choose

$$\underline{V}(x,z) = \frac{k}{1+\psi(x)e^{-\mu_1(\epsilon)z}}, \quad \underline{U}(x,z) \text{ is the function determined by (2) of Lemma 3.7,} \tag{3.20}$$

where $\psi(x)$ is the L-periodic eigenfunction of the operator $\mathcal{L}_\mu\psi$ with respect to $\mu_1(\epsilon)$.

Corollary 3.12 *The speed c of the bistable traveling wave of (1.6) satisfies $c \geq \epsilon > 0$ provided that*

$$2d_2(x)\left(\mu_1(\epsilon) - \frac{\psi'}{\psi}\right)^2 + J_4(x,z) \geq 0, \tag{3.21}$$

where

$$J_4(x,z) = -\frac{\Lambda_2(x)}{1-\frac{\underline{V}}{k}} + \frac{(1-\underline{V})\big[a_{21}(x)p(x)\underline{U} - a_{22}(x)q(x)\underline{V}\big]}{\underline{V}(1-\frac{\underline{V}}{k})^2}.$$

Proof A substitution of (3.20) into the right of the second equation in (1.6) leads to

$$
\begin{aligned}
H_2(\underline{U},\underline{V}) &= \underline{V}(1-\frac{\underline{V}}{k})\Big\{d_2(x)\Big[-\frac{\psi''}{\psi}+2\Big(\frac{\psi'}{\psi}\Big)^2-\Big(2\mu_1+2\frac{q'(x)}{q(x)}\Big)\frac{\psi'}{\psi}+\mu_1^2+2\mu_1\frac{q'(x)}{q(x)}\Big]\\
&\quad -\epsilon\mu_1-2d_2(x)\frac{\underline{V}}{k}\Big(\mu_1-\frac{\psi'}{\psi}\Big)^2\Big\}+(1-\underline{V})\big[a_{21}(x)p(x)\underline{U}-a_{22}(x)q(x)\underline{V}\big]\\
&= \underline{V}(1-\frac{\underline{V}}{k})\Big\{2d_2(x)\Big(\mu_1-\frac{\psi'}{\psi}\Big)^2(1-\frac{\underline{V}}{k})-\Lambda_2(x)\Big\}\\
&\quad +(1-\underline{V})\big[a_{21}(x)p(x)\underline{U}-a_{22}(x)q(x)\underline{V}\big]\\
&= \underline{V}(1-\frac{\underline{V}}{k})^2\Big\{2d_2(x)\Big(\mu_1-\frac{\psi'}{\psi}\Big)^2-\frac{\Lambda_2(x)}{1-\frac{\underline{V}}{k}}+\frac{(1-\underline{V})\big[a_{21}(x)p(x)\underline{U}-a_{22}(x)q(x)\underline{V}\big]}{\underline{V}(1-\frac{\underline{V}}{k})^2}\Big\}.
\end{aligned}
$$

It follows from (3.21) that $H_2(\underline{U},\underline{V}) \geq 0$, which implies the pair of functions $(\underline{U},\underline{V})(x,z)$ defined in (3.20) is a lower solution. By making use of Theorem 3.6, the proof is complete. □

Remark 3.13 If $p=1$ and $k=1$, then the conditions (3.19) and (3.21) become

$$
-2d_1(x)\Big(\lambda_1(\epsilon)+\frac{\phi'}{\phi}\Big)^2+a_{12}(x)q(x)\frac{\underline{V}-\underline{U}}{(1-\underline{U})\underline{U}}\geq 0 \tag{3.22}
$$

and

$$
2d_2(x)\Big(\mu_1(\epsilon)-\frac{\psi'}{\psi}\Big)^2+a_{21}(x)p(x)\frac{\underline{U}-\underline{V}}{(1-\underline{V})\underline{V}}\geq 0, \tag{3.23}
$$

respectively.

4 Explicit Formulas for the Sign of the Bistable-Wave Speed

In this section, we concentrate on establishing explicit formulas for the sign of bistable-wave speed via choosing specific functions $\overline{U}(x,z)$, $\overline{V}(x,z)$, $\underline{U}(x,z)$ and $\underline{V}(x,z)$ in Corollaries 3.9–3.12. These formulas enable us to predict the direction the bistable traveling wave moves to.

We first construct irregular upper solutions to obtain sufficient conditions under which the bistable-wave speed is negative. Let $\overline{U}(x,z)$ be the function defined in (3.14), and define an irregular upper solution

$$
\overline{V}(x,z)=\min_{(x,z)\in\mathbb{R}^2}\{1,\eta_1\overline{U}(x,z)\} \tag{4.1}
$$

for some η_1 to be chosen. By virtue of the relations:

$$\begin{aligned}
&\overline{U}_z = \lambda_1 \overline{U}(1-\overline{U}),\ \overline{U}_x = \frac{\phi'}{\phi}\overline{U}(1-\overline{U}),\\
&\overline{U}_{zz} = \lambda_1^2 \overline{U}(1-\overline{U})(1-2\overline{U}),\ \ \overline{U}_{zx} = \lambda_1 \frac{\phi'}{\phi}\overline{U}(1-\overline{U})(1-2\overline{U}),\\
&\overline{U}_{xx} = \frac{\phi''\phi-\phi'^2}{\phi^2}\overline{U}(1-\overline{U}) + \left(\frac{\phi'}{\phi}\right)^2 \overline{U}(1-\overline{U})(1-2\overline{U}),
\end{aligned} \tag{4.2}$$

we have the following conclusion.

Theorem 4.1 *The speed of the bistable traveling wave of* (1.6) *is negative if*

$$\begin{aligned}
1 &< \frac{a_{21}(x)p(x)}{a_{22}(x)q(x)-d_2(x)Q_1(x)}\\
&< \min_{x\in[0,L]}\left\{\frac{2d_1(x)\big(\lambda_1+\frac{\phi'}{\phi}\big)^2}{a_{12}(x)q(x)},\ \frac{2\big(\lambda_1+\frac{\phi'}{\phi}\big)^2}{Q_1(x)}\right\},
\end{aligned} \tag{4.3}$$

where

$$Q_1(x) = \frac{\phi''}{\phi} + \left(2\lambda_1 + 2\frac{q'(x)}{q(x)}\right)\frac{\phi'}{\phi} + \lambda_1^2 + 2\lambda_1\frac{q'(x)}{q(x)}.$$

Proof From (4.3), we can take η_1 such that it satisfies

$$\begin{aligned}
1 &< \frac{a_{21}(x)p(x)}{a_{22}(x)q(x)-d_2(x)Q_1(x)} < \eta_1\\
&< \min_{x\in[0,L]}\left\{\frac{2d_1(x)\big(\lambda_1+\frac{\phi'}{\phi}\big)^2}{a_{12}(x)q(x)},\ \frac{2\big(\lambda_1+\frac{\phi'}{\phi}\big)^2}{Q_1(x)}\right\}.
\end{aligned} \tag{4.4}$$

We shall prove such $(\overline{U}, \overline{V})$ defined in (4.1) is an irregular upper solution. Indeed, it is easy to check that

$$J_1(x,z) = \begin{cases} \dfrac{1}{\overline{U}} \le \eta_1, & (x,z) \in \left\{(x,z) : U \ge \dfrac{1}{\eta_1}\right\},\\[2ex] \dfrac{\eta_1 - 1}{1-\overline{U}} < \eta_1, & (x,z) \in \left\{(x,z) : U \le \dfrac{1}{\eta_1}\right\}. \end{cases}$$

Therefore, by the right inequality in (4.4), we have

$$-2d_1(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)\eta_1 < 0, \tag{4.5}$$

which follows from (3.15) that $H_1(\overline{U}, \overline{V}) \leq 0$. Similarly, in the case of $(x, z) \in \{(x, z) : U \geq \frac{1}{\eta_1}\}$, it is obvious that $H_2(\overline{U}, \overline{V}) = 0$. In the case of $(x, z) \in \{(x, z) : U \leq \frac{1}{\eta_1}\}$, a substitution of $\overline{V}(x, z) = \eta_1 \overline{U}(x, z)$ enables us to obtain $H_2(\overline{U}, \overline{V}) = \eta_1 \overline{U} F_1(\overline{U})$, where

$$\begin{aligned} F_1(\overline{U}) = (1 - \overline{U})\Big\{ & d_2(x)\Big[\frac{\phi''}{\phi} + \Big(2\lambda_1 + 2\frac{q'(x)}{q(x)}\Big)\frac{\phi'}{\phi} + \lambda_1^2 + 2\lambda_1\frac{q'(x)}{q(x)}\Big] + \epsilon\lambda_1 \\ & - 2d_2(x)\overline{U}\Big(\lambda_1 + \frac{\phi'}{\phi}\Big)^2\Big\} + \Big[\frac{a_{21}(x)p(x)}{\eta_1} - a_{22}(x)q(x)\Big](1 - \eta_1\overline{U}). \end{aligned}$$

It is easy to see that $F_1''(\overline{U}) \geq 0$. In addition, let $\epsilon \to 0^+$ and take a view at (4.4), we know that

$$F_1(0) < 0 \quad \text{and } F_1\left(\frac{1}{\eta_1}\right) < 0.$$

Therefore, we have $F_1(\overline{U}) < 0$, which further implies that $H_2(\overline{U}, \overline{V}) \leq 0$. As a result, we have proved that $(\overline{U}, \overline{V})$ is an upper solution. By Theorem 3.5, the proof is finished. □

Next, we are going to construct a new regular upper solution to get sufficient conditions such that the bistable-wave speed is negative. Define

$$\overline{U} = \frac{1}{1 + \frac{e^{-\lambda_1(-\epsilon)z}}{\phi(x)}}, \quad \overline{V} = \overline{U}(a + (1 - a)\overline{U}), \quad a \geq 1. \tag{4.6}$$

here $\phi(x)$ is the eigenfunction of the operator $\mathcal{L}_\lambda\phi$ with respect to the eigenvalue $\lambda_1(-\epsilon)$. It is easy to verify that

$$\begin{aligned} &(\overline{U}^2)_z = 2\lambda_1\overline{U}^2(1 - \overline{U}), \ (\overline{U}^2)_x = 2\frac{\phi'}{\phi}\overline{U}^2(1 - \overline{U}), \\ &(\overline{U}^2)_{zz} = 2\lambda_1^2\overline{U}^2(1 - \overline{U})(2 - 3\overline{U}), \ (\overline{U}^2)_{zx} = 2\lambda_1\frac{\phi'}{\phi}\overline{U}^2(1 - \overline{U})(2 - 3\overline{U}), \\ &(\overline{U}^2)_{xx} = 2\frac{\phi''\phi - \phi'^2}{\phi^2}\overline{U}^2(1 - \overline{U}) + 2\Big(\frac{\phi'}{\phi}\Big)^2\overline{U}^2(1 - \overline{U})(2 - 3\overline{U}). \end{aligned} \tag{4.7}$$

We have the following theorem.

Theorem 4.2 *Assume that*

$$-2d_1(x)\Big(\lambda_1 + \frac{\phi'}{\phi}\Big)^2 + a_{12}(x)q(x)(a - 1) \leq 0, \tag{4.8}$$

and

$$6d_2(x)\Big(\lambda_1 + \frac{\phi'}{\phi}\Big)^2 - a_{22}(x)q(x)(a - 1) \geq 0. \tag{4.9}$$

The speed of the bistable traveling wave of (1.6) *is negative provided that*

$$\begin{aligned} &aQ_1(x)d_2(x) + a_{21}(x)p(x) - aa_{22}(x)q(x) \le 0, \\ &d_2(x)\left[aQ_1(x) + 2(2a-3)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2\right] + 2d_2(x)(1-a)Q_2(x) \\ &\quad + (2-a)(a_{21}(x)p(x) - a_{22}(x)q(x)) \le 0, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} Q_1(x) &= \frac{\phi''}{\phi} + 2\left(\lambda_1 + \frac{q'(x)}{q(x)}\right)\frac{\phi'}{\phi} + \lambda_1^2 + 2\lambda_1\frac{q'(x)}{q(x)}, \\ Q_2(x) &= \frac{\phi''}{\phi} + \frac{\phi'^2}{\phi^2} + 2\left(2\lambda_1 + \frac{q'(x)}{q(x)}\right)\frac{\phi'}{\phi} + 2\lambda_1^2 + 2\lambda_1\frac{q'(x)}{q(x)}. \end{aligned}$$

Proof By the choice of $(\overline{U}, \overline{V})$ as in (4.6), the condition (3.15) thus becomes

$$-2d_1(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)(a-1) \le 0. \tag{4.11}$$

That is true by condition (4.8). In addition, by a direct calculation and the use of (4.2) and (4.7), we can rewrite $H_2(\overline{U}, \overline{V}) = \overline{U}(1-\overline{U})F_2(\overline{U})$, with $F_2(\overline{U})$ given by

$$\begin{aligned} F_2(\overline{U}) =& aQ_1(x)d_2(x) + a\epsilon\lambda_1 + a_{21}(x)p(x) - aa_{22}(x)q(x) + \Bigg[-2ad_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 \\ &+ 2(1-a)Q_2(x)d_2(x) + (1-a)(a_{21}(x)p(x) - aa_{22}(x)q(x)) - (1-a)a_{22}(x)q(x) \\ &+ 2\epsilon(1-a)\lambda_1\Bigg]\overline{U} + \left[6d_2(x)(a-1)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 - a_{22}(x)q(x)(a-1)^2\right]\overline{U}^2. \end{aligned}$$

Letting $\epsilon \to 0^+$, then the condition (4.9) implies that $F_2(\overline{U})$ is a concave upward function. The condition (4.10) ensures that at the two endpoints 0 and 1, there hold $F_2(0) \le 0$ and $F_2(1) \le 0$. Hence, it follows that $F_2(\overline{U}) \le 0$, $\overline{U} \in (0,1)$. This in turn indicates $H_1(\overline{U}, \overline{V}) \le 0$. As a result, $(\overline{U}, \overline{V})$ defined in (4.6) is a regular upper solution. In view of Theorem 3.5, the proof is complete. □

From the discussion of Theorem 4.2, we can also deduce the following proposition.

Proposition 4.3 *Assume that*

$$-2d_1(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x)(a-1) \le 0, \tag{4.12}$$

and

$$6d_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 - a_{22}(x)q(x)(a-1) < 0. \tag{4.13}$$

The speed of the bistable traveling wave of (1.6) is negative provided that

$$\begin{aligned} & aQ_1(x)d_2(x) + a_{21}(x)p(x) - aa_{22}(x)q(x) \leq 0, \\ & -2ad_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 \\ & + (1-a)[2Q_2(x)d_2(x) + a_{21}(x)p(x) - (a+1)a_{22}(x)q(x)] \leq 0, \end{aligned} \tag{4.14}$$

or

$$\begin{aligned} & aQ_1(x)d_2(x) + a_{21}(x)p(x) - aa_{22}(x)q(x) - 2ad_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 \\ & \quad + (1-a)[2Q_2(x)d_2(x) + a_{21}(x)p(x) - (a+1)a_{22}(x)q(x)] \leq 0, \\ & -2ad_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 \\ & + (1-a)[2Q_2(x)d_2(x) + a_{21}(x)p(x) - (a+1)a_{22}(x)q(x)] > 0. \end{aligned} \tag{4.15}$$

Remark 4.4 We can illustrate an application of Theorem 4.2 to the following constant-coefficient system

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} + u_1(1 - u_1 - a_1 u_2), \\ \frac{\partial u_2}{\partial t} &= d\frac{\partial^2 u_2}{\partial x^2} + ru_2(1 - a_2 u_1 - u_2),\ t > 0, \quad x \in \mathbb{R}, \end{aligned} \tag{4.16}$$

See, e.g., Girardin and Nadin (2015), Huang and Han (2011), Huang (2001). As a result, by taking $a = 2$, we obtain that the speed of the bistable traveling wave of (4.16) is negative provided that

$$\begin{aligned} & a_1 \geq 2,\ 1 < a_2 < 2, \\ & \frac{2d(a_1 - 1)}{2 - a_2} \leq r \leq 6d(1 - a_1). \end{aligned} \tag{4.17}$$

Alternatively, besides the choice of the function $\overline{V}(x, z)$ in (3.16) as an upper solution, we can try upper solutions by other functions that are hetero-clinic connections of 0 and 1. To be exact, we define $V(x, z)$ as the solution of

$$\begin{aligned} & V_x = \frac{\phi'}{\phi}V(1 - V),\ V_z = \frac{1}{2}\lambda_1 V(1 - V), \\ & V(x, -\infty) = 0,\ V(x, \infty) = 1, \end{aligned} \tag{4.18}$$

where $\phi(x)$ is the L-periodic eigenfunction of the operator $\mathcal{L}_\lambda\phi$ with respect to $\lambda_1(-\epsilon)$. The existence of V can be given as

$$V = \frac{1}{1 + \frac{e^{-\frac{1}{2}\lambda_1(-\epsilon)z}}{\phi(x)}}.$$

Meanwhile, redefine $U = V^2$. Consequently, we have the following result.

Theorem 4.5 *Assume that*

$$\begin{aligned} &a_{21}(x)p(x) - 2d_2(x)\left(\frac{1}{2}\lambda_1 + \frac{\phi'}{\phi}\right)^2 \le 0, \\ &d_2(x)Q_3(x) - a_{22}(x)q(x) \le 0, \end{aligned} \tag{4.19}$$

or

$$\begin{aligned} &a_{21}(x)p(x) - 2d_2(x)\left(\frac{1}{2}\lambda_1 + \frac{\phi'}{\phi}\right)^2 > 0, \\ &d_2(x)Q_3(x) + \Lambda_2(x) - 2d_2(x)\left(\frac{1}{2}\lambda_1 + \frac{\phi'}{\phi}\right)^2 \le 0, \end{aligned} \tag{4.20}$$

are valid. Then, the speed of the bistable traveling wave of (1.6) *is negative provided that*

$$\begin{aligned} &a_{11}(x)p(x) - \frac{3}{2}d_1(x)\left(\lambda_1 + \frac{2\phi'}{\phi}\right)^2 \le 0, \\ &d_1(x)Q_4(x) + \Lambda_1(x) \le 0, \end{aligned} \tag{4.21}$$

or

$$\begin{aligned} &a_{11}(x)p(x) - \frac{3}{2}d_1(x)\left(\lambda_1 + \frac{2\phi'}{\phi}\right)^2 > 0, \\ &d_1(x)Q_4(x) + \Lambda_1(x) + a_{11}(x)p(x) - \frac{3}{2}d_1(x)\left(\lambda_1 + \frac{2\phi'}{\phi}\right)^2 \le 0, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} Q_3(x) &= \frac{\phi''}{\phi} + \left(\lambda_1 + 2\frac{q'(x)}{q(x)}\right)\frac{\phi'}{\phi} + \frac{1}{4}\lambda_1^2 + \lambda_1\frac{q'(x)}{q(x)}, \\ Q_4(x) &= \frac{2\phi''}{\phi} + \left(2\lambda_1 + 4\frac{p'(x)}{p(x)}\right)\frac{\phi'}{\phi} + \lambda_1^2 + 2\lambda_1\frac{p'(x)}{p(x)} + 2\frac{\phi'^2}{\phi^2}. \end{aligned}$$

Proof Putting $(U, V) := (V^2, V)$ into the second equation of (1.6) gives

$$\begin{aligned} H_2(U, V) = V(1-V)\Bigg\{ & d_2(x)\left[\frac{\phi''}{\phi} + \left(\lambda_1 + 2\frac{q'(x)}{q(x)}\right)\frac{\phi'}{\phi} + \frac{1}{4}\lambda_1^2 + \lambda_1\frac{q'(x)}{q(x)}\right] + \frac{1}{2}\lambda_1\epsilon \\ & - a_{22}(x)q(x) + V\left[a_{21}(x)p(x) - 2d_2(x)\left(\frac{1}{2}\lambda_1 + \frac{\phi'}{\phi}\right)^2\right]\Bigg\}. \end{aligned} \tag{4.23}$$

By letting $\epsilon \to 0^+$, it follows from (4.19) and (4.20) that $H_2(U, V) \le 0$.

Plugging (4.6) into the first equation of (1.6) leads to

$$\begin{aligned} H_1(U, V) = U(1-U^{\frac{1}{2}})\Bigg\{ & d_1(x)\left[\frac{2\phi''}{\phi} + \left(2\lambda_1 + 4\frac{p'(x)}{p(x)}\right)\frac{\phi'}{\phi} + \lambda_1^2 + 2\lambda_1\frac{p'(x)}{p(x)} + 2\frac{\phi'^2}{\phi^2}\right] \\ & + \lambda_1\epsilon + \Lambda_1(x) + U^{\frac{1}{2}}\left[a_{11}(x)p(x) - \frac{3}{2}d_1(x)\left(\lambda_1 + \frac{2\phi'}{\phi}\right)^2\right]\Bigg\}. \end{aligned} \tag{4.24}$$

By letting $\epsilon \to 0^+$, it is easy to see that (4.21) or (4.22) makes $H_1(U, V) \le 0$. Consequently, $(U, V) := (V^2, V)$ defined in (4.18) is a regular upper solution. By Theorem 3.5, the proof is complete. □

Now, we are in a position to investigate the positivity of the speed of the bistable traveling wave of (1.6) through establishing various lower solutions.

Theorem 4.6 *The speed of the bistable traveling wave of* (1.6) *is positive if*

$$\begin{aligned} 1 &< \frac{a_{12}(x)q(x)}{a_{11}(x)p(x) + d_1(x)Q_5(x) - 2d_1(x)\Delta(x)} \\ &< \min_{x\in[0,L]}\left\{\frac{2\Delta(x)}{2\Delta(x) - Q_5(x)}, \frac{2d_2(x)\Delta(x)}{a_{21}(x)p(x)}\right\}, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} Q_5(x) = & -\frac{\psi''}{\psi} + 2\left(\frac{\psi'}{\psi}\right)^2 - \left(2\mu_1 + 2\frac{p'(x)}{p(x)}\right)\frac{\psi'}{\psi} \\ & + \mu_1^2 + 2\mu_1\frac{p'(x)}{p(x)}, \quad \Delta(x) = \left(\mu_1 - \frac{\psi'}{\psi}\right)^2. \end{aligned}$$

Proof Thanks to (4.25), one can choose η_2 satisfying

$$\begin{aligned} 1 &< \frac{a_{12}(x)q(x)}{a_{11}(x)p(x) + d_1(x)Q_5(x) - 2d_1(x)\Delta(x)} < \eta_2 \\ &< \min_{x\in[0,L]}\left\{\frac{2\Delta(x)}{2\Delta(x) - Q_5(x)}, \frac{2d_2(x)\Delta(x)}{a_{21}(x)p(x)}\right\}. \end{aligned} \tag{4.26}$$

In (3.20), we redefine

$$\underline{V}(x, z) = \frac{1}{1 + \psi(x)e^{-\mu_1(\epsilon)z}}, \quad \underline{U}(x, z) = \max\{0, 1 - \eta_2 + \eta_2\underline{V}(x, z)\}. \tag{4.27}$$

Next, we shall verify that under condition (4.26), the pair of functions $(\underline{U}, \underline{V})(x, z)$ is an irregular lower solution. Note that

$$J_4(x,z) = \frac{\underline{U}-\underline{V}}{(1-\underline{V})\underline{V}} = \begin{cases} \dfrac{-1}{1-\underline{V}} \geq -\eta_2, & (x,z) \in \{(x,z) : \underline{V} \leq \dfrac{\eta_2 - 1}{\eta_2}\}, \\ \dfrac{1-\eta_2}{\underline{V}} > -\eta_2, & (x,z) \in \{(x,z) : \underline{V} > \dfrac{\eta_2 - 1}{\eta_2}\}. \end{cases}$$

Hence, by the right side of (4.26), we conclude that (3.23) is satisfied. This implies that $H_2(\underline{U}, \underline{V}) \geq 0$. When $(x, z) \in \{(x, z) : \underline{V} \leq \frac{\eta_2-1}{\eta_2}\}$, it is easy to see that $H_1(\underline{U}, \underline{V}) = 0$. When $(x, z) \in \{(x, z) : \underline{V} > \frac{\eta_2-1}{\eta_2}\}$, we have $H_1(\underline{U}, \underline{V}) = \eta_2(1-\underline{V})F_3(\underline{V})$, where

$$\begin{aligned} F_3(\underline{V}) = \underline{V}\Big\{ d_1(x)\Big[-\frac{\psi''}{\psi} + 2\left(\frac{\psi'}{\psi}\right)^2 - \left(2\mu_1 + 2\frac{p'(x)}{p(x)}\right)\frac{\psi'}{\psi} + \mu_1^2 + 2\mu_1\frac{p'(x)}{p(x)}\Big] - \epsilon\mu_1 \\ - 2d_1(x)\underline{V}\left(\mu_1 - \frac{\psi'}{\psi}\right)^2 \Big\} + (1-\eta_2+\eta_2\underline{V})\left(a_{11}(x)p(x) - \frac{a_{12}(x)q(x)}{\eta_2}\right). \end{aligned}$$

By letting $\epsilon \to 0^+$ and using (4.26), we obtain

$$F_3''(\underline{V}) < 0, \;\; F_3(\frac{\eta_2-1}{\eta_2}) > 0, \;\; F_3(1) > 0.$$

This means $F_3(\underline{V}) > 0$, which further implies that $H_1(\underline{U}, \underline{V}) \geq 0$. By Theorem 3.6, the proof is complete. □

Theorem 4.7 *The speed of the bistable traveling wave of* (1.6) *is positive if*

$$\begin{aligned} &\frac{a_{22}(x)q(x) - d_2(x)Q_1(x) + 2d_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2}{a_{21}(x)p(x)} \\ &\quad < \min_{x\in[0,L]}\left\{1 - \frac{2d_1(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2}{a_{11}(x)p(x)}\right\}. \end{aligned} \tag{4.28}$$

Proof Set

$$\underline{U}(x,z) = \frac{p}{1 + \frac{e^{-\lambda_1(\epsilon)z}}{\phi(x)}}, \quad \underline{V}(x,z) = \frac{1}{p}\underline{U}(x,z), \tag{4.29}$$

where p is a constant satisfying

$$\frac{a_{22}(x)q(x) - d_2(x)Q_1(x) + 2d_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2}{a_{21}(x)p(x)} < p < 1 - \frac{2d_1(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2}{a_{11}(x)p(x)}.$$

From (3.19), it follows that

$$H_1(\underline{U},\underline{V}) \geq \frac{1}{p}\underline{U}^2(1-\frac{\underline{U}}{p})\left\{-2d_1(x)\left(\lambda_1+\frac{\phi'}{\phi}\right)^2+a_{11}(x)p(x)(1-p)\right\}>0, \tag{4.30}$$

which is confirmed from (4.28) by letting $\epsilon \to 0^+$. On the other side, by a substitution of (4.29) and letting $\epsilon \to 0^+$, we obtain

$$\begin{aligned} H_2(\underline{U},\underline{V}) &= \underline{V}(1-\underline{V})\Big\{d_2(x)\Big[\frac{\phi''}{\phi}+\Big(2\lambda_1+2\frac{q'(x)}{q(x)}\Big)\frac{\phi'}{\phi}+\Big(\lambda_1^2+2\lambda_1\frac{q'(x)}{q(x)}\Big)\Big] \\ &\quad -\epsilon\lambda_1-2d_2(x)\underline{V}\left(\lambda_1+\frac{\phi'}{\phi}\right)^2+\big[a_{21}(x)p(x)p-a_{22}(x)q(x)\big]\Big\} \\ &\geq \underline{V}(1-\underline{V})\left\{d_2(x)Q_1(x)-2d_2(x)\left(\lambda_1+\frac{\phi'}{\phi}\right)^2+\big[a_{21}(x)p(x)p-a_{22}(x)q(x)\big]\right\} \\ &> 0. \end{aligned} \tag{4.31}$$

The last inequality follows from $0<\underline{V}<1$ and (4.28). Also, (4.30), combined with (4.31), shows that $(\underline{U},\underline{V})$ redefined in (4.29) is a lower solution. By Theorem 3.6, the proof is complete. □

To seek more sufficient conditions under which the bistable traveling wave spreads to the left, we define, with $0<p<1$ and $0<\epsilon$,

$$\underline{U}=\frac{p}{1+\frac{e^{-\lambda_1(\epsilon)z}}{\phi(x)}},\quad \underline{V}=\frac{\underline{U}}{p}(2-\frac{\underline{U}}{p}). \tag{4.32}$$

Theorem 4.8 *Assume that*

$$6d_2(x)\left(\lambda_1+\frac{\phi'}{\phi}\right)^2-a_{22}(x)q(x)<0. \tag{4.33}$$

The speed of the bistable traveling wave of (1.6) is positive provided that

$$\begin{aligned} &\frac{2a_{22}(x)q(x)-2Q_1(x)d_2(x)}{a_{21}(x)p(x)} \\ &\quad <\min_{x\in[0,L]}\left\{1+\frac{a_{12}(x)q(x)-2d_1(x)\left(\lambda_1+\frac{\phi'}{\phi}\right)^2}{a_{11}(x)p(x)}\right\}. \end{aligned} \tag{4.34}$$

Proof In view of (4.34), we first choose p satisfying

$$\frac{2a_{22}(x)q(x) - 2Q_1(x)d_2(x)}{a_{21}(x)p(x)} < p < 1 + \frac{a_{12}(x)q(x) - 2d_1(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2}{a_{11}(x)p(x)}. \tag{4.35}$$

A substitution of (4.32) into (3.19) leads to

$$-2d_1(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 + a_{12}(x)q(x) + a_{11}(x)p(x)(1-p) \geq 0, \tag{4.36}$$

which holds by virtue of the right side of (4.35). Additionally, a substitution of (4.32) enables us to rewrite the left of the V-equation as $H_2(\underline{U}, \underline{V}) := \frac{\underline{U}}{p}\left(1 - \frac{\underline{U}}{p}\right) F_4\left(\frac{\underline{U}}{p}\right)$, with $F_4(\frac{\underline{U}}{p})$ given by

$$\begin{aligned} F_4\left(\frac{\underline{U}}{p}\right) &= 2Q_1(x)d_2(x) + 2\epsilon\lambda_1 + a_{21}(x)p(x)p - 2a_{22}(x)q(x) \\ &\quad + \Bigg[-4d_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 - 2Q_2(x)d_2(x) - a_{21}(x)p(x)p \\ &\quad + 3a_{22}(x)q(x) - 2\epsilon\lambda_1\Bigg]\frac{\underline{U}}{p} \\ &\quad + \left[6d_2(x)\left(\lambda_1 + \frac{\phi'}{\phi}\right)^2 - a_{22}(x)q(x)\right]\left(\frac{\underline{U}}{p}\right)^2. \end{aligned}$$

It is easy to see that $F_4(1) = 0$. From the condition (4.33), we know that $F_4\left(\frac{\underline{U}}{p}\right)$ is a concave downward function. This combined with the left side of (4.35) implies that $F_4(0) > 0$. Therefore, $F_4\left(\frac{\underline{U}}{p}\right) > 0$ for $\frac{\underline{U}}{p} \in (0, 1)$. Hence, $(\underline{U}, \underline{V})$ is a regular lower solution, by Theorem 3.6, the proof is complete. □

5 Interval Estimation of the Pulsating Wave Speed

By the usual classical partial order, we denote $[\mathbf{u_1}, \mathbf{u_2}]_{\mathcal{C}}$ as the set $\{\mathbf{u} \in \mathcal{C} : \mathbf{u_1} \leq \mathbf{u} \leq \mathbf{u_2}\}$. Moreover, we denote the bistable steady states by $\{\mathbf{o}, \beta\}$ and other unstable steady states of (1.3) by α. Let $c_-^*(\alpha, \beta)$ be the leftward spreading speed of (1.6) and $c_+^*(\mathbf{o}, \alpha)$ be the rightward spreading speed. Fang and Zhao (2015) indicated that if and only if $c \geq c_-^*(\alpha, \beta)$, the system (1.6) has a traveling wave connecting α and β; if and only if $c \leq -c_+^*(\mathbf{o}, \alpha)$, the system (1.6) has a traveling wave connecting $\mathbf{o}$ and α. Once the bistable wave exists with a speed c, we want to establish a relationship among all above speeds.

Theorem 5.1 *The speed c of the bistable traveling wave solution to (1.6) satisfies*

$$-c_+^*(\mathbf{o}, \alpha) \leq c \leq c_-^*(\alpha, \beta). \tag{5.1}$$

Proof We first prove $c \leq c_-^*(\alpha, \beta)$. Choose a pair of continuous, non-decreasing functions $(v_1, v_2)(0, x)$ as initial function to the system (1.3) satisfying

$$(v_1, v_2)(0, x) = \begin{cases} 0, & x \leq -M, \\ 1 - \varepsilon, & x \geq M, \end{cases}$$

where $M > 0$ and $\varepsilon \in (0, 1)$ are constants. On the one hand, we have from Lemma 3.4 that

$$\begin{aligned} v_1(t, x) \geq & U(x, x + ct + \xi^- - \sigma_0 \delta(1 - e^{-\beta_0 t})) \\ & -\delta p_1(x, x + ct + \xi^- - \sigma_0 \delta(1 - e^{-\beta_0 t}))e^{-\beta_0 t}. \end{aligned} \tag{5.2}$$

On the other hand, suppose $(U_1, V_1)(x, x + c_-^*(\alpha, \beta)t)$ is the monostable traveling wave solution, connecting α to β, to the system (1.6). Hence, by taking appropriate M, ε and shifting if it is necessary, the initial function $(U_1, V_1)(x, x)$ can be assumed to dominate $(v_1, v_2)(0, x)$, namely

$$(U_1, V_1)(x, x) \geq (v_1, v_2)(0, x).$$

By the comparison principle, we have

$$(U_1, V_1)(x, x + c_-^*(\alpha, \beta)t) \geq (v_1, v_2)(t, x). \tag{5.3}$$

Fix $x + c_-^*(\alpha, \beta)t = z_0$ such that $U_1(x, z_0) < 1$. Then, at the line $z = z_0$, a combination of (5.2) and (5.3) enables us to get

$$\begin{aligned} U_1(x, z_0) \geq & U(x, z_0 + (c - c_-^*(\alpha, \beta))t + \xi^- - \sigma_0 \delta(1 - e^{-\beta_0 t})) \\ & - \delta p_1(x, z_0 + (c - c_-^*(\alpha, \beta))t + \xi^- - \sigma_0 \delta(1 - e^{-\beta_0 t}))e^{-\beta_0 t}. \end{aligned} \tag{5.4}$$

If $c > c_-^*(\alpha, \beta)$, as $t \to +\infty$, we obtain from (5.4) that

$$U_1(x, z_0) \geq 1,$$

which gives a contradiction. Thus, $c \leq c_-^*(\alpha, \beta)$, this completes the proof of right part of (5.1). As for the proof of the left part, one can carry out a similar analysis, and we omit it here. □

6 Numerical Simulations

In this section, we perform numerical simulations to demonstrate the solution patterns (moving wave patterns) to the system.

By choosing the set of periodic coefficients in (1.1) as follows

$$\begin{aligned} &d_1(x) = 1,\ d_2(x) = 1,\\ &b_1(x) = 0.5\cos(0.2x) + 1,\ b_2(x) = 0.6\sin(0.2x) + 1,\\ &a_{11}(x) = 0.3\cos(0.2x) + 1,\ a_{12}(x) = 0.5\sin(0.2x) + 6,\\ &a_{21}(x) = 0.5\sin(0.2x) + 1.2,\ a_{22}(x) = 0.5\sin(0.2x) + 1, \end{aligned} \tag{6.1}$$

we find that

$$\lambda(d_1(x), 0, b_1(x)) = 1.4502 > 0,\ \lambda(d_2(x), 0, b_2(x)) = 1.3000 > 0$$

and

$$\begin{aligned} \lambda(d_1(x), 2d_1(x)\frac{p'(x)}{p(x)}, \Lambda_1(x)) &= -0.0571 < 0,\ \lambda(d_2(x), 2d_2(x)\frac{q'(x)}{q(x)},\\ &\quad -\Lambda_2(x)) = -3.6172 < 0. \end{aligned}$$

Therefore, the conditions (H1) and (H2) are satisfied. In addition, it can be verified numerically that (4.28) holds true. By Theorem 4.7, the waves are then assumed to propagate to the left. This can be justified in Fig. 1.

Similarly, we can take

$$\begin{aligned} &d_1(x) = 1,\ d_2(x) = 0.2,\\ &b_1(x) = 0.03\cos(0.2x) + 1.5,\ b_2(x) = 0.02\sin(0.2x) + 4,\\ &a_{11}(x) = 0.02\cos(0.2x) + 2.6,\ a_{12}(x) = 0.02\sin(0.2x) + 0.6,\\ &a_{21}(x) = 0.02\sin(0.2x) + 0.6,\ a_{22}(x) = 0.02\sin(0.2x) + 0.5. \end{aligned} \tag{6.2}$$

We can verify that the conditions in Theorem 4.1 are satisfied. Numerical solution stabilizes to a wave pattern that propagates to the right, and this is consistent with our theorem.

In the simulation, we also find that under the bistable nonlinearities (H1) and (H2), traveling wave moving to the right still exists, even though the conditions in Theorems 4.1–4.2 or Theorem 4.5 are not satisfied. For example, we choose

$$\begin{aligned} &d_1(x) = 1,\ d_2(x) = 1,\\ &b_1(x) = 0.3\cos(0.2x) + 1.5,\ b_2(x) = 1.5\sin(0.2x) + 4,\\ &a_{11}(x) = 1.5\cos(0.2x) + 2.6,\ a_{12}(x) = 0.8\sin(0.2x) + 0.6,\\ &a_{21}(x) = 0.3\sin(0.2x) + 0.6,\ a_{22}(x) = 0.2\sin(0.2x) + 0.5. \end{aligned} \tag{6.3}$$

The evolutions of $u(t, x)$ and $v(t, x)$ are shown in Fig. 2. The same situation also occurs for Theorems 4.6–4.7. One of the examples is given by

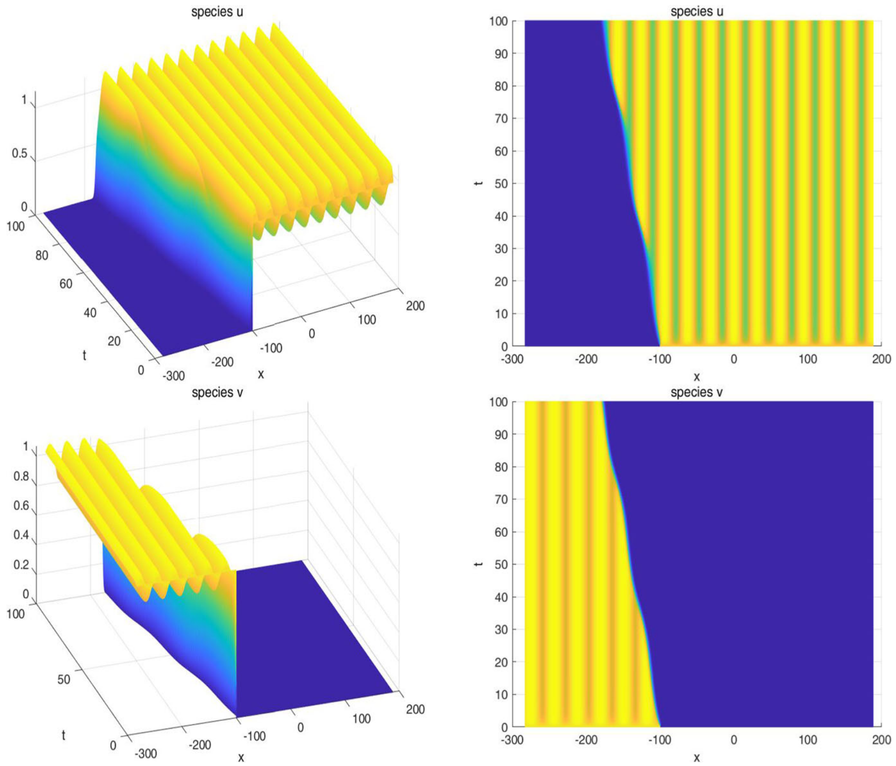

Fig. 1 Positive wave speed (moving to the left). The figures of $u(t,x)$ and $v(t,x)$ are depicted when (6.1) holds. The right panels are the ones viewed from the top

$$
\begin{aligned}
&d_1(x) = 1,\ d_2(x) = 30,\\
&b_1(x) = 0.6\cos(0.2x) + 1,\ b_2(x) = 0.01\sin(0.2x) + 0.5,\\
&a_{11}(x) = 0.5\cos(0.2x) + 1,\ a_{12}(x) = 0.5\sin(0.2x) + 5,\\
&a_{21}(x) = 0.6\sin(0.2x) + 1.02,\ a_{22}(x) = 0.1\sin(0.2x) + 0.5.
\end{aligned}
\tag{6.4}
$$

The above observations indicate that the conditions obtained in Sect. 4 are the sufficient conditions, not the necessary conditions. In other words, this means that the conditions in these theorems are not complete and there is possibility to improve the above results in the future by choosing some new upper or lower solutions or by developing new methods. We will revisit these problems in our further work.

7 Conclusion

In this paper, we establish criteria for determining the speed sign of a Lotka–Volterra competition model in a periodic habitat. Our results show that the speed of the bistable wave is negative, if we can find an upper solution of the model with negative near-zero speed. Similarly, the speed of the bistable wave must be positive as long as the

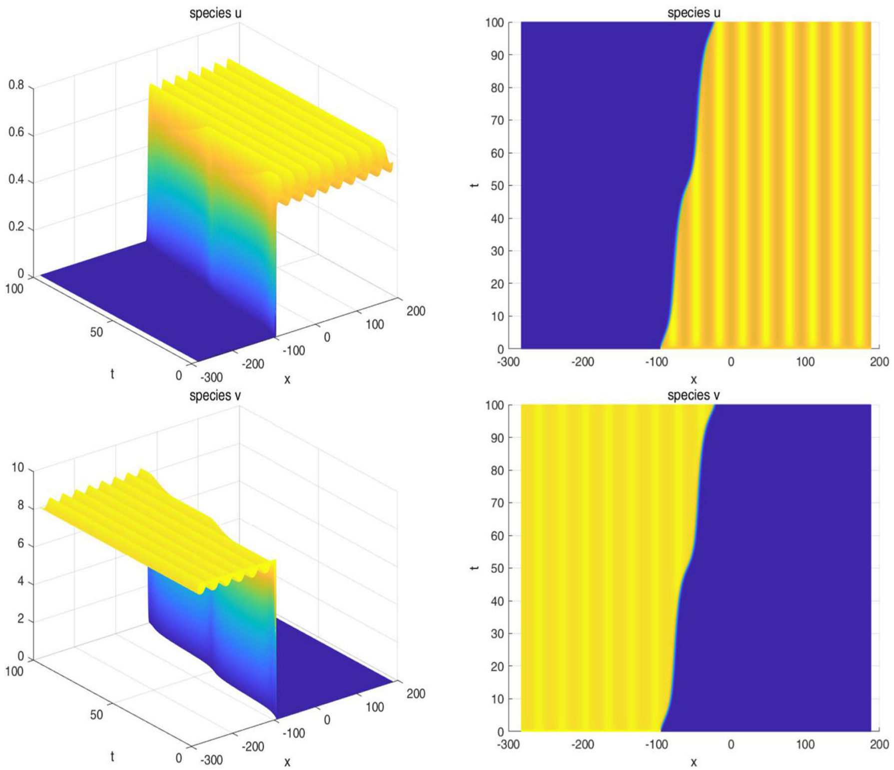

Fig. 2 Negative wave speed (moving to the right). The figures of $u(t,x)$ and $v(t,x)$ are depicted when (6.3) holds. The right panels are the ones viewed from the top

system has a lower solution with positive near-zero speed. We have constructed various upper or lower solutions as testing functions to obtain the sign of the wave speed based on explicit formulas. Biologically, our idea provides insight for an effective approach to find or control the direction of wave propagation for a competitive system in heterogeneous environments. The obtained results are also significantly meaningful in constructing front-like entire wave patterns (see, e.g., Du et al. 2019).

Finally, we need mention that our methods can be extended to more reaction–advection–diffusion competition model in a periodic habitat or with time-space-periodic coefficients.

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