MATH 2050 **Assignment 2** **Fall 2018**

Due: Friday, September 28

[5] 1. Let $\vec{u} = \begin{bmatrix} -3 \\ 4 \\ 0 \\ 12 \end{bmatrix}$. Find a unit vector in the direction of $\vec{u}$ and a vector of length 4 in the direction opposite to $\vec{u}$.

Solution: $\|\vec{u}\| = \sqrt{(-3)^2 + 4^2 + 0^2 + 12^2} = \sqrt{169} = 13$

$\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{13}\begin{bmatrix} -3 \\ 4 \\ 0 \\ 12 \end{bmatrix}$ is a unit vector in the direction of $\vec{u}$.

A vector of norm 4 in the direction opposite to $\vec{u}$ would be

$$-\frac{4}{13}\vec{u} = -\frac{4}{13}\begin{bmatrix} -3 \\ 4 \\ 0 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{12}{13} \\ -\frac{16}{13} \\ 0 \\ -\frac{48}{13} \end{bmatrix}.$$

[5] 2. Find all vectors $\vec{u}$ that are parallel to $\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$ and satisfy $\|\vec{u}\|^2 = 2\|\vec{v}\|^2$.

Solution: Let $\vec{u} = k\vec{v}$, so that $\vec{u}$ is parallel to $\vec{v}$. Then $\|\vec{u}\|^2 = k^2\|\vec{v}\|^2$. To get what we want, we need $k = \pm\sqrt{2}$

Therefore $\vec{u} = \sqrt{2}\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 4\sqrt{2} \end{bmatrix}$ or $\vec{u} = -\sqrt{2}\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ 2\sqrt{2} \\ -4\sqrt{2} \end{bmatrix}$.

[10] 3. (a) Let $\vec{u}$ and $\vec{v}$ be vectors of magnitude 2 and 5, respectively, and suppose that $\vec{u} \cdot \vec{v} = -3$. Find $(\vec{u} - \vec{v}) \cdot (2\vec{u} - 3\vec{v})$ and $\|\vec{u} + \vec{v}\|$.

(b) The two vectors $3\vec{u} + \vec{v}$ and $\vec{u} - 4\vec{v}$ are perpendicular. Find the angle between $\vec{u}$ and $\vec{v}$ if $\|\vec{u}\| = 2\|\vec{v}\|$.

Solution: (a)

$$\begin{aligned}(\vec{u} - \vec{v}) \cdot (2\vec{u} - 3\vec{v}) &= 2\|\vec{u}\|^2 - 5\vec{u} \cdot \vec{v} + 3\|\vec{v}\|^2 \\ &= 2(4) - 5(-3) + 3(25) = 98.\end{aligned}$$

From $\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 = 2^2 + 2(-3) + 5^2 = 23$,
Therefore $\|\vec{u} + \vec{v}\| = \sqrt{23}$.

(b) $(3\vec{u}+\vec{v})\cdot(\vec{u}-4\vec{v}) = 3\|\vec{u}\|^2 - 11\vec{u}\cdot\vec{v} - 4\|\vec{v}\|^2 = 0$

Since $\|\vec{u}\| = 2\|\vec{v}\|$, then we have $8\|\vec{v}\|^2 - 11\vec{u}\cdot\vec{v} = 0$.

$\Longrightarrow \vec{u}\cdot\vec{v} = \frac{8}{11}\|\vec{v}\|^2$ and $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{\frac{8}{11}\|\vec{v}\|^2}{2\|\vec{v}\|\|\vec{v}\|} = \frac{4}{11}, \Longrightarrow \theta = \cos^{-1}(\frac{4}{11}) \approx 1.20$ radians.

[5] 4. Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2k-3 \\ 3k-k^2 \\ 3 \end{bmatrix}$. Determine all values of k for which $\vec{u}$ and $\vec{v}$ are orthogonal.

Solution: $\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2k-3 \\ 3k-k^2 \\ 3 \end{bmatrix} = 0 \Longleftrightarrow 2k-3-(3k-k^2)-9 = 0 \Longleftrightarrow k^2-k-12 = 0 \Longleftrightarrow k = 4, k = -3$

[5] 5. Find all real numbers x such that $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$.

Solution: $\cos\frac{\pi}{3} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{2-x+2}{\sqrt{6}\sqrt{1+x^2+4}}$

$\frac{1}{2} = \frac{4-x}{\sqrt{6}\sqrt{5+x^2}} \Longrightarrow x^2+16x-17 = 0 \Longrightarrow x = -17, x = 1$

[5] 6. Let $\vec{u} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Show that $\vec{u}$ is orthogonal to $\vec{v}-\vec{w}$.

(b) Show that $\vec{u}$ is orthogonal to $a\vec{v}+b\vec{w}$ for any scalars a and b.

Solution: (a) $\vec{v}-\vec{w} = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}$. Since $\vec{u}\cdot(\vec{v}-\vec{w}) = 0+12-12 = 0$, $\vec{u}$ is orthogonal to $\vec{v}-\vec{w}$.

(b) $a\vec{v}+b\vec{w} = \begin{bmatrix} a+b \\ 2b \\ 3b-a \end{bmatrix}$.

$\Longrightarrow \vec{u}\cdot(a\vec{v}+b\vec{w}) = 3(a+b)-6(2b)+3(3b-a) = 0$

Therefore $\vec{u}$ is orthogonal to $a\vec{v}+b\vec{w}$.

Alternatively, $\vec{u}\cdot\vec{v} = 3+0-3 = 0$, $\vec{u}\cdot\vec{w} = 3-12+9 = 0$, hence $\vec{u}\cdot(a\vec{v}+b\vec{w}) = a(\vec{u}\cdot\vec{v})+b(\vec{u}\cdot\vec{w}) = a(0)+b(0) = 0$.

[5] 7. Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}$ be a unit vector in the plane. What are the possible values of $\|\vec{u}+\vec{v}\|$? Give a unit vector $\vec{v}$ such that $\|\vec{u}+\vec{v}\| = \sqrt{3}$.

Solution: We can compute $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2+2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2 = 2+2\vec{u}\cdot\vec{v}$, since $\vec{u}$ and $\vec{v}$ are unit vectors. From the Cauchy-Schwarz inequality, we have that $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\| = 1$. Thus, $-1 \leq \vec{u}\cdot\vec{v} \leq 1$, and $0 \leq \|\vec{u}+\vec{v}\|^2 \leq 4$. This gives

$$0 \leq \|\vec{u}+\vec{v}\| \leq 2.$$

Note that both extremes can occur: if $\vec{v} = -\vec{u}$, $\|\vec{u}+\vec{v}\| = 0$, and if $\vec{v} = \vec{u}$, then $\|\vec{u}+\vec{v}\| = 2$.

If $\|\vec{u}+\vec{v}\|^2 = 3$, then $\vec{u}\cdot\vec{v} = 1/2$. Since $\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}$, this gives that $v_1 = 1/2$; in order for $\vec{v}$ to be a unit vector, we must have $v_2^2 = 1 - v_1^2 = 3/4$, giving $\vec{v} = \begin{bmatrix}1/2\\\sqrt{3}/2\end{bmatrix}$ or $\vec{v} = \begin{bmatrix}1/2\\-\sqrt{3}/2\end{bmatrix}$

[5] 8. Give vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ such that $\vec{u}\cdot\vec{v} = 0$ and $\vec{v}\cdot\vec{w} = 0$, but $\vec{u}\cdot\vec{w} \neq 0$.

Solution: There are lots of possibilities. One is to take $\vec{v} = \vec{0}$, and $\vec{u} = \vec{w} = \vec{\imath}$. For any nonzero vector, $\vec{u}$, choosing $\vec{v}$ so that $\vec{u}\cdot\vec{v} = 0$ and $\vec{w} = k\vec{u}$ for any nonzero constant k will work.

[5] 9. Given unit vector $\vec{u}$, is it possible to find a vector $\vec{v}$ such that $\vec{u}\cdot\vec{v} = -3$ and $\|\vec{v}\| = 2$? Give an example or explain why this can't be done.

Solution: The Cauchy-Schwarz inequality states that $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$. Given the information in the problem, this would require that $3 \leq 2$, which is not true. Thus, no such vector $\vec{v}$ can be found.

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