MEMORIAL UNIVERSITY

DEPARTMENT OF MATHEMATICS & STATISTICS

TEST 2 **Math 2050** **FALL 2018**

Last Name: First name: **Student ID:**

[8] 1. Assume $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ x+2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ x^2 \end{bmatrix}$. For what value of x, are the vectors $\vec{u}$,$\vec{v}$ and $\vec{w}$ linearly dependent?

Sol^n: The vectors are linearly dependent if and only if the system $A\vec{x}=\vec{0}$, with these vectors as columns of the matrix A has non-zero solutions. So, we put A into row echelon for

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & x+2 & x^2 \end{pmatrix} \xrightarrow[r_3 \leftarrow r_3 - r_1]{r_2 \leftarrow r_2 + r_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & x+1 & x^2 \end{pmatrix} \xrightarrow{r_3 \leftarrow r_3 - (x+1)\frac{r_2}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & x^2 - \frac{x+1}{2} \end{pmatrix}$$

To have non-zero solutions, we need

$x^2 - \frac{x+1}{2} = 0 \Rightarrow x_1 = 1$ or $x_2 = -\frac{1}{2}$.

Thus, the vectors are linearly dependent for $x_1 = 1$ or $x_2 = -\frac{1}{2}$.

[9] 2. Find conditions on a and b such that the system

$$\begin{cases} -x + 3y + 2z = -8 \\ x + z = 2 \\ 3x + 3y + az = b + 1 \end{cases}$$

has a unique solution; no solution; infinitely many solutions.

Sol^n: $\left(\begin{array}{ccc|c} -1 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b+1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ -1 & 3 & 2 & -8 \\ 3 & 3 & a & b+1 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 - 3r_1]{r_2 \leftarrow r_2 + r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & 6 \\ 0 & 3 & a-3 & b-5 \end{array}\right)$

$\xrightarrow{r_3 \leftarrow r_3 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & a-6 & b+1 \end{array}\right)$

(1) Unique solution: $a-6 \neq 0 \Leftrightarrow a \neq 6$.

(2) No solution: $a-6=0$ and $b+1 \neq 0 \Leftrightarrow a=6$ and $b \neq -$

(c) Infinitely many solutions: $a-6=0$ and $b+1=0$

$\Leftrightarrow a=6$ and $b=-1$.

[10] 3. Solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - 2x_4 = 0 \\ -3x_1 + 6x_2 + x_3 = 4 \\ -3x_1 + 6x_2 + 6x_3 - 3x_4 = 6 \end{cases}$$

Express the solution of in the form $X = X_p + X_h$ where X_p is a particular solution and X_h is a solution to the corresponding homogeneous system.

Sol^n: We have

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 4 \\ -3 & 6 & 6 & -3 & 6 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 + 3r_1]{r_2 \leftarrow r_2 + 3r_1} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ 0 & 0 & 10 & -6 & 4 \\ 0 & 0 & 15 & -9 & 6 \end{array}\right)$$

$$\xrightarrow[r_3 \leftarrow \frac{r_3}{3}]{r_2 \leftarrow \frac{r_2}{2}} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ 0 & 0 & 5 & -3 & 2 \\ 0 & 0 & 5 & -3 & 2 \end{array}\right) \xrightarrow{r_3 \leftarrow r_3 - r_2} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ 0 & 0 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, choose $x_2 = s$, $x_4 = t$.

Then $5x_3 - 3x_4 = 2 \Rightarrow x_3 = \frac{2+3x_4}{5} = \frac{2}{5} + \frac{3}{5}t$

$$x_1 - 2x_2 + 3x_3 - 2x_4 = 0 \Rightarrow x_1 = 2x_2 - 3x_3 + 2x_4$$

$$= 2s - 3 \cdot \frac{2+3t}{5} + 2t$$

$$= 2s + \frac{1}{5}t - \frac{6}{5}$$

Therefore, $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2s + \frac{1}{5}t - \frac{6}{5} \\ s \\ \frac{3}{5}t + \frac{2}{5} \\ t \end{pmatrix} = \underbrace{s\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/5 \\ 0 \\ 3/5 \\ 1 \end{pmatrix}}_{x_h} + \underbrace{\begin{pmatrix} -6/5 \\ 0 \\ 2/5 \\ 0 \end{pmatrix}}_{x_p}.$$

[8] 4. (a) Find the inverse of $A = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 6 \\ 1 & 0 & 1 \end{bmatrix}$

[3] (b) Solve the matrix equation $AX = b$ where A is the matrix of part (a) and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(You may use A^{-1} from part (a).)

Sol^n: (a) Method I: $[A|I]$:

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 4 & 1 & 0 & 0 \\ 4 & 1 & 6 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 4 & 1 & 6 & 0 & 1 & 0 \\ 3 & 1 & 4 & 1 & 0 & 0 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 - 3r_1]{r_2 \leftarrow r_2 - 4r_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -4 \\ 0 & 1 & 1 & 1 & 0 & -3 \end{array}\right) \xrightarrow{r_3 \leftarrow r_3 - r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -4 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[r_1 \leftarrow r_1 + r_3]{r_2 \leftarrow r_2 + 2r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right) \xrightarrow{r_3 \leftarrow (-1) r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

Thus, $A^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & -2 \\ -1 & 1 & -1 \end{pmatrix}$

Method II: $\det(A) = 1$. $\operatorname{adj}(A) = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 2 & -2 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & -2 \\ -1 & 1 & -1 \end{pmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & -2 \\ -1 & 1 & -1 \end{pmatrix}.$$

(b): $A\cancel{X} = b$ with A^{-1} exist. Thus,

$$X = A^{-1}b = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & -2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-2+6 \\ 2-2-6 \\ -1+2-3 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ -2 \end{pmatrix}$$

[6] 5. (a) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Find the matrix X if $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} XA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}$

Sol'n: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} XA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} X [A - I] = \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}$

$\Leftrightarrow \quad X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix} [A - I]^{-1}$

$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix} \cdot \left(-\frac{1}{2}\right) \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$

$= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \cdot \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \left(-\frac{1}{2}\right) \begin{bmatrix} -2 & 0 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

[6] (b) Express the matrix $B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ as a product of elementary matrices.

Sol'n: $B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

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$\xrightarrow{r_1 \Leftrightarrow r_2} \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\xrightarrow{r_2 \leftarrow r_2 - 3r_1} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \; E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$\xrightarrow{r_1 \leftarrow r_1 + r_2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; E_3^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\xrightarrow{r_2 \leftarrow (-1) \cdot r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; E_4^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

That is, $(E_4 E_3 E_2 E_1) B = I$

Solution is not unique.

$\Rightarrow B = (E_4 E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.