MATH 2050 Assignment 4 Fall 2018

Due: Thursday

[5] 1. Let $\vec{u} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Find the projection of $\vec{u}$ onto $\vec{v}$; and the projection of $\vec{v}$ onto $\vec{u}$ respectively.

ANS: The projection of $\vec{u}$ onto $\vec{v}$ is $Proj_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$. Note that $\vec{u}\cdot\vec{v} = (2\times 1) + (2\times 1) - (3\times 1) = 1$ and $\|\vec{v}\|^2 = 1+1+1 = 3$. Hence,

$$Proj_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Similarly, as $\|\vec{u}\|^2 = 4+4+9 = 17$, one has,

$$Proj_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u} = \frac{1}{17}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

[10] 2. (a) Find two orthogonal vectors in the plane $x+y-2z=0$.

ANS: First of all, let $z=0$, one gets $x+y=0$. Let $y=-1$ and then $x=1$. That is, vector $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is in the plane $x+y-2z=0$.

Second, let $y=0$, one gets $x-2z=0$. Let $z=1$ and then $x=2$. That is, vector $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is in the plane $x+y-2z=0$.

We can check that $\vec{u}\cdot\vec{v} = 2 \neq 0$, and hence $\vec{u}$ is not orthogonal to $\vec{v}$. We now calculate the projection of $\vec{v}$ onto $\vec{u}$ as follows:

$$Proj_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u} = \frac{2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

as $\|\vec{u}\|^2 = 1+1 = 2$. Let $\vec{f} = \vec{v} - Proj_{\vec{u}}(\vec{v}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{e} = \vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. One can check that $\vec{e}$ and $\vec{f}$ are in the plane $x+y-2z=0$ and are orthogonal to each other.

(b) Find the projection of $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ onto the plane $x + y - 2z = 0$.

ANS: Method 1: The projection of $\vec{u}$ onto the plane $\pi : x + y - 2z = 0$ is

$$Proj_{\pi}(\vec{u}) = \frac{\vec{u} \cdot \vec{e}}{\|\vec{e}\|^2}\vec{e} + \frac{\vec{u} \cdot \vec{f}}{\|\vec{f}\|^2}\vec{f} = \begin{bmatrix} 5/2 \\ 3/2 \\ 2 \end{bmatrix},$$

where $\|\vec{e}\|^2 = 2, \vec{e} \cdot \vec{u} = 1, \vec{f} \cdot \vec{u} = 6$ and $\|\vec{f}\|^2 = 3$.

Method 2: The plane π has normal vector $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. We now calculate the projection of $\vec{u}$ onto $\vec{n}$ as follows:

$$Proj_{\vec{n}}(\vec{u}) = \frac{\vec{u} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \frac{-3}{6}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix},$$

as $\|\vec{n}\|^2 = 1 + 1 + 4 = 6$ and $\vec{u} \cdot \vec{n} = -3$. So, the projection of $\vec{u}$ onto the plane π is

$$Proj_{\pi}(\vec{u}) = \vec{u} - Proj_{\vec{n}}(\vec{u}) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/2 \\ 2 \end{bmatrix}$$

[5] 3. Calculate the distance from point $P(1, 2, 1)$ to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ 2-2t \\ 2t-1 \end{bmatrix}$.

ANS: The line has the direction $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and passes one point, say $Q = (1, 2, -1)$ (by letting $t = 0$).

Let $\vec{v} = \vec{PQ} = \begin{bmatrix} 1-1 \\ 2-2 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$. Now let us calculate

$$Proj_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\vec{u} = \begin{bmatrix} -4/9 \\ 8/9 \\ -8/9 \end{bmatrix} \quad \& \quad \vec{v} - Proj_{\vec{u}}(\vec{v}) = \begin{bmatrix} 4/9 \\ -8/9 \\ -10/9 \end{bmatrix}.$$

So the distance from point $P(1, 2, 1)$ to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ 2-2t \\ 2t-1 \end{bmatrix}$ is the length of $\vec{v} - Proj_{\vec{u}}(\vec{v})$, which is, $\sqrt{180}/9 = \sqrt{20}/3$.

[5] 4. Calculate the distance from point $P(1,2,1)$ to the plane $2x - y + 2z = 1$.

ANS: A normal vector for the plane $2x - y + 2z = 1$ is $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and one point in the plane is, say $Q = (0, -1, 0)$.

Let $\vec{v} = \vec{PQ} = \begin{bmatrix} 0-1 \\ -1-2 \\ 0-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$. Now let us calculate

$$Proj_{\vec{n}}(\vec{v}) = \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} -2/9 \\ 1/9 \\ -2/9 \end{bmatrix}.$$

So the distance from point $P(1,2,1)$ to the plane $2x - y + 2z = 1$ is the length of $Proj_{\vec{n}}(\vec{v})$, which is, $1/3$.

[5] 5. Are vectors $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ linearly independent?

ANS: Let c_1, c_2, c_3 and c_4 be such that $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} + c_4\vec{x} = 0$. That is, c_1, c_2, c_3 and c_4 satisfy the following equations:

$$\begin{cases} c_1 - c_2 - c_4 = 0; \\ -c_1 + c_2 + c_3 = 0; \\ c_1 + 2c_2 + c_3 + c_4 = 0. \end{cases}$$

Solve the above equations by elimination method: the first equation plus the second equation yields

$$c_3 = c_4;$$

put this into the third equation to have

$$c_1 + 2c_2 + 2c_4 = 0.$$

Combining this equation with the first equation (first subtracting, then adding them) implies that

$$c_2 + c_4 = 0 \quad \text{and} \quad c_1 = 0.$$

Hence, one gets $c_1 = 0$ and $c_2 = -c_4$. (Note that this system has 3 equations with 4 variables, and hence there is at least one free variable, say c_4.)

In conclusion, the solution for the above equations are $c_4 = c_3 = -c_2$ and $c_1 = 0$. In particular, one can let $c_4 = 1$ and we get the following $-\vec{v} + \vec{w} + \vec{x} = 0$, and hence vectors $\vec{u}, \vec{v}, \vec{w}$, and $\vec{x}$ are linearly dependent.

[10] 6. Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Show that $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly independent.

ANS: Let c_1, c_2 and c_3 be such that $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = 0$. That is, c_1, c_2 and c_3 satisfy the following equations:

$$\begin{cases} c_1 + c_2 + c_3 = 0; \\ c_2 + c_3 = 0; \\ c_1 + c_3 = 0. \end{cases}$$

Solve the above equations by elimination method: the first equation minus the second equation yields

$$c_1 = 0.$$

the first equation minus the third equation yields

$$c_2 = 0.$$

By the first equation, one has $c_3 = 0$.
So the equation $c_1\vec{u}+c_2\vec{v}+c_3\vec{w} = 0$ only has solution $c_1 = c_2 = c_3 = 0$, and hence vectors $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly independent.

(b) Show that any 3-dimensional vector can be written as a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$.

ANS: Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any 3-dimensional vector. Suppose that $\vec{x} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$, and hence one has the following equations (treat x_1, x_2, x_3 as constant numbers):

$$\begin{cases} c_1 + c_2 + c_3 = x_1; \\ c_2 + c_3 = x_2; \\ c_1 + c_3 = x_3. \end{cases}$$

Solve the above equations by elimination method: the first equation minus the second equation yields

$$c_1 = x_1 - x_2;$$

the first equation minus the third equation yields

$$c_2 = x_1 - x_3.$$

The first equation then implies $c_3 = x_2 + x_3 - x_1$.
In conclusion, the vector $\vec{x}$ has the following form

$$\vec{x} = (x_1 - x_2)\vec{u} + (x_1 - x_3)\vec{v} + (x_2 + x_3 - x_1)\vec{w},$$

a linear combination of of $\vec{u}$, $\vec{v}$ and $\vec{w}$.

[10] 7. Let $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(a) Are $\vec{u}$, $\vec{v}$ and $\vec{w}$ linearly independent?

ANS: Let c_1, c_2 and c_3 be such that $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = 0$. That is, c_1, c_2 and c_3 satisfy the following equations:

$$\begin{cases} -2c_1 + c_2 - c_3 = 0; \\ c_1 - 2c_2 - c_3 = 0; \\ 2c_1 - c_2 + c_3 = 0. \end{cases}$$

Note the the first equation is identical to the third one. So in fact, we have only two equations with three variables:

$$\begin{cases} c_1 - 2c_2 - c_3 = 0; \\ 2c_1 - c_2 + c_3 = 0. \end{cases}$$

Solve the above equations by elimination method: the second equation plus the first equation yields

$$c_1 - c_2 = 0.$$

The first equation then implies $c_2 = -c_3$. By letting $c_1 = 1$, one gets $c_2 = 1$ and $c_3 = -1$ is a solution for the desired system of linear equations. Hence, $\vec{u} + \vec{v} - \vec{w} = 0$ and they are linearly dependent.

(b) Can any 3-dimensional vector be written as a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$? If yes, please show your reason; if no, please provide one example.

ANS: NO. For example, the cross product of $\vec{u}$ and $\vec{v}$ is not the linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$. That is, $\vec{u} \times \vec{v} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$ can not be written as a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$. Suppose yes, then there are constants c_1, c_2, c_3 such that $\vec{u} \times \vec{v} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$, i.e., the following equation

$$\begin{cases} -2c_1 + c_2 - c_3 = 3; \\ c_1 - 2c_2 - c_3 = 0; \\ 2c_1 - c_2 + c_3 = 3. \end{cases}$$

The first equation plus the third equation yields $0 = 6$ which is not possible for any c_1, c_2 and c_3.

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