[10] 1. Find the determinant by reducing to triangular form for the following matrices.

(a) $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{bmatrix}$.

ANS: We perform the Gaussian Elimination on A by the following process:

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{bmatrix} \to^{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 5 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{bmatrix} \to^{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & -6 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\to^{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 5 \\ 0 & -1 & 2 \\ 0 & 3 & -6 \end{bmatrix} \to^{R_3 \leftarrow R_3 + 3R_2} \begin{bmatrix} 1 & -1 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

Note that matrix U is a triangular matrix and hence determinant of U is 0, the product of its diagonal elements.

Also note that interchange rows only effect the determinant by a sign (i.e., multiply by -1), and operation $R \leftarrow R - cR'$ does not change the determinant, and $R \leftarrow cR$ effect the determinant by a multiplication of c. Hence, $\det(A) = \det(U) = 0$.

(b) $A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & 5 & 3 & 1 \\ -1 & 0 & 2 & -1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$.

ANS: We perform the Gaussian Elimination on A by the following process:

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & 5 & 3 & 1 \\ -1 & 0 & 2 & -1 \\ 3 & 1 & 2 & 0 \end{bmatrix} \to^{R_2 \leftarrow R_2 - 2R_1}_{R_3 \leftarrow R_3 + R_1, R_4 \leftarrow R_4 - 3R_1} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 7 & -1 & 5 \\ 0 & -1 & 4 & -3 \\ 0 & 4 & -4 & 6 \end{bmatrix}$$

$$\to^{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 7 & -1 & 5 \\ 0 & 4 & -4 & 6 \end{bmatrix}$$

$$\to^{R_4 \leftarrow R_4 + 4R_1}_{R_3 \leftarrow R_3 + 7R_1} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 27 & -16 \\ 0 & 0 & 12 & -6 \end{bmatrix} \to^{R_4 \leftarrow R_4/12} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 27 & -16 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

$$\rightarrow_{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 27 & -16 \end{bmatrix} \rightarrow_{R_4 \leftarrow R_4 - 27R_3} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -5/2 \end{bmatrix} = U.$$

Note that matrix U is a triangular matrix and hence determinant of U is $5/2$, the product of its diagonal elements. The determinant of A is 30.

[10] 2. Find the matrix M of minors and the matrix C of cofactors, compute the product AC^T and $C^T A$, calculate the determinant of A, and find A^{-1} if possible, for the following matrices

(a): $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

ANS: The matrix M of minors for matrix A is $M = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, and the matrix C of cofactors for matrix A is $C = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$, and

$$AC^T = C^T A = 5 \cdot I_2.$$

Hence, determinant of A is 5, and the matrix A has inverse to be

$$A^{-1} = \frac{1}{det(A)} C^T = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}.$$

(b): $A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 5 & 2 \\ 4 & 8 & 9 \end{bmatrix}$.

ANS: The matrix M of minors for matrix A is $M = \begin{bmatrix} 29 & 19 & 4 \\ -22 & -29 & -16 \\ -21 & -17 & -11 \end{bmatrix}$,

and the matrix C of cofactors for matrix A is $C = \begin{bmatrix} 29 & -19 & 4 \\ 22 & -29 & 16 \\ -21 & 17 & -11 \end{bmatrix}$, and

$$AC^T = C^T A = -47 \cdot I_3.$$

Hence, determinant of A is -47, and the matrix A does have inverse

$$A^{-1} = \frac{1}{\det(A)} C^T = -\frac{1}{47} \begin{bmatrix} 29 & 22 & -21 \\ -19 & -29 & 17 \\ 4 & 16 & -11 \end{bmatrix}.$$

[10] 3. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 5 \\ 6 & 4 & -1 \end{bmatrix}$.

(a): Find the determinant of A with a Laplace expansion down the first column.

ANS: The $(1,1)$-cofactor is $(-3)\times(-1)-4\times 5 = 3-20 = -17$ and the $(1,3)$-cofactor is $(-1)^{1+3}[2\times 5-0\times(-3)] = 10$. Hence, the determinant of A with a Laplace expansion down the first column is

$$1\times(-17)+6\times 10 = 43.$$

(b): Find the determinant of A with a Laplace expansion down the second row.

ANS: The $(2,2)$-cofactor is $-1\times 1-6\times 0 = -1$, and $(3,2)$-cofactor is $(-1)^{2+3}(1\times 4-2\times 6) = 8$. Hence, the determinant of A with a Laplace expansion down the second row is

$$-1\times(-3)+5\times 8 = 43.$$

[5] 4. Let A and B be 4×4 matrices with $\det(A) = 3$ and $\det(B) = 2$. Find the following determinants:

(a) $\det(B^{-1})$, $\det(A^3)$, and $\det(-4B)$;
ANS: $\det(B^{-1}) = 1/\det(B) = 1/2$.
$\det(A^3) = \det(A)\det(A)\det(A) = [\det(A)]^3 = 27$.
$\det(-4B) = (-4)^4\det(B) = 2\times 4^4 = 512$.

(b) $\det(3B^{-1}A^2B^3A^{-1}B^t)$.
ANS: We have

$$\begin{aligned}\det(3B^{-1}A^2B^3A^{-1}B^t) &= 3^4\det(B^{-1})\det(A^2)\det(B^3)\det(A^{-1})\det(B^t)\\ &= 3^4\det(A)[\det(B)]^3\\ &= 3^5\times 2^3\\ &= 1944.\end{aligned}$$

[5] 5. Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$. Given that $det(A) = -5$, evaluate the following determinant:

$$\begin{vmatrix} a+2x & a+p & 2p+x \\ b+2y & b+q & 2q+y \\ c+2z & c+r & 2r+z \end{vmatrix}.$$

ANS: We have

$$
\begin{aligned}
\det\begin{vmatrix} a+2x & a+p & 2p+x \\ b+2y & b+q & 2q+y \\ c+2z & c+r & 2r+z \end{vmatrix} &= \det\begin{vmatrix} a & a+p & 2p+x \\ b & b+q & 2q+y \\ c & c+r & 2r+z \end{vmatrix} + 2\det\begin{vmatrix} x & a+p & 2p+x \\ y & b+q & 2q+y \\ z & c+r & 2r+z \end{vmatrix} \\
&= \det\begin{vmatrix} a & a & 2p+x \\ b & b & 2q+y \\ c & c & 2r+z \end{vmatrix} + 2\det\begin{vmatrix} x & a & 2p+x \\ y & b & 2q+y \\ z & c & 2r+z \end{vmatrix} \\
&\quad + \det\begin{vmatrix} a & p & 2p+x \\ b & q & 2q+y \\ c & r & 2r+z \end{vmatrix} + 2\det\begin{vmatrix} x & p & 2p+x \\ y & q & 2q+y \\ z & r & 2r+z \end{vmatrix} \\
&= 2\det\begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + 4\det\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \\
&\quad + \det\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} + 2\det\begin{vmatrix} a & p & p \\ b & q & q \\ c & r & r \end{vmatrix} \\
&\quad +2\det\begin{vmatrix} x & p & x \\ y & q & y \\ z & r & z \end{vmatrix} + 4\det\begin{vmatrix} x & p & p \\ y & q & q \\ z & r & r \end{vmatrix} \\
&= 4\det\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} + \det\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \\
&= 4\det\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} + \det A^T \\
&= -4\det\begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix} + \det A \\
&= 4\det\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \det A \\
&= 5\det A \\
&= -25.
\end{aligned}
$$

[5] 6. Calculate the determinant of $\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 6 & 5 & 14 & 8 \\ 13 & 12 & 12 & 13 & 2 \\ 4 & 5 & 8 & 3 & 5 \\ 5 & 10 & 15 & 20 & 0 \end{bmatrix}$.

ANS: Note that the fifth row is five times the first row, therefore the determinant of this matrix is zero (recall that if one row of a matrix is a scalar multiple of another row of the same matrix, then the determinant of this matrix is zero – Property 6 of the Textbook).

Or, you could argument that

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 6 & 5 & 14 & 8 \\ 13 & 12 & 12 & 13 & 2 \\ 4 & 5 & 8 & 3 & 5 \\ 5 & 10 & 15 & 20 & 0 \end{bmatrix} = 5 \times \det\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 6 & 5 & 14 & 8 \\ 13 & 12 & 12 & 13 & 2 \\ 4 & 5 & 8 & 3 & 5 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} = 5 \times 0 = 0. \text{ because if I}$$

have two equal rows in a matrix, the determinant of this matrix is zero.

Another way to solve is using Gaussian elimination, you will come up with an entire row of zeros, therefore, determinant equals to zero.

If you are up to it, Laplace expansion or the definition would also work.

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[45]