Exact Solitary Wave Solutions for a Coupled gKdV-NLS System for Symmetry, Invariants, and their Applications: A Celebration of Peter Olvers 70th Birthday

S. Anco[1], J. Hornick[1], R. Zhao[2], T. Wolf[1]

[1]Department of Mathematics and Statistics
Brock University, St Catharines, Canada
[2]Department of Mathematics
Queen's University, Kingston, Canada

August 5, 2022

Introduction

Coupled KdV-NLS systems

$$\begin{aligned} u_t + \alpha u u_x + \beta u_{xxx} &= \gamma(|\psi|^2)_x \\ i\psi_t + \kappa\psi_{xx} &= \sigma u \psi \end{aligned} \tag{1}$$

$u(x,t)$ is real-valued and $\psi(x,t)$ is complex-valued
$\alpha, \beta, \gamma, \kappa, \sigma$ are real non-zero constants

- electron propagation along deformable molecular chain (Davydov, Gaididei, Zolotaryuk)
- general model of energy transfer in an anharmonic crystal material (Cisneros-Ake & Peláez)
- also models electron propagation coupled to nonlinear ion-acoustic waves in a collisionless plasma
- existence of coherent propagating structures: solitary wave solutions of different shapes and subsonic/supersonic speeds

ψ = electron wave function
u = deformation of molecule/material; ion wave amplitude

Introduction continued

We study a coupled gKdV-NLS system with a power nonlinearity $p > 0$:

$$\begin{aligned} u_t + \alpha u^p u_x + \beta u_{xxx} &= \gamma(|\psi|^2)_x \\ i\psi_t + \kappa\psi_{xx} &= \sigma u\psi \end{aligned} \tag{2}$$

$p = 1$ is KdV case; quadratic nonlinearity
$p = 2$ is mKdV case; cubic nonlinearity
Strong interest in exact solutions which describe frequency-modulated solitary waves

$$u = U(\xi), \quad \psi = e^{i\omega t}\Psi(\xi), \quad \xi = x - ct \tag{3}$$

$c =$ wave speed; $\omega =$ modulated frequency

- some solutions known for $p = 1$ using an ansatz (Cisneros-Ake & Pelaez, Physica D 2017)

Introduction continued

We study a coupled gKdV-NLS system with a power nonlinearity $p > 0$:

$$\begin{aligned} u_t + \alpha u^p u_x + \beta u_{xxx} &= \gamma(|\psi|^2)_x \\ i\psi_t + \kappa\psi_{xx} &= \sigma u\psi \end{aligned} \tag{2}$$

$p = 1$ is KdV case; quadratic nonlinearity
$p = 2$ is mKdV case; cubic nonlinearity
Strong interest in exact solutions which describe frequency-modulated solitary waves

$$u = U(\xi), \quad \psi = e^{i\omega t}\Psi(\xi), \quad \xi = x - ct \tag{3}$$

$c =$ wave speed; $\omega =$ modulated frequency

- some solutions known for $p = 1$ using an ansatz (Cisneros-Ake & Pelaez, Physica D 2017)
- no exact solutions found to-date for $p = 2$ (Cisneros-Ake et al, Physics Letters A 2018)

Introduction continued

We study a coupled gKdV-NLS system with a power nonlinearity $p > 0$:

$$\begin{aligned} u_t + \alpha u^p u_x + \beta u_{xxx} &= \gamma(|\psi|^2)_x \\ i\psi_t + \kappa\psi_{xx} &= \sigma u\psi \end{aligned} \tag{2}$$

$p = 1$ is KdV case; quadratic nonlinearity
$p = 2$ is mKdV case; cubic nonlinearity
Strong interest in exact solutions which describe frequency-modulated solitary waves

$$u = U(\xi), \quad \psi = e^{i\omega t}\Psi(\xi), \quad \xi = x - ct \tag{3}$$

c = wave speed; ω = modulated frequency

- some solutions known for $p = 1$ using an ansatz (Cisneros-Ake & Pelaez, Physica D 2017)
- no exact solutions found to-date for $p = 2$ (Cisneros-Ake et al, Physics Letters A 2018)
- nothing known for higher nonlinearities $p \geq 3$

Summary of our results

- Use new, more powerful method to solve travelling wave ODE system

Summary of our results

- Use new, more powerful method to solve travelling wave ODE system
- Derive exact solutions for $p = 1, 2, 3, 4$

Summary of our results

- Use new, more powerful method to solve travelling wave ODE system
- Derive exact solutions for $p = 1, 2, 3, 4$
- Solitary waves exhibit a wide range of features: bright and dark peaks; single-peaked and multi-peaked; zero and non-zero backgrounds

Dimensionless form

Convenient to work with dimensionless form of the gKdV-NLS system:

$$u_t + s_1 u^p u_x + u_{xxx} + s_2(|\psi|^2)_x = 0 \tag{4}$$

$$i\psi_t + \psi_{xx} + ku\psi = 0 \tag{5}$$

where

$$s_1 = \begin{cases} 1, & p = \text{odd} \\ \pm 1, & p = \text{even} \end{cases}, \quad s_2 = \begin{cases} \pm 1, & p = \text{odd} \\ 1, & p = \text{even} \end{cases}, \quad k = \text{const.} \neq 0 \tag{6}$$

ODE system for frequency-modulated travelling waves

$$u = U(\xi), \quad \psi = e^{i\omega t}\Psi(\xi), \quad \xi = x - ct$$

Substitute into the coupled equations (4)–(5) $\Rightarrow$ nonlinear ODE system

$$\begin{aligned} U''' + (s_1 U^p - c)U' + s_2(|\Psi|^2)' &= 0 \\ \Psi'' - ic\Psi' + (kU - w)\Psi &= 0 \end{aligned} \tag{7}$$

Go to amplitude-phase variables

$$\Psi = A\exp(i\Phi)$$

which yields

$$U''' + (s_1 U^p - c)U' + 2s_2 AA' = 0 \tag{8a}$$

$$A'' + (kU + c\Phi' - \Phi'^2 - w)A = 0 \tag{8b}$$

$$A\Phi'' + (2\Phi' - c)A' = 0 \tag{8c}$$

ODE system for frequency-modulated travelling waves cont'ed

- one third-order ODE (8a) and two second-order ODEs (8b) and (8c)

ODE system for frequency-modulated travelling waves cont'ed

- one third-order ODE (8a) and two second-order ODEs (8b) and (8c)
- general solution requires a total of 7 integrations

ODE system for frequency-modulated travelling waves cont'ed

- one third-order ODE (8a) and two second-order ODEs (8b) and (8c)
- general solution requires a total of 7 integrations
- inspection suggests that only a few explicit integrations are possible by basic methods (Cisneros-Ake & Pelaez, Physica D 2017; Cisneros-Ake et al, Physics Letters A 2018)

ODE system for frequency-modulated travelling waves cont'ed

- one third-order ODE (8a) and two second-order ODEs (8b) and (8c)
- general solution requires a total of 7 integrations
- inspection suggests that only a few explicit integrations are possible by basic methods (Cisneros-Ake & Pelaez, Physica D 2017; Cisneros-Ake et al, Physics Letters A 2018)
- Basic method consists of looking for integrating factors by inspection.

ODE system for frequency-modulated travelling waves cont'ed

- one third-order ODE (8a) and two second-order ODEs (8b) and (8c)
- general solution requires a total of 7 integrations
- inspection suggests that only a few explicit integrations are possible by basic methods (Cisneros-Ake & Pelaez, Physica D 2017; Cisneros-Ake et al, Physics Letters A 2018)
- Basic method consists of looking for integrating factors by inspection.
- Can miss complicated/unexpected forms for integrating factors

ODE system for frequency-modulated travelling waves cont'ed

- one third-order ODE (8a) and two second-order ODEs (8b) and (8c)
- general solution requires a total of 7 integrations
- inspection suggests that only a few explicit integrations are possible by basic methods (Cisneros-Ake & Pelaez, Physica D 2017; Cisneros-Ake et al, Physics Letters A 2018)
- Basic method consists of looking for integrating factors by inspection.
- Can miss complicated/unexpected forms for integrating factors
- Difficult to handle general nonlinearity powers $p \neq 1$

New method

Main steps:

- obtain first integrals by use of multi-reduction symmetry theory (Olver 1986, Anco & Gandarias, CNSNS 2020);

New method

Main steps:

- obtain first integrals by use of multi-reduction symmetry theory (Olver 1986, Anco & Gandarias, CNSNS 2020);
- apply a hodograph transformation which leads to a triangular decoupled system;

New method

Main steps:

- obtain first integrals by use of multi-reduction symmetry theory (Olver 1986, Anco & Gandarias, CNSNS 2020);
- apply a hodograph transformation which leads to a triangular decoupled system;
- introduce a power-balance ansatze for solutions of the base ODE;

New method

Main steps:

- obtain first integrals by use of multi-reduction symmetry theory (Olver 1986, Anco & Gandarias, CNSNS 2020);
- apply a hodograph transformation which leads to a triangular decoupled system;
- introduce a power-balance ansatze for solutions of the base ODE;
- characterize conditions under which solutions yield solitary waves;

New method

Main steps:

- obtain first integrals by use of multi-reduction symmetry theory (Olver 1986, Anco & Gandarias, CNSNS 2020);
- apply a hodograph transformation which leads to a triangular decoupled system;
- introduce a power-balance ansatze for solutions of the base ODE;
- characterize conditions under which solutions yield solitary waves;
- solve an algebraic system for the coefficients in the ansatz under those conditions

Step (i) — symmetry multi-reduction I

Symmetry characterizing the form of frequency-modulated travelling waves

$$x \to x + c\epsilon, \quad t \to t + \epsilon, \quad \psi \to e^{i\omega\epsilon}\psi \quad \Leftrightarrow \quad \partial_t + c\partial_x + i\omega\psi\partial_\psi - i\omega\bar{\psi}\partial_{\bar{\psi}}$$

Step (i) — symmetry multi-reduction I

Symmetry characterizing the form of frequency-modulated travelling waves

$$x \to x + c\epsilon, \quad t \to t + \epsilon, \quad \psi \to e^{i\omega\epsilon}\psi \quad \Leftrightarrow \quad \partial_t + c\partial_x + i\omega\psi\partial_\psi - i\omega\bar{\psi}\partial_{\bar{\psi}}$$

Galilean transformation combined with a phase rotation invariants $U = u$, $\Psi = e^{-i\omega t}\psi$, $\xi = x - ct$

Step (i) — symmetry multi-reduction I

Symmetry characterizing the form of frequency-modulated travelling waves

$$x \to x + c\epsilon, \quad t \to t + \epsilon, \quad \psi \to e^{i\omega\epsilon}\psi \quad \Leftrightarrow \quad \partial_t + c\partial_x + i\omega\psi\partial_\psi - i\omega\bar{\psi}\partial_{\bar{\psi}}$$

Galilean transformation combined with a phase rotation invariants $U = u$, $\Psi = e^{-i\omega t}\psi$, $\xi = x - ct$

Find all conservation laws that are invariant under this symmetry

Step (i) — symmetry multi-reduction I

Symmetry characterizing the form of frequency-modulated travelling waves

$$x \to x + c\epsilon, \quad t \to t + \epsilon, \quad \psi \to e^{i\omega\epsilon}\psi \quad \Leftrightarrow \quad \partial_t + c\partial_x + i\omega\psi\partial_\psi - i\omega\bar{\psi}\partial_{\bar{\psi}}$$

Galilean transformation combined with a phase rotation invariants $U = u$, $\Psi = e^{-i\omega t}\psi$, $\xi = x - ct$

Find all conservation laws that are invariant under this symmetry

Carried out systematically by method of symmetry multi-reduction

Step (i) — symmetry multi-reduction II

Yields four conserved integrals:

$$M = \int_{\mathbb{R}} u\, dx \quad \text{mass}$$

$$J = \int_{\mathbb{R}} \tfrac{1}{2}|\psi|^2\, dx \quad \text{charge}$$

$$E = \int_{\mathbb{R}} \tfrac{1}{2}u^2 - (s_2/k)|\psi|^2 \arg(\psi)_x\, dx \quad \text{elastic energy}$$

$$H = \int_{\mathbb{R}} \tfrac{1}{2}u_x^2 - s_1 \tfrac{1}{(p+1)(p+2)} u^{p+2} + (s_2/k)|\psi_x|^2 - s_2 u|\psi|^2\, dx \text{ energy}$$

Inherited as first integrals of the ODE system; only three are functionally independent

Step (i) — symmetry multi-reduction II

$\Rightarrow$ ODE system is reduced to one second-order ODE and two first-order ODEs

$$U'' = cU - s_1 \tfrac{1}{p+1} U^{p+1} - s_2 A^2 + C_1 \tag{9a}$$

$$\tfrac{1}{2} k U'^2 + s_2 A'^2 = k C_1 U + \tfrac{1}{2} k c U^2 - s_1 \tfrac{1}{(p+2)(p+1)} k U^{p+2} - s_2 k U A^2 - s_2 (\tfrac{1}{4} c^2 + \omega) A^2 - s_2 {C_2}^2 A^{-2} - C_3 \tag{9b}$$

$$\Phi' = \tfrac{1}{2} c + C_2 A^{-2} \tag{9c}$$

C_1, C_2, C_3 are free constants.

$\Rightarrow$ General solution now requires 4 integrations

Step (ii) — decoupling into triangular form I

Apply hodograph transformation $A = F(U)$ which yields

$$U'' = cU - s_1 \tfrac{1}{p+1} U^{p+1} - s_2 F(U)^2 + C_1 \tag{10a}$$

$$U'^2 = \frac{1}{F(U)^2(\frac{1}{2}k + s_2 F'(U)^2)} \Big(\big(kC_1 U + \tfrac{1}{2}kcU^2 - \tfrac{s_1 k}{(p+2)(p+1)} U^{p+2} - s_2((\tfrac{1}{4}c^2 + \omega) + kU)F(U)^2 - C_3\big) F(U)^2 - s_2 {C_2}^2 \Big) \tag{10b}$$

$$\Phi' = \tfrac{1}{2}c + C_2 F(U)^{-2} \tag{10c}$$

Step (ii) — decoupling into triangular form I

Apply hodograph transformation $A = F(U)$ which yields

$$U'' = cU - s_1 \tfrac{1}{p+1} U^{p+1} - s_2 F(U)^2 + C_1 \tag{10a}$$

$$U'^2 = \frac{1}{F(U)^2(\frac{1}{2}k + s_2 F'(U)^2)}\Big(\big(kC_1 U + \tfrac{1}{2}kcU^2 - \tfrac{s_1 k}{(p+2)(p+1)} U^{p+2} - s_2((\tfrac{1}{4}c^2 + \omega) + kU)F(U)^2 - C_3\big)F(U)^2 - s_2 {C_2}^2\Big) \tag{10b}$$

$$\Phi' = \tfrac{1}{2}c + C_2 F(U)^{-2} \tag{10c}$$

Second-order ODE (10a) is nonlinear oscillator equation

Step (ii) — decoupling into triangular form I

Apply hodograph transformation $A = F(U)$ which yields

$$U'' = cU - s_1 \tfrac{1}{p+1} U^{p+1} - s_2 F(U)^2 + C_1 \tag{10a}$$

$$U'^2 = \frac{1}{F(U)^2(\frac{1}{2}k + s_2 F'(U)^2)} \Big(\big(kC_1 U + \tfrac{1}{2}kcU^2 - \tfrac{s_1 k}{(p+2)(p+1)} U^{p+2}$$
$$- s_2((\tfrac{1}{4}c^2 + \omega) + kU)F(U)^2 - C_3\big) F(U)^2 - s_2 {C_2}^2 \Big) \tag{10b}$$

$$\Phi' = \tfrac{1}{2}c + C_2 F(U)^{-2} \tag{10c}$$

Second-order ODE (10a) is nonlinear oscillator equation
It possesses the first integral

$$U'^2 = -s_2 2G(U) - s_1 \tfrac{2}{(p+1)(p+2)} U^{p+2} + cU^2 + 2C_1 U + 2C_4,$$
$$G(U) = \textstyle\int F(U)^2 \, dU, \qquad C_4 \text{ is a free constant}$$

Step (ii) — decoupling into triangular form I

Apply hodograph transformation $A = F(U)$ which yields

$$U'' = cU - s_1 \tfrac{1}{p+1} U^{p+1} - s_2 F(U)^2 + C_1 \quad (10a)$$

$$U'^2 = \frac{1}{F(U)^2(\frac{1}{2}k + s_2 F'(U)^2)} \Big(\big(kC_1 U + \tfrac{1}{2}kcU^2 - \tfrac{s_1 k}{(p+2)(p+1)} U^{p+2} - s_2((\tfrac{1}{4}c^2 + \omega) + kU)F(U)^2 - C_3\big) F(U)^2 - s_2 {C_2}^2 \Big) \quad (10b)$$

$$\Phi' = \tfrac{1}{2}c + C_2 F(U)^{-2} \quad (10c)$$

Second-order ODE (10a) is nonlinear oscillator equation
It possesses the first integral

$$U'^2 = -s_2 2G(U) - s_1 \tfrac{2}{(p+1)(p+2)} U^{p+2} + cU^2 + 2C_1 U + 2C_4,$$

$$G(U) = \int F(U)^2\, dU, \qquad C_4 \text{ is a free constant}$$

Substitute into ODE (10b) and clear the denominator

Step (ii) — decoupling into triangular form II

⇒ Result is a second-order ODE for $G(U)$

$$\frac{1}{2}\left(s_2 G(U) + s_1 \frac{1}{(p+1)(p+2)} U^{p+2} - \frac{1}{2} c U^2 - C_1 U - \tilde{C}_4\right) G''(U)^2 =$$
$$(kU + \tfrac{1}{4} c^2 + \omega) G'(U)^2 - \left(kG(U) - s_2(\tilde{C}_3 + k\tilde{C}_4)\right) G'(U) + {C_2}^2 \tag{11}$$

Step (ii) — decoupling into triangular form II

$\Rightarrow$ Result is a second-order ODE for $G(U)$

$$\frac{1}{2}\Big(s_2 G(U) + s_1 \tfrac{1}{(p+1)(p+2)} U^{p+2} - \tfrac{1}{2} cU^2 - C_1 U - \tilde{C}_4\Big) G''(U)^2 =$$
$$(kU + \tfrac{1}{4}c^2 + \omega) G'(U)^2 - \Big(kG(U) - s_2(\tilde{C}_3 + k\tilde{C}_4)\Big) G'(U) + {C_2}^2 \quad (11)$$

Theorem Any solution $G(U)$ of (11) yields a solution $(U(\xi), A(\xi), \Phi(\xi))$ of the reduced ODE system (9a)–(9c) by quadratures. Hence, a frequency-modulated travelling wave is obtained for gKdV-NLS system.

Step (ii) — decoupling into triangular form II

$\Rightarrow$ Result is a second-order ODE for $G(U)$

$$\frac{1}{2}\Big(s_2 G(U) + s_1 \frac{1}{(p+1)(p+2)} U^{p+2} - \frac{1}{2} c U^2 - C_1 U - \tilde{C}_4\Big) G''(U)^2 =$$
$$(kU + \frac{1}{4}c^2 + \omega) G'(U)^2 - \Big(kG(U) - s_2(\tilde{C}_3 + k\tilde{C}_4)\Big) G'(U) + {C_2}^2 \tag{11}$$

Theorem Any solution $G(U)$ of (11) yields a solution $(U(\xi), A(\xi), \Phi(\xi))$ of the reduced ODE system (9a)–(9c) by quadratures. Hence, a frequency-modulated travelling wave is obtained for gKdV-NLS system.

$\Rightarrow$ *Original ODE system (requiring 7 integrations) has been reduced to a single second-order ODE (requiring only 2 integrations)*

Step (ii) — decoupling into triangular form II

$\Rightarrow$ Result is a second-order ODE for $G(U)$

$$\tfrac{1}{2}\Big(s_2 G(U) + s_1 \tfrac{1}{(p+1)(p+2)} U^{p+2} - \tfrac{1}{2} c U^2 - C_1 U - \tilde{C}_4\Big) G''(U)^2 =$$
$$(kU + \tfrac{1}{4}c^2 + \omega) G'(U)^2 - \Big(kG(U) - s_2(\tilde{C}_3 + k\tilde{C}_4)\Big) G'(U) + {C_2}^2 \tag{11}$$

Theorem Any solution $G(U)$ of (11) yields a solution $(U(\xi), A(\xi), \Phi(\xi))$ of the reduced ODE system (9a)–(9c) by quadratures. Hence, a frequency-modulated travelling wave is obtained for gKdV-NLS system.

$\Rightarrow$ *Original ODE system (requiring 7 integrations) has been reduced to a single second-order ODE (requiring only 2 integrations)*

Observation: ODE (11) is polynomial in G'', G', G, and U

Step (iii) — power-balance ansatz for solutions

Consider $G(U)$ to be a polynomial in U whose degree is determined by balancing powers in the ODE (11)
$\Rightarrow$

$$G(U) = -s_1 s_2 \frac{1}{(p+1)(p+2)} U^{p+2} + g_3 U^3 + g_2 U^2 + g_1 U + g_0 \quad (12)$$

g_0, g_1, g_2, g_3 are constant parameters which will be determined

Steps (iv) and (v) — overdetermined algebraic system

Substitution of $G(U)$
$\Rightarrow$ several exponents like $p-2, p-1, p, p+1, p+2, ..$

Steps (iv) and (v) — overdetermined algebraic system

Substitution of $G(U)$
$\Rightarrow$ several exponents like $p-2, p-1, p, p+1, p+2, ..$
$\Rightarrow$ many combinations of terms that might cancel each other

Steps (iv) and (v) — overdetermined algebraic system

Substitution of $G(U)$
$\Rightarrow$ several exponents like $p-2, p-1, p, p+1, p+2, ..$
$\Rightarrow$ many combinations of terms that might cancel each other
$\Rightarrow$ CA package CRACK generating these cases each resulting in an overdetermined algebraic system for g_0, g_1, g_2, g_3 and p.

Steps (iv) and (v) — overdetermined algebraic system

Substitution of $G(U)$
$\Rightarrow$ several exponents like $p-2, p-1, p, p+1, p+2, ..$
$\Rightarrow$ many combinations of terms that might cancel each other
$\Rightarrow$ CA package CRACK generating these cases each resulting in an overdetermined algebraic system for g_0, g_1, g_2, g_3 and p.

Solve by standard methods (CRACK and again using Maple command RIFSIMP via case splitting).

Steps (iv) and (v) — overdetermined algebraic system

Substitution of $G(U)$
$\Rightarrow$ several exponents like $p-2, p-1, p, p+1, p+2, ..$
$\Rightarrow$ many combinations of terms that might cancel each other
$\Rightarrow$ CA package CRACK generating these cases each resulting in an overdetermined algebraic system for g_0, g_1, g_2, g_3 and p.

Solve by standard methods (CRACK and again using Maple command RIFSIMP via case splitting).

Impose condition that $U(\xi)$ has asymptotic exponential decay as $|\xi| \to \infty$ (so that it will describe a solitary wave)

Solitary waves

Theorem Solutions exist for $p = 1, 2, 3, 4$ and take the form

$$U(\xi) = b + s_3 h \operatorname{sech}^2\left(\sqrt{gh/2}\,\xi\right)$$

$$A(\xi) = \sqrt{g_1 + 2g_2 U(\xi) + 3g_3 U(\xi)^2 - s_1 s_2 \tfrac{1}{p+1} U(\xi)^{p+1}}$$

$$\Phi(\xi) = \frac{c}{2}\xi + C_1 \int_0^{\xi} \frac{d\xi}{A(\xi)^2} + \phi$$

where

$$s_3 = s_2 \operatorname{sgn}(g_3) = \pm 1, \quad g = |g_3|, \quad b = -\tfrac{1}{3}(s_3 h + (g_2 - s_2 \tfrac{1}{2} c)/g_3),$$

$$h = \tfrac{1}{2}\sqrt{d}/g, \quad d = (c - s_2 2 g_2)^2 + 12 g_3 (s_2 C_1 - g_1) > 0$$

Solitary waves

Theorem Solutions exist for $p = 1, 2, 3, 4$ and take the form

$$U(\xi) = b + s_3 h \operatorname{sech}^2\big(\sqrt{gh/2}\,\xi\big)$$

$$A(\xi) = \sqrt{g_1 + 2g_2 U(\xi) + 3g_3 U(\xi)^2 - s_1 s_2 \tfrac{1}{p+1} U(\xi)^{p+1}}$$

$$\Phi(\xi) = \frac{c}{2}\xi + C_1 \int_0^{\xi} \frac{d\xi}{A(\xi)^2} + \phi$$

where

$$s_3 = s_2 \operatorname{sgn}(g_3) = \pm 1, \quad g = |g_3|, \quad b = -\tfrac{1}{3}(s_3 h + (g_2 - s_2 \tfrac{1}{2} c)/g_3),$$
$$h = \tfrac{1}{2}\sqrt{d}/g, \quad d = (c - s_2 2 g_2)^2 + 12 g_3 (s_2 C_1 - g_1) > 0$$

- wave profile U is similar to a KdV soliton

Solitary waves

Theorem Solutions exist for $p = 1, 2, 3, 4$ and take the form

$$U(\xi) = b + s_3 h\,\mathrm{sech}^2\left(\sqrt{gh/2}\,\xi\right)$$

$$A(\xi) = \sqrt{g_1 + 2g_2 U(\xi) + 3g_3 U(\xi)^2 - s_1 s_2 \tfrac{1}{p+1} U(\xi)^{p+1}}$$

$$\Phi(\xi) = \frac{c}{2}\xi + C_1 \int_0^{\xi} \frac{d\xi}{A(\xi)^2} + \phi$$

where

$$s_3 = s_2\,\mathrm{sgn}(g_3) = \pm 1, \quad g = |g_3|, \quad b = -\tfrac{1}{3}(s_3 h + (g_2 - s_2 \tfrac{1}{2} c)/g_3),$$
$$h = \tfrac{1}{2}\sqrt{d}/g, \quad d = (c - s_2 2g_2)^2 + 12 g_3 (s_2 C_1 - g_1) > 0$$

- wave profile U is similar to a KdV soliton
- profile for A can have several different shapes depending on p and (k, c, ω, b, h) and the signs s_3, s_2, s_1

Solitary waves

Theorem Solutions exist for $p = 1, 2, 3, 4$ and take the form

$$U(\xi) = b + s_3 h \operatorname{sech}^2\left(\sqrt{gh/2}\,\xi\right)$$

$$A(\xi) = \sqrt{g_1 + 2g_2 U(\xi) + 3g_3 U(\xi)^2 - s_1 s_2 \tfrac{1}{p+1} U(\xi)^{p+1}}$$

$$\Phi(\xi) = \frac{c}{2}\xi + C_1 \int_0^{\xi} \frac{d\xi}{A(\xi)^2} + \phi$$

where

$$s_3 = s_2 \operatorname{sgn}(g_3) = \pm 1, \quad g = |g_3|, \quad b = -\tfrac{1}{3}(s_3 h + (g_2 - s_2 \tfrac{1}{2} c)/g_3),$$
$$h = \tfrac{1}{2}\sqrt{d}/g, \quad d = (c - s_2 2g_2)^2 + 12 g_3 (s_2 C_1 - g_1) > 0$$

- wave profile U is similar to a KdV soliton
- profile for A can have several different shapes depending on p and (k, c, ω, b, h) and the signs s_3, s_2, s_1
- parameters obey a set of equations and inequalities, which differ for each case $p = 1, 2, 3, 4$

Concluding remarks

- *Large multi-parameter families of solitary waves are obtained*

Concluding remarks

- *Large multi-parameter families of solitary waves are obtained*
- Investigation of properties is underway

Concluding remarks

- *Large multi-parameter families of solitary waves are obtained*
- Investigation of properties is underway
- Applications to electron propagation along molecular chains and to coupling of electrons to ion waves in plasmas

Concluding remarks

- *Large multi-parameter families of solitary waves are obtained*
- Investigation of properties is underway
- Applications to electron propagation along molecular chains and to coupling of electrons to ion waves in plasmas
- New method is applicable to many other systems with power nonlinearities

The End

Happy Birthday Peter
with many Returns!