Introduction to Moving Frames à la Cartan

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Outline

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Origin of Curvature for Planar Curves

Let $\gamma : \mathbb{R} \to \mathbb{E}^2$ be a curve in the Euclidean plane, acted on by group $ASO(2)$ of rotations and translations. WLOG, assume it's a graph

$$\gamma(x) = \begin{bmatrix} x \\ f(x) \end{bmatrix}$$

We use the group to normalize Taylor series of γ at $x = x_0$, by moving it into a 'standard position'.

- first, translate to origin

$$\tilde{\gamma}(x) = \begin{bmatrix} x - x_0 \\ f(x) - f(x_0) \end{bmatrix}$$

- apply a unique rotation so $\tilde{\gamma}'(x_0)$ points along the positive x-axis

$$\tilde{\gamma}(x) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} x - x_0 \\ f(x) - f(x_0) \end{bmatrix}, \qquad \theta = \arctan f'(x_0).$$

- group motions are used up, so *remaining Taylor series coefficients are geometric invariants*

Regard this 'standardized' curve as a new graph

$$\begin{bmatrix} \overline{x} \\ \overline{f}(\overline{x}) \end{bmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} x - x_0 \\ f(x) - f(x_0) \end{bmatrix}.$$

Using implicit diff, find Taylor series coeffs at $\overline{x} = 0$:

$$\overline{f}(0) = 0, \quad \overline{f}'(0) = 0, \quad \overline{f}''(0) = \frac{f''(x_0)}{(1 + f'(x_0)^2)^{3/2}}, \quad \ldots$$

- second derivative gives curvature of original curve at $(x_0, f(x_0))$.

Is the curvature function $\kappa(x) = \dfrac{f''(x)}{(1 + f'(x)^2)^3/2}$ enough to identify two congruent planar curves?

Focus on the mapping into the group!

For convenience, write the action of $G = ASO(2)$ on $\mathbb{R}^2$ in linear form:

$$\begin{pmatrix} 1 & 0 \\ \mathbf{b} & A \end{pmatrix} \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 \\ A\mathbf{x} + \mathbf{b} \end{bmatrix}, \qquad \mathbf{b}, \mathbf{x} \in \mathbb{R}^2, A \in SO(2).$$

Consider the *inverse* transformation, moving a graph with horizontal tangent at the origin back to $(x, f(x))$:

$$\mathbf{b} = \begin{bmatrix} x \\ f(x) \end{bmatrix}, \quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta(x) = \arctan f'(x).$$

Regard this as a mapping $x \mapsto g \in G$, and apply left-invt 1-forms to compute its derivative in the Lie algebra $\mathfrak{g}$:

$$g^{-1}\frac{dg}{dx} = \sec\theta \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\kappa(x) \\ 0 & \kappa(x) & 0 \end{pmatrix}$$

Reparametrize the mapping by arclength $s = \int \sec\theta \, dx$. Then two curves γ_1, γ_2 are congruent via action of G if and only if their curvature functions match (up to shift in s) as functions of arclength:

$$\kappa_1(s+c) = \kappa_2(s), \qquad c \in \mathbb{R}.$$

Why's it called a moving frame?

En français: *repére mobile*

Let $x \to g(x)$ be our mapping into $ASO(2)$, and let $\mathbf{i}, \mathbf{j}$ be unit vectors at the origin $\mathbf{0}$, tangent to the axes. Then

$$\mathbf{e}_1(x) = g(x)_* \mathbf{i}, \quad \mathbf{e}_2(x) = g(x)_* \mathbf{j}$$

gives a 1-parameter family of orthonormal pairs $(\mathbf{e}_1, \mathbf{e}_2)$, comprising a basis (or 'frame') for the tangent space to $\mathbb{R}^2$ at $\gamma(x) = g(x)\mathbf{0}$ which 'moves' so that $\mathbf{e}_1(x)$ always tangent to the curve.

In this way, g can be identified with a lift *of γ into the frame bundle of $\mathbb{E}^2$.*

The General Linear Frame Bundle of $\mathbb{R}^n$

Let $\mathcal{F}_{\mathbb{R}^n} = \{(\mathbf{b}, \mathbf{e}_1, \ldots, \mathbf{e}_n)\}$ where $\mathbf{b}$ is a point in $\mathbb{E}^n$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is any basis for $T_{\mathbf{b}}\mathbb{R}^n$. Thus, $\mathcal{F}_{\mathbb{R}^n}$ is a bundle over $\mathbb{R}^n$ with basepoint map

$$\pi : (\mathbf{b}, \mathbf{e}_1, \ldots, \mathbf{e}_n) \mapsto \mathbf{b}$$

and fiber $GL(n)$. Regard $\mathbf{b}, \mathbf{e}_i$ as $\mathbb{R}^n$-valued fns on $\mathcal{F}$, and resolve their exterior derivatives in terms of the basis $\{\mathbf{e}_i\}$:

$$d\mathbf{b} = \mathbf{e}_i \otimes \omega^i, \quad d\mathbf{e}_j = \mathbf{e}_i \otimes \varphi^i_j,$$

defining n canonical 1-forms ω^i and n^2 connection 1-forms φ^i_j on $\mathcal{F}$. (Drop tensor signs from now on.) These independent 1-forms satisfy *structure equations*

$$d\omega^i = -\phi^i_j \wedge \omega^j, \qquad d\varphi^i_j = -\varphi^i_k \wedge \varphi^k_j.$$

The Euclidean frame bundle $\mathcal{F}_{\mathbb{E}^n}$ is the sub-bundle where bases $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ are oriented and orthonormal. The same structure equations hold, but only $\binom{n}{2}$ of the connection forms are independent, since $\varphi^i_j = -\varphi^j_i$ and the fiber is $SO(n)$.

(Likewise, any Riemannian manifold has an orthonormal frame bundle with canonical & connection forms, but the 2nd structure equation also has a curvature term.)

Moving Frame as Lift into the Frame Bundle

When $n = 2$, $\dim \mathcal{F}_{\mathbb{E}^2} = 3$, with a global coframe $(\omega^1, \omega^2, \varphi^2_1)$ defined by

$$d\mathbf{b} = \mathbf{e}_1\omega^1 + \mathbf{e}_2\omega^2, \qquad d\mathbf{e}_1 = \mathbf{e}_2\varphi^2_1, \qquad d\mathbf{e}_2 = \mathbf{e}_1\varphi^1_2 = -\mathbf{e}_1\varphi^2_1.$$

For $\gamma : \mathbb{R} \to \mathbb{E}^2$ as before, the map $x \mapsto g(x)$ gives a *lift* $\Gamma : \mathbb{R} \to \mathcal{F}_{\mathbb{E}^2}$ with

$$\mathbf{b} = \gamma(x), \qquad \mathbf{e}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Differentiating and comparing with our calculation of $g^{-1}dg/dx$ shows that this lift satisfies (and is uniquely characterized by)

$$\Gamma^*\omega^1 = ds, \quad \Gamma^*\omega^2 = 0, \quad \Gamma^*\varphi^2_1 = \kappa\, ds.$$

Cartan's method of moving frame consists of using the freedom of the group to simplify as much as possible the values of the pullbacks of the canonical and connection 1-forms with the goal of obtaining a unique lift into the frame bundle.

Overview

- Next, want to illustrate Cartan's method of moving frames in the setting of a larger transformation group, still acting on planar curves
- proceed by requiring more and more specific adaptations of the moving frame, with goal of identifying differential invariants
- at each step, all such frames are sections of a reduction $\mathcal{F}_k$ of the general frame bundle to smaller fiber group
- keep track of which canonical and connection forms are zero on $\mathcal{F}_k$, and which are semibasic (i.e., zero on vertical vectors)
- each semibasic is a multiple of some canonical form(s) on $\mathcal{F}_k$, and the fiber group acts on these coefficients
- if action is trivial, get an invariant function on the base; if not, we use the group to normalize, leading to further reductions

The (Equi)affine Plane

Let $ASL(2)$ be the group of (equi)affine transformations of $\mathbb{R}^2$:

$$\begin{pmatrix} 1 & 0 \\ \mathbf{b} & A \end{pmatrix} \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 \\ A\mathbf{x} + \mathbf{b} \end{bmatrix}, \qquad \mathbf{x}, \mathbf{b} \in \mathbb{R}^2, A \in SL(2, \mathbb{R}).$$

By acting on the standard basis at the origin, we again identify $ASL(2)$ with a sub-bundle $\mathcal{F}_{\mathbb{A}^2} \subset \mathcal{F}_{\mathbb{R}^2}$,

$$\mathbf{b} = \text{basepoint}, \quad \mathbf{e}_1 = A_1 \text{ (first column of } A\text{)}, \quad \mathbf{e}_2 = A_2.$$

Because $\det A = 1$, on this sub-bundle the connection forms satisfy

$$\varphi_1^1 + \varphi_2^2 = 0$$

and the fiber group is 3-dimensional $SL(2, \mathbb{R})$.

Given a regular immersion $\gamma : \mathbb{R} \to \mathbb{R}^2$, can we define a unique lift into $\mathcal{F}_{\mathbb{A}^2}$?

What are necessary and sufficient conditions for two such curves to be congruent by the $ASL(2)$ action?

First Adaptation

Given regular γ, let $\mathcal{F}_0 = \gamma^* \mathcal{F}_{\mathbb{A}^2}$ (i.e., frames with basepoint on γ).
Define a sub-bundle $\mathcal{F}_1 \subset \mathcal{F}_0$ consisting of frames such that $\mathbf{e}_1$ is tangent to γ.
Since on $\mathcal{F}_0$

$$d\mathbf{b} = \mathbf{e}_1 \omega^1 + \mathbf{e}_2 \omega^2$$

then ω^2 pulls back to be zero on $\mathcal{F}_1$.

- fiber of $\mathcal{F}_1$ is 2-dimensional, isomorphic to an upper triangular subgroup $G_1 \subset SL(2, \mathbb{R})$ which acts on the fiber by

$$\mathbf{e}_1 \mapsto a\mathbf{e}_1, \quad \mathbf{e}_2 \mapsto a^{-1}\mathbf{e}_2 + b\mathbf{e}_1, \qquad a, b \in \mathbb{R}, a \neq 0.$$

(For $g \in G_1$, let R_g denote this action.)
Thus, the *direction* of $\mathbf{e}_1$ is fixed as we move along the fiber. Since

$$d\mathbf{e}_1 = \mathbf{e}_1 \varphi_1^1 + \mathbf{e}_2 \varphi_1^2$$

this means that on $\mathcal{F}_1$, φ_1^2 is semibasic (i.e., zero on vectors tangent to the fiber).
Hence there is a function u on $\mathcal{F}_1$ such that

$$\varphi_1^2 = u\omega^1.$$

How does this function vary along the fiber?

From above,

$$R_g^* \mathbf{e}_1 = a\mathbf{e}_1.$$

For a fixed a, b, differentiate both sides and sub in for $d\mathbf{e}_1$ and $d\mathbf{e}_2$.

$$a\mathbf{e}_1 R_g^* \varphi_1^1 + (a^{-1}\mathbf{e}_2 + b\mathbf{e}_1) R_g^* \varphi_1^2 = a(\mathbf{e}_1\varphi_1^1 + \mathbf{e}_2\varphi_1^2)$$

Equating $\mathbf{e}_2$ coefficients implies

$$R_g^* \varphi_1^2 = a^2 \varphi_1^2.$$

On the other hand, since $d\mathbf{b} = \omega^1 \mathbf{e}_1$ and the basepoint $\mathbf{b}$ is fixed by R_g, then

$$R_g^* \omega^1 = a^{-1}\omega^1.$$

Since $\varphi_1^2 = u\omega^1$, then

$$R_g^* u = a^3 u.$$

<u>Remark</u> We can get the same information by differentiating $\varphi_1^2 = u\omega^1$. Using the structure equations (and $\varphi_1^1 + \varphi_2^2 = 0$ and $\omega^2 = 0$) yields

$$(du - 3u\omega_1^1) \wedge \omega^1 = 0$$

i.e., $d\log|u| \equiv 3\omega_1^1$ modulo semibasic terms. Hence, u scales like the cube of $\mathbf{e}_1$ along the fiber.

Second Adaptation

Either $u = 0$ or $u \neq 0$ on entire fibers of $\mathcal{F}_1$. But

$$d\mathbf{e}_1 = \mathbf{e}_1 \varphi_1^1 + u \mathbf{e}_2 \omega^1$$

shows that when $u \neq 0$ the direction $\mathbf{e}_1$ of the tangent line is always changing as we move along the base. Hence, points where $u = 0$ are *inflection points* of the curve.

Assume that γ is free of inflection points; then there is a sub-bundle $\mathcal{F}_2 \subset \mathcal{F}_1$ where $u = 1$ identically.

- fiber of $\mathcal{F}_2$ is isomorphic to a 1-dimensional subgroup $G_2 \subset G_1$ acting by

$$\mathbf{e}_1 \mapsto \mathbf{e}_1, \qquad \mathbf{e}_2 \mapsto \mathbf{e}_2 + b\mathbf{e}_1.$$

Now $R_g^* \omega^1 = \omega^1$ is a well-defined 1-form along γ, the (equi)affine arclength differential. For a generic parametrized curve $\gamma(x)$ we can compute the affine arclength

$$s = \int \omega^1 = \int \det(\gamma'(x), \gamma''(x))^{1/3} dx.$$

Since $\mathbf{e}_1$ is fixed along the fiber, φ_1^1 is semibasic, so there is a function v on $\mathcal{F}_2$ such that

$$\varphi_1^1 = v\, \omega^1.$$

How does v vary along the fiber of $\mathcal{F}_2$?

Third (and last) Adaptation

The earlier calculation (with $a = 1$) implies that

$$R_g^* \varphi_1^1 = \varphi_1^1 - b\omega^1,$$

so that $R_g^* v = v - b$. Thus, along each fiber of $\mathcal{F}_2$ there is a unique point where $v = 0$.

Thm A generic curve in the equiaffine plane has a unique lift Γ into $\mathcal{F}_{\mathbb{A}^2}$ such that

$$\Gamma^*\omega^1 = ds, \quad \Gamma^*\omega^2 = 0, \quad \Gamma^*\varphi_1^2 = \omega^1, \quad \Gamma^*\varphi_1^1 = 0.$$

The *(equi)affine curvature* function $\varkappa$ is defined by

$$\Gamma^*\varphi_2^1 = -\varkappa\, ds \qquad \text{(Blaschke, 1923)}$$

- two curves are congruent under $ASL(2, \mathbb{R})$ if and only if their curvature functions coincide (up to a shift in s)
- for a curve parametrized by equiaffine arclength, $\gamma'''(s) + \varkappa\,\gamma'(s) = 0$
- hence, curves with constant $\varkappa$ are *conics*: parabolas, ellipses, and hyperbolas, depending on whether $\varkappa = 0$, $\varkappa > 0$ or $\varkappa < 0$ respectively

The Conformal 3-Sphere

On $\mathbb{R}^5$, defined an indefinite quadratic form

$$\langle \mathbf{x}, \mathbf{x} \rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$$

Let $\mathcal{N}$ be the cone of nonzero null vectors: $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, $\mathbf{x} \neq 0$

- under projectivization $\pi : \mathbb{R}^5 \to \mathbb{R}\mathrm{P}^4$ the image of $\mathcal{N}$ is S^3.

Proof For points on $\mathcal{N}$, $x_0 \neq 0$, and projective coordinates $y_i = x_i / x_0$ satisfy

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1.$$

- the group of proper linear tx preserving $\langle\ ,\ \rangle$ is $SO(4,1)$

This 10-dimensional group acts on $S^3 = \pi(\mathcal{N})$ by *Möbius transformations*.

- preserve angles, take spheres to spheres and circles to circles
- conformal moving frames reveal what else is preserved

It's convenient to make a linear change of coords on $\mathbb{R}^5$ so that

$$\langle \mathbf{z}, \mathbf{z} \rangle = z_1^2 + z_2^2 + z_3^2 - 2 z_0 z_4.$$

Let $K \simeq SO(4,1)$ be the group preserving this form.

Conformal Frames for S^3

A matrix g belongs to K iff $\det(g) = 1$ and its columns $(\mathbf{e}_0, \ldots, \mathbf{e}_4)$ satisfy

$$\langle \mathbf{e}_0, \mathbf{e}_0 \rangle = \langle \mathbf{e}_4, \mathbf{e}_4 \rangle = 0, \qquad \langle \mathbf{e}_0, \mathbf{e}_4 \rangle = -1,$$
$$\langle \mathbf{e}_i, \mathbf{e}_0 \rangle = \langle \mathbf{e}_i, \mathbf{e}_4 \rangle = 0, \qquad \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \qquad i, j = 1, 2, 3.$$

- in particular, first and last columns are null vectors ...
- *want to have frames in a fixed vector space*

For $p \in S^3$ let a *conformal frame* at p be any 5-tuple of vectors in $\mathbb{R}^5$ satisfying the above relations, such that

$$\pi(\mathbf{e}_0) = p$$

Then the mapping $p : g \mapsto \pi(\mathbf{e}_0)$ lets us identify K as the conformal frame bundle $\mathcal{F}$ of S^3. As before, define 1-forms ω^a_b on $\mathcal{F}$ such that

$$d\mathbf{e}_a = \mathbf{e}_b \omega^b_a, \qquad a, b = 0 \ldots 4$$

- these satisfy $d\omega^a_b = -\omega^a_c \wedge \omega^c_b$ (Maurer-Cartan) and

$$\omega^0_4 = 0, \qquad \omega^4_0 = 0, \qquad \omega^4_4 = -\omega^0_0,$$
$$\omega^4_i = \omega^i_0, \qquad \omega^i_4 = \omega^0_i, \qquad \omega^j_i = -\omega^i_j, \qquad i, j = 1, 2, 3.$$

We'll write $\omega^1, \omega^2, \omega^3$ for semibasics $\omega^1_0, \omega^2_0, \omega^3_0$ respectively.

Adapted Conformal Frames for Curves

Let $\gamma : \mathbb{R} \to S^3$ be regular, and $\mathcal{F}_0 = \gamma^* \mathcal{F}$, with 7-dimensional fibers.

Let $\mathcal{F}_1 \subset \mathcal{F}_0$ be frames such that $p_* \mathbf{e}_1$ is tangent to γ; this has 5-dim'l fibers. Since

$$d\mathbf{e}_0 = \mathbf{e}_0 \omega_0^0 + \mathbf{e}_1 \omega^1 + \mathbf{e}_2 \omega^2 + \mathbf{e}_3 \omega^3,$$

then $\omega^2 = \omega^3 = 0$ on $\mathcal{F}_1$, while

$$d\mathbf{e}_1 = \mathbf{e}_0 \omega_1^0 + \mathbf{e}_2 \omega_1^2 + \mathbf{e}_3 \omega_1^3 + \mathbf{e}_4 \omega^1$$

shows that ω_1^2, ω_1^3 are semibasic (i.e., multiples of ω^1) on $\mathcal{F}_1$.

- adding multiples of $\mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{e}_4$ lets us absorb these terms

Let $\mathcal{F}_2 \subset \mathcal{F}_1$ be frames such that $\omega_1^2 = \omega_1^3 = 0$.

- fibers are 3-dimensional, acted on by

$$\mathbf{e}_0 \mapsto \lambda \mathbf{e}_0, \qquad \mathbf{e}_1 \mapsto \mathbf{e}_1 + \mu \mathbf{e}_0, \qquad \mathbf{e}_4 \mapsto \lambda^{-1}(\mathbf{e}_4 + \mu \mathbf{e}_1 + \tfrac{1}{2}\mu^2 \mathbf{e}_0),$$

plus rotations $\mathbf{e}_2 \mapsto \cos\theta\, \mathbf{e}_2 + \sin\theta\, \mathbf{e}_3$, $\mathbf{e}_3 \mapsto -\sin\theta\, \mathbf{e}_2 + \cos\theta\, \mathbf{e}_3$

- 1-forms $\omega_4^2 = \omega_2^0$ and $\omega_4^3 = \omega_3^0$ are semibasic on $\mathcal{F}_2$

Moreover $\mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_4\}$ is invariant along each fiber of $\mathcal{F}_2$.

- projects under π to give the 'osculating plane' to the curve
- curves in S^3 for which this is *constant* lie in a fixed affine plane in $\mathbb{R}^4$, i.e., they are (not necessarily great) circles in S^3

Assume that nowhere does γ have higher-order contact with its osculating plane. Then we can use the remaining fiber group to make ω_4^2 equal to ω^1 and $\omega^0 = \omega_4^3 = 0$, resulting in a unique moving frame. Let $\omega^1 = ds$ (conformal arclength differental); the remaining semibasics ω_1^0, ω_2^3 give invariants:

$$\frac{d\mathbf{e}_0}{ds} = \mathbf{e}_1, \quad \frac{d\mathbf{e}_1}{ds} = \kappa\mathbf{e}_0 + \mathbf{e}_4, \quad \frac{d\mathbf{e}_2}{ds} = \mathbf{e}_0 + \tau\mathbf{e}_3, \quad \frac{d\mathbf{e}_3}{ds} = -\tau\mathbf{e}_2, \quad \frac{d\mathbf{e}_4}{ds} = \kappa\mathbf{e}_1 + \mathbf{e}_2$$

- curves with $\tau = 0$ lie on spheres in S^3
- curves with $\kappa = 0$ and $\tau = 0$ are conformally equivalent to *logarithmic spirals* in the plane (under stereographic projection)

Adapted Conformal Frames for Surfaces

Let $\Sigma \subset S^3$ be an surface, and let $\mathcal{F}_0 = \mathcal{F}|_\Sigma$.

First adaptation: Let $\mathcal{F}_1 \subset \mathcal{F}_0$ be sub-bundle of frames such that $\pi_* \mathbf{e}_1, \pi_* \mathbf{e}_2$ span the tangent space of Σ. This has 5-dimensional fibers. Since

$$d\mathbf{e}_0 = \mathbf{e}_0 \omega^0 + \mathbf{e}_1 \omega^1 + \mathbf{e}_2 \omega^2 + \mathbf{e}_3 \omega^3$$

then $\omega^3 = 0$ and ω^1, ω^2 span the semibasics on $\mathcal{F}_1$. Differentiating $\omega^3 = 0$ gives

$$0 = \omega^3_1 \wedge \omega^1 + \omega^3_2 \wedge \omega^2.$$

By Cartan's Lemma, there are functions a, b, c on $\mathcal{F}_1$ such that

$$\omega^3_1 = a\omega^1 + b\omega^2, \qquad \omega^3_2 = b\omega^1 + c\omega^2.$$

▶ usual second fund. form $a(\omega^1)^2 + c(\omega^2)^2 + 2b\omega^1\omega^2$ is not conformally invariant! However, by computing how a, b, c vary along the fiber, we see that the third fund. form

$$\mathcal{T} = b(\omega^1)^2 - b(\omega^2)^2 + (c - a)\omega^1\omega^2$$

is well-defined (up to scaling) on fibers of $\mathcal{F}_1$.

Conformal Invariants

- lines of curvature (which are null directions for $\mathcal{T}$) and umbilic points are conformally invariant!

- 2-form $\Omega = (b^2 + \frac{1}{4}(a-c)^2)\omega^1 \wedge \omega^2$ is invariant along the fibers of $\mathcal{F}_1$, so is well-defined on Σ, giving the Willmore functional

$$\mathcal{W}(\Sigma) = \int_\Sigma \Omega$$

- critical surfaces are known as *Willmore immersions*.

Thm (Codá Marques & Neves, 2014) Among all immersed tori in S^3, the Willmore functional is minimized by the Clifford torus.

Second Adaptation: Let $\mathcal{F}_2 \subset \mathcal{F}_1$ be where $a + c = 0$. Then ω^0_3 is semibasic on $\mathcal{F}_2$, and $\mathbf{e}_3$ is fixed along the 4-d fibers. This defines a *conformal Gauss map*

$$\Gamma : \Sigma \to Q, \qquad \Gamma = \pi \circ \mathbf{e}_3$$

where $Q^4 \subset \mathbb{R}^5$ is the quadric defined by $\langle \mathbf{z}, \mathbf{z} \rangle = 1$.

- $\mathcal{W}(\Sigma) =$ area of the Gauss image

Third Adaptation:

Assume Σ is free of umbilic points. Then there is a unique conformal moving frame such that $b = 0$, $\omega_3^0 = 0$ and $\Omega = \omega^1 \wedge \omega^2$.

- frame vector $\mathbf{e}_4$ maps into null cone $\mathcal{N}$, & projectivizing gives a well-defined *dual* surface in S^3

Thm (Bryant) If $X : M \to S^3$ is a Willmore immersion, with non-umbilic points forming an open dense subset $U \subset M$, then its dual immersion $\widehat{X} : U \to S^3$ is also a Willmore immersion.

Remark Other choices of adaptations are useful in other situations:

Problem Classify surfaces $\Sigma \subset \mathbb{R}^3$ foliated by two orthogonal families of circles.

By adapting sections of $\mathcal{F}_2$ to point along these circles, one can prove:

Thm (I–) The circles must be lines of curvature (hence Σ is a *cyclide of Dupin*) unless tangents to the circles are bisected by the lines of curvature, in which case they are also circles, and Σ is conformally equivalent to a Clifford torus.