Symmetry and Explicit Integrability of Control Systems: Cascade Feedback Linearization

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What to cover

Control Systems

Definition

Let M be a manifold with local coordinates $(t, \boldsymbol{x}, \boldsymbol{u})$, where $\boldsymbol{x} = (x^1, \ldots, x^n)$ and $\boldsymbol{u} = (u^1, \ldots, u^m)$. A ***control system*** *on M is an underdetermined system of ordinary differential equations,*

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}),$$

where $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}) = (f^1(t, \boldsymbol{x}, \boldsymbol{u}), \ldots, f^n(t, \boldsymbol{x}, \boldsymbol{u}))$. The coordinate t will denote time, and the variables $\boldsymbol{x}$ and $\boldsymbol{u}$ are the ***state variables*** *and* ***control variables*** *respectively. Additionally, denote $\boldsymbol{X}(M) \cong \mathbb{R}^n$ to be the* ***state space*** *of M with the states $\boldsymbol{x}$ as local coordinates on $\boldsymbol{X}(M)$.*

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A Desired Property: Given any two points A and B in $\mathbf{X}(M)$ find $\mathbf{u}(t)$ such that $(\mathbf{x}(t), \mathbf{u}(t))$ a solution to the control system with $\mathbf{x}(t_0) = A$ and $\mathbf{x}(t_f) = B$. This is known broadly as controllability.

Appearances in STEM

Control systems are ubiquitous in engineering and the sciences. Some notable examples are

- Autonomous vehicles (drones, self-driving cars, submersibles, etc.)

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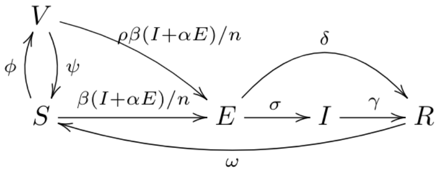
- ...Many More!

Example: SVE(R)IRS Epidemiology Model

Joint with M. Chyba, Y. Mileyko, and C. Shandbrom.

$$\frac{dS}{dt} = -\beta S(I+\alpha E)/n + \omega R - \phi S + \psi V,$$
$$\frac{dE}{dt} = \beta S(I+\alpha E)/n - (\sigma+\delta)E + \rho\beta V(I+\alpha E)/n,$$
$$\frac{dI}{dt} = \sigma E - \gamma I,$$
$$\frac{dR}{dt} = \delta E + \gamma I - \omega R,$$
$$\frac{dV}{dt} = -\rho\beta V(I+\alpha E)/n + \phi S - \psi V.$$



S-Susceptible, E-Exposed/Asymptomatic, I-Infected, R-Recovered, V-Vaccinated. Nine Paramters: $n, \alpha, \beta, \gamma, \delta, \rho, \sigma, \phi, \psi$, and ω. Notice we can reduce by $R = n - S - E - I - V$.

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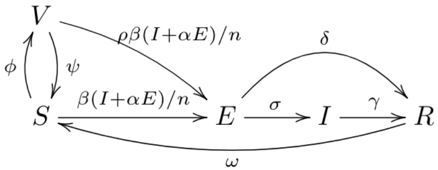

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"Solution"

$$S(t) = g(t)$$
$$E(t) = \frac{1}{nl}(-\gamma\omega f(t) - (\gamma+\omega)\dot{f}(t) - \ddot{f}(t) + \sigma n^2\omega\gamma),$$
$$I(t) = \frac{1}{l}(-\omega f(t) - \dot{f}(t) + n^2\omega\sigma),$$
$$V(t) = \frac{1}{nl}(((\omega+\gamma)\sigma + \gamma(\omega+\delta))f(t) + (\sigma+\gamma+\delta+\omega)\dot{f}(t) + \ddot{f}(t), + n\sigma(\delta\omega - \gamma\delta - \gamma\sigma)g(t) - n^2\sigma\omega(\sigma+\gamma+\delta)),$$

$$u_1(t) = \frac{1}{nS(t)}(-u_2(t)S(t)(\alpha E(t) + I(t)) + n\psi V(t) + n\omega R(t) - n\dot{g}(t)),$$
$$u_2(t) = \frac{1}{D_2(t)}(n\omega^3\sigma R(t) + C_{1,E}E(t) + C_{1,I}I(t) - \dddot{f}(t)),$$
$$D_2(t) = l(S(t) + \rho V(t))(\alpha E(t) + I(t)),$$

Explicitly Integrable

Definition

A controllable control system is called ***explicitly integrable*** *(EI) if generic solutions may be written as*

$$\boldsymbol{x}(t) = \boldsymbol{A}(t, f_i(t), \dot{f}_i(t), \ldots, f_i^{(s_i)}(t)),$$
$$\boldsymbol{u}(t) = \boldsymbol{B}(t, f_i(t), \dot{f}_i(t), \ldots, f_i^{(r_i)}(t)),$$

for $1 \leq i \leq m$ *where* m *is the number of controls,* $f_i(t)$ *are arbitrary smooth functions, and* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are smooth functions.*

No integration of arbitrary functions! Everything is in terms of arbitrary functions and their derivatives. Useful for trajectory planning.

Brunovský Normal Form

Definition

The ***Brunovský normal form*** *is a linear control system*

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x} + B\boldsymbol{u},$$

such that matrix A is of the form

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & A_m \end{bmatrix}, \qquad A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix B has 1 on a diagonal position if it aligns with a row of zeros of A. All other entries of B are zero.

Static Feedback Equivalence

Definition

Two control systems

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}) \quad \text{and} \quad \frac{d\boldsymbol{y}}{dt} = \boldsymbol{g}(t, \boldsymbol{y}, \boldsymbol{v})$$

are ***static feedback equivalent (SFE)*** *if there is a local change coordinates of the form* $(t, \boldsymbol{y}(t, \boldsymbol{x}), \boldsymbol{v}(t, \boldsymbol{x}, \boldsymbol{u}))$ *such that solutions to a control system in one coordinate system are transformed to solutions of the other control system via this change of coordinates (and vice verse).*

Static Feedback Linearization

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Definition

A control system is static feedback linearizable (SFL) if and only if it is SFE to a Brunovský normal form.

The problem of recognizing when a control system is SFL was completely solved by P. Vassiliou and the proof gives an explicit construction of such transformations. For SFL systems without the time dependence, the question was answered by Gardner, Shadwick, and Wilkens.

Control Systems as Distributions/Pfaffian Systems

Definition

Let M be an open manifold such that $M \subset \mathbb{R} \times \boldsymbol{X}(M) \times \boldsymbol{U}(M)$, where $\mathbb{R}$ has t as a local coordinate, $\boldsymbol{X}(M)$ is a manifold of dimension n with local coordinates $\boldsymbol{x} = (x^1, \ldots, x^n)$, and $\boldsymbol{U}(M)$ is a manifold of dimension m with local coordinates $\boldsymbol{u} = (u^1, \ldots, u^m)$. A control system on M may be written as the linear Pfaffian system

$$\omega = \langle dx^1 - f^1(t, \boldsymbol{x}, \boldsymbol{u})\, dt, \ldots, dx^n - f^n(t, \boldsymbol{x}, \boldsymbol{u})\, dt \rangle,$$

with independence condition dt. In the language of distributions, a control system is given by the rank $m+1$ distribution $\mathcal{V} = \operatorname{ann} \omega$, which in local coordinates is given by

$$\mathcal{V} = \{\partial_t + f^1(t, \boldsymbol{x}, \boldsymbol{u})\, \partial_{x^1} + \cdots + f^n(t, \boldsymbol{x}, \boldsymbol{u})\, \partial_{x^n}, \partial_{u^1}, \ldots, \partial_{u^m}\}.$$

Additionally, we require that the Cauchy bundles of ω and $\mathcal{V}$ be trivial.

Partial Prolongation/Mixed Order Jet Space

Definition

Let $\kappa = \langle \rho_1, \ldots, \rho_k \rangle$ be a list of nonegative integers with $\rho_k \neq 0$. Then we define a ***partial prolongation*** *of $J^1(\mathbb{R}, \mathbb{R}^m)$ to be*

$$J^\kappa(\mathbb{R}, \mathbb{R}^m) := \left(\prod_{i \in I} J^i(\mathbb{R}, \mathbb{R}^{\rho_i}) \right) / \sim,$$

$$\boldsymbol{\beta}^\kappa_m := \bigoplus_{i \in I} \boldsymbol{\beta}^i_{\rho_i},$$

with $I = \{1 \leq a \leq k \mid \rho_a \neq 0\}$ and each $\boldsymbol{\beta}^i_{\rho_i}$ is the Brunovký form on jet space $J^i(\mathbb{R}, \mathbb{R}^{\rho_i})$. The equivalence relation '$\sim$' in (7) is defined by

$$\pi_i \left(J^i(\mathbb{R}, \mathbb{R}^{\rho_i}) \right) = \pi_j \left(J^j(\mathbb{R}, \mathbb{R}^{\rho_j}) \right),$$

for all $1 \leq i, j \leq k$, where π_i, π_j are source projection maps (i.e. projection on to the t-coordinate on $\mathbb{R}$). Define $\mathcal{C}_\kappa = \mathrm{ann} \boldsymbol{\beta}^\kappa_m$.

Generalized Goursat Bundle

Definition

A distribution $\mathcal{V}$ is a ***generalized Goursat bundle*** *if it is diffeomorphism equivalent $\mathcal{C}_\kappa$ for some κ.*

If $\mathcal{V}$ represents a control system and the diffeomorphism is a static feedback map, then we'll say that $\mathcal{V}$ is SFL. Peter Vassiliou completely characterized nondegenerate generalized Goursat bundles in terms of integer invariants and associated integrable subbundles derived from $\mathcal{V}$. The variable parameterizing integral curves is determined in this process.

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Control Admissible Symmetries

Definition

Let ω be a control system on a manifold M. We say a finite dimensional local Lie group G acting freely and regularly on M with action $\mu : M \times G \to M$ is a **control symmetry group** if:

1. ω is invariant under the action i.e. $\mu_g^*\omega = \omega$ for all $g \in G$,
2. the function t is invariant,
3. $\operatorname{rank}(d\boldsymbol{p}(\Gamma)) = \dim G$, where $\boldsymbol{p}$ is the projection $p : M \to \mathbb{R} \times \boldsymbol{X}(M)$ given by $\boldsymbol{p}(t, \mathbf{x}, \mathbf{u}) = (t, \mathbf{x})$ where Γ is the Lie algebra of infinitesimal generators of the action of G on M.

The group G is an **admissible control symmetry group** if $\dim G < \dim \boldsymbol{X}(M)$ and G acts strongly transversely i.e.

$$\Gamma \cap \mathcal{V}^{(1)} = \{0\}$$

where $\mathcal{V} = \operatorname{ann} \omega$ and $\mathcal{V}^{(1)}$ is the first derived system.

SFL Quotient Systems

Theorem (J. De DonÁ , N. Tehseen, P. J. Vassiliou)

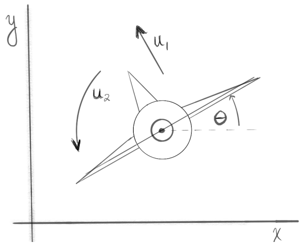
If $\mathcal{V}$ is a control system on M with admissible control symmetry G with Lie algebra of infinitesimal gnerators Γ, then $\mathcal{V}/G$ is SFL if $\mathcal{V} \oplus \Gamma$ satisfies some easy to check rank and integrability conditions related to the characterization of Goursat bundles.

Observation: Many control systems admitting control admissible symmetry groups have quoteint systems that are SFL!

PVTOL System

The system for a Planar Vertical Take-Off and Landing (PVTOL) vehicle system was introduced by Hauser, Sastry, and Meyer (1992)

$$\dot{x} = x_1 \qquad \dot{x}_1 = hu_2\cos(\theta) - u_1\sin(\theta)$$
$$\dot{y} = y_1 \qquad \dot{y}_1 = u_1\cos(\theta) + hu_2\sin(\theta) - 1$$
$$\dot{\theta} = \theta_1 \qquad \dot{\theta}_1 = u_2$$



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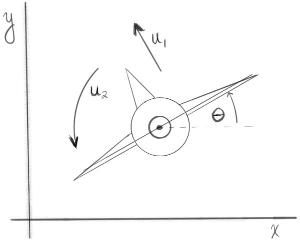


The PVTOL system is NOT SFL!

PVTOL Explicitly Integrable

In 1996 Martin, Devasia, and Paden discovered that if one added the extra dynamics

$$\ddot{z} = -v_1 \sin(\theta) + v_2 \cos(\theta) + z\dot{\theta}^2$$

with new controls v_1 and v_2 related to the previous controls by

$$u_1 = z + \dot{\theta}^2 \text{ and } u_2 = -\frac{1}{z}(v_1 \cos(\theta) + v_2 \sin(\theta) + 2\dot{z}\dot{\theta})$$

then the PVTOL control system with the above modifications/additions is SFL! But determining the above involved guesswork and tricky physics.

Dynamic Feedback Linearization

Definition

*A control system is **dynamic feedback linearizable** (DFL) if there exists an augmented system of the form*

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}), \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u} \in \mathbb{R}^m, \\
\dot{\boldsymbol{y}} &= \boldsymbol{g}(t, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}), \boldsymbol{y} \in \mathbb{R}^k, \boldsymbol{w} \in \mathbb{R}^q, \\
\boldsymbol{u} &= \boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}),
\end{aligned}$$

such that the control system

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})), \\
\dot{\boldsymbol{y}} &= \boldsymbol{g}(t, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}),
\end{aligned}$$

is SFL. There is also a resularity condition on the lift of integral curves of the original system to the larger manifold for the augmented system.

Symmetry of PVTOL

Theorem

The PVTOL control system has an eight dimensional control admissible symmetry algebra Γ_{alg} isomorphic to a Lie algebra with Levi decomposition

$$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{s}_{5,9},$$

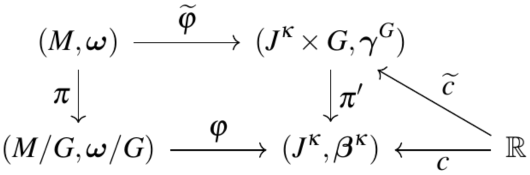
where $\mathfrak{s}_{5,9}$ is a 5-dimensional solvable Lie algebra.*

Proposition

Every 2-dimensional Lie subalgebra Λ_{alg} of Γ_{alg} such that $\Lambda \cap \mathcal{V}^{(2)} = \{0\}$ leads to a SFL quotient of the PVTOL control system.

*- This Lie algebra can be found in the appendix of *Classification and identification of Lie algebras*, by Libor Šnobl and Pavel Winternitz.

Contact sub-connection



Theorem (P.J. Vassiliou)

Let $\varphi : M/G \to J^\kappa$ be a SFL of ω/G. Then as a local principle G-bundle the local trivializtion $\widetilde{\varphi}$ covering φ yields a Pfaffian system γ^G of the form

$$\gamma^G = \boldsymbol{\beta}^\kappa \oplus \Theta^G$$

with

$$\Theta^G = \mathrm{span}\{\eta^a - p^a(t, \boldsymbol{z}^\kappa)\, dt\}_{a=1}^r,$$

where each η^a is an entry of the right Maurer-Cartan form of G, i.e. $\eta^a(R_b) = \delta^a_b$, where δ^a_b is the kronecker delta.

PVTOL Quotient

The following choice of admissible Lie subalgebra of the control symmetry algebra of the PVTOL system

$$\mathbf{\Gamma} = \operatorname{span}\{X_1, X_2\}$$

where

$$\begin{aligned} X_1 =& h\sin^2(\theta)\cos(\theta)\partial_x + h\theta_1(3\cos^2(\theta)-1)\sin(\theta)\partial_{x_1} + \sin^2(\theta)\partial_\theta \\ &+ (x - h\sin(\theta)\cos^2(\theta))\partial_z + \theta_1\sin(2\theta)\partial_{\theta_1} \\ &+ (x_1 + 2h\theta_1\cos(\theta) - 3h\theta_1\cos^3(\theta))\partial_{z_1} \\ &+ \cos(\theta)\sin(\theta)(5h\theta_1^2 - u_1)\partial_{u_1} + (2\theta_1^2\cos(2\theta) + u_2\sin(2\theta))\partial_{u_2} \\ X_2 =& \partial_z \end{aligned}$$

leads us to the contact sub-connection on $J^2(\mathbb{R}, \mathbb{R}^2) \times G$

$$\gamma^G = \beta^{\langle 0,2\rangle} \oplus \operatorname{span}\left\{ d\epsilon_1 - \frac{1+z_1}{w_1}\,dt, d\epsilon_2 - \frac{w_1(z-w_1)+w(1+z_1)}{w_1}\,dt \right\},$$

$(z, z_1, z_2, w, w_1, w_2)$ make up the coordinates on J^2.

Partial Contact Curve Reduction

Observe that any Brunovský normal form $\boldsymbol{\beta}^{\kappa}$ on $J^{\kappa}(\mathbb{R}, \mathbb{R}^m)$ may be decomposed into $\boldsymbol{\beta}^{\nu} \oplus \boldsymbol{\beta}^{\nu^{\perp}}$ where $\kappa = \nu + \nu^{\perp}$ entrywise and $\boldsymbol{\beta}^{\nu}$ and $\boldsymbol{\beta}^{\nu^{\perp}}$ are Brunovský normal forms on $J^{\nu}(\mathbb{R}, \mathbb{R}^{m_{\nu}})$ and $J^{\nu}(\mathbb{R}, \mathbb{R}^{m_{\nu^{\perp}}})$ respectively where $m = m^{\nu} + m^{\nu^{\perp}}$. For example,

$$\begin{aligned}\boldsymbol{\beta}^{\kappa} &= \langle dz_0^1 - z_1^1\,dt, dz_0^2 - z_1^2\,dt, dz_1^2 - z_2^2\,dt, dz_0^3 - z_1^3\,dt, dz_1^3 - z_2^3\,dt\rangle \\ &= \langle dz_0^1 - z_1^1\,dt, dz_0^2 - z_1^2\,dt, dz_1^2 - z_2^2\,dt\rangle \oplus \langle dz_0^3 - z_1^3\,dt, dz_1^3 - z_2^3\,dt\rangle \\ &= \boldsymbol{\beta}^{\nu} \oplus \boldsymbol{\beta}^{\nu^{\perp}}.\end{aligned}$$

where $\kappa = \langle 1, 2\rangle = \langle 1, 1\rangle + \langle 0, 1\rangle$ with $\nu = \langle 1, 1\rangle$ and $\nu^{\perp} = \langle 0, 1\rangle$.

Partial Contact Curve Reduction (cont.)

Definition

We say that a submanifold $\Sigma^{\nu}_{f} \subset J^{\kappa} \times G$ is a ***codimension s partial contact curve*** *of $\boldsymbol{\beta}^{\kappa} = \boldsymbol{\beta}^{\nu} \oplus \boldsymbol{\beta}^{\nu^{\perp}}$ if Σ^{ν}_{f} is an integral manifold of $\boldsymbol{\beta}^{\nu}$ and s is the sum of the entries in ν. It is the image of a map*

$$\mathbf{c}^{\nu}_{f} : J^{\nu^{\perp}} \times G \to J^{\kappa} \times G$$

defined by

$$\mathbf{c}^{\nu}_{f}(t, \boldsymbol{z}^{\nu^{\perp}}, \varepsilon) = (t, j^{\nu}_{f}(t), \boldsymbol{z}^{\nu^{\perp}}, \varepsilon),$$

where $\boldsymbol{z}^{\nu^{\perp}}$ represents the local contact coordinates on $J^{\nu^{\perp}}$, $j^{\nu}_{f}(t)$ represents the integral curve of $\boldsymbol{\beta}^{\nu}$ corresponding to the jet of some smooth function $f : \mathbb{R} \to \mathbb{R}^{m_{\nu}}$, and ε represents local coordinates on G. We refer to a system $\boldsymbol{\gamma}^{G}$ restricted to the family of such submanifolds as f ranges over the space of generic smooth functions as a ***partial contact curve reduction*** *of $\boldsymbol{\gamma}^{G}$ and denote it by $\bar{\boldsymbol{\gamma}}^{G}$.*

PVTOL Partial Contact Curve Reduction

Let us consider the family of maps $\boldsymbol{c}_f^{\langle 0,1\rangle} = J^{\langle 0,1\rangle}\times G \to J^{\langle 0,1\rangle}\times G$ parameterized by generic smooth fs such that

$$\left(\mathbf{c}_f^{\langle 0,1\rangle}\right)^* \operatorname{span}\{dw - dw_1\,dt, dw_1 - w_2\,dt\} = \{0\},$$

i.e. we've restricted to the family of submanifolds defiend by $(w = f(t), w_1 = \dot{f}(t), w_2 = \ddot{f}(t))$. Then

$$\begin{aligned}\bar{\boldsymbol{\gamma}}^G &= \left(\mathbf{c}_f^{\langle 0,1\rangle}\right)^* \boldsymbol{\gamma}^G,\\ &= \boldsymbol{\beta}^{\langle 0,1\rangle} \oplus \operatorname{span}\left\{d\varepsilon_1 - \frac{1+z_1}{\dot{f}(t)}\,dt, d\varepsilon_2 - \frac{\dot{f}(t)(z-\dot{f}(t)) + f(t)(1+z_1)}{\dot{f}(t)}\,dt\right\}\end{aligned}$$

Remarkably, $\bar{\boldsymbol{\gamma}}^G$ is SFL for generic $f(t)$!

Cascade Feedback Linearization

Definition

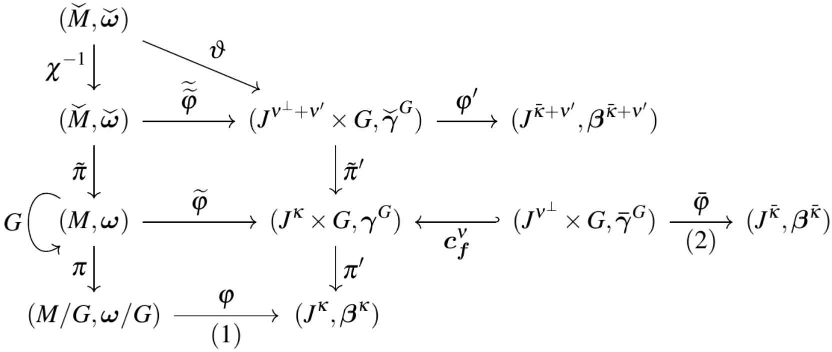
A control system ω is cascade feedback linearizable if it possess a SFL quotient system by an admissible control symmetry group and the associated contact sub-connection admits a SFL partial contact curve reduction.

CFL systems are explicitly integrable! Moreover,

Theorem (J Clelland, T. Klotz, P.J. Vassiliou)

A CFL control system yields explicit formulas for constructing a DFL.

Cascade Feedback Linearization and Dynamic Feedback Linearization



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- Are there global topological constraints on closed orbits of nonlinear control systems? I.e. an Arnold type conjecture.
- Applications!
- Explore equivalence problems concerning the contact sub-connections i.e. when are γ^H and γ^G ESF equivalent?

Thanks!

References

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