On structures of the complete point symmetry group and the maximal Lie invariance algebra of an ultraparabolic Fokker–Planck equation

Serhii Koval, Alex Bihlo, Roman O. Popovych

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Koval S.D., Bihlo A. and Popovych R.O., Extended symmetry analysis of remarkable (1+2)-dimensional Fokker–Planck equation, arXiv:2205.13526

The remarkable Fokker–Planck equation

$$u_t + xu_y = u_{xx}. \tag{1}$$

The most general form of the class $\bar{\mathcal{F}}$ of two-dimensional ultraparabolic degenerate Fokker–Planck equations is

$$\begin{gathered} u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y) \\ \text{with} \quad A^2 \neq 0, \quad B_x \neq 0, \quad \bar{\theta} := (B, A^2, A^1, A^0, C) \in \mathcal{S}_{\bar{\mathcal{F}}}. \end{gathered} \tag{2}$$

Davydovych V., Preliminary group classification of $(2+1)$-dimensional linear ultraparabolic Kolmogorov–Fokker–Planck equations, *Ufimsk. Mat. Zh.* **7** (2015), 39–49.

González-López A., Kamran N. and Olver P.J., Lie algebras of vector fields in the real plane, *Proc. London Math. Soc. (3)* **64** (1992), 339–368.

The equation (1) corresponds to $B = x$, $A^2 = 1$ and $A^1 = A^0 = C = 0$. $\dim \mathfrak{g}^{\rm ess} = 8$. which is the maximum within the class $\bar{\mathcal{F}}$. Moreover, it is, up to the point equivalence, a unique equation from $\bar{\mathcal{F}}$ that has this property.

The fundamental solution of (1) is

$$F(t,y,x,t',y',x') = \frac{\sqrt{3}H(t-t')}{2\pi(t-t')^2}\exp\left(-\frac{(x-x')^2}{4(t-t')} - \frac{3\big(y-y'-\frac{1}{2}(x+x')(t-t')\big)^2}{(t-t')^3}\right)$$

Kolmogoroff A., Zufällige Bewegungen (zur Theorie der Brownschen Bewegung), *Ann. of Math. (2)* **35** (1934), 116–117.

A preliminary study of symmetry properties of (1) was carried out in

Güngör F., Equivalence and symmetries for variable coefficient linear heat type equations. II. Fundamental solutions, *J. Math. Phys.* **59** (2018), 061507.

Kovalenko S.S., Kopas I.M. and Stogniy V.I., Preliminary group classification of a class of generalized linear Kolmogorov equations, *Research Bulletin of the National Technical University of Ukraine "Kyiv Polytechnic Institute"* **4** (2013), 67–72.

The maximal Lie invariance algebra of (1) is

$$\mathfrak{g} := \langle \mathcal{P}^t,\, \mathcal{D},\, \mathcal{K},\, \mathcal{P}^3,\, \mathcal{P}^2,\, \mathcal{P}^1,\, \mathcal{P}^0,\, \mathcal{I}, \mathcal{Z}(f)\rangle,$$

where

$$\mathcal{P}^t = \partial_t, \quad \mathcal{D} = 2t\partial_t + x\partial_x + 3y\partial_y - 2u\partial_u,$$
$$\mathcal{K} = t^2\partial_t + (tx + 3y)\partial_x + 3ty\partial_y - (x^2 + 2t)u\partial_u,$$
$$\mathcal{P}^3 = 3t^2\partial_x + t^3\partial_y + 3(y - tx)u\partial_u, \quad \mathcal{P}^2 = 2t\partial_x + t^2\partial_y - xu\partial_u, \quad \mathcal{P}^1 = \partial_x + t\partial_y,$$
$$\mathcal{P}^0 = \partial_y, \quad \mathcal{I} = u\partial_u, \quad \mathcal{Z}(f) = f(t,x,y)\partial_u.$$

Here the parameter function f runs through the solution set of the equation (1).

$\mathfrak{g}^{\rm lin} := \{\mathcal{Z}(f)\}$ is an infinite-dimensional abelian ideal of $\mathfrak{g}$.
$\mathfrak{g}^{\rm ess} := \langle \mathcal{P}^t,\, \mathcal{D},\, \mathcal{K},\, \mathcal{P}^3,\, \mathcal{P}^2,\, \mathcal{P}^1,\, \mathcal{P}^0,\, \mathcal{I}\rangle$.
$\mathfrak{g} = \mathfrak{g}^{\rm ess} \Subset \mathfrak{g}^{\rm lin}$.

The nonzero commutation relations between the basis vector fields of $\mathfrak{g}^{\rm ess}$:

$$
\begin{aligned}
&[\mathcal{P}^t,\mathcal{D}]=2\mathcal{P}^t,\quad [\mathcal{P}^t,\mathcal{K}]=\mathcal{D},\quad [\mathcal{D},\mathcal{K}]=2\mathcal{K},\\
&[\mathcal{P}^t,\mathcal{P}^3]=3\mathcal{P}^2,\quad [\mathcal{P}^t,\mathcal{P}^2]=2\mathcal{P}^1,\quad [\mathcal{P}^t,\mathcal{P}^1]=\mathcal{P}^0,\\
&[\mathcal{P}^3,\mathcal{D}]=-3\mathcal{P}^3,\quad [\mathcal{P}^2,\mathcal{D}]=-\mathcal{P}^2,\quad [\mathcal{P}^1,\mathcal{D}]=\mathcal{P}^1,\quad [\mathcal{P}^0,\mathcal{D}]=3\mathcal{P}^0,\\
&[\mathcal{P}^2,\mathcal{K}]=\mathcal{P}^3,\quad [\mathcal{P}^1,\mathcal{K}]=2\mathcal{P}^2,\quad [\mathcal{P}^0,\mathcal{K}]=3\mathcal{P}^1,\\
&[\mathcal{P}^1,\mathcal{P}^2]=-\mathcal{I},\quad [\mathcal{P}^0,\mathcal{P}^3]=3\mathcal{I}.
\end{aligned}
$$

The algebra $\mathfrak{g}^{\rm ess}$ is nonsolvable.
Its Levi decomposition is given by $\mathfrak{g}^{\rm ess}=\mathfrak{f}\mathbin{\ltimes}\mathfrak{r}$.
The subalgebra $\mathfrak{r}=\langle\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0,\mathcal{I}\rangle$ is both radical and nilradical, $\mathfrak{r}\simeq{\rm h}(2,\mathbb{R})$. The Levi factor $\mathfrak{f}=\langle\mathcal{P}^t,\mathcal{D},\mathcal{K}\rangle$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to ${\rm sl}(2,\mathbb{R})$.

The real representation of the Levi factor $\mathfrak{f}$ on the radical $\mathfrak{r}$ coincides, in the basis $(\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0,\mathcal{I})$, with the real representation $\rho_3\oplus\rho_0$ of ${\rm sl}(2,\mathbb{R})$.[1]

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[1] $\rho_n(\mathcal{D})_{ij}=(n-2i+2)\delta_{ij}$, $\rho_n(\mathcal{K})_{ij}=(i-1)\delta_{i-1,j}$, $\rho_n(\mathcal{P}^t)_{ij}=(n-i)\delta_{i,j-1}$, where $i,j\in\{1,2,\dots,n+1\}$, $n\in\mathbb{N}\cup\{0\}$, and δ_{kl} is the Kronecker delta, i.e., $\delta_{kl}=1$ if $k=l$ and $\delta_{kl}=0$ otherwise, $k,l\in\mathbb{Z}$.

Equivalence groupoid of the class $\bar{\mathcal{F}}$

$$u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y) \qquad (2)$$

Theorem

The class $\bar{\mathcal{F}}$ is normalized. Its equivalence (pseudo)group $G^{\sim}_{\bar{\mathcal{F}}}$ consist of the admissible transformations with the components

$$\tilde t = T(t,y), \quad \tilde x = X(t,x,y), \quad \tilde y = Y(t,y), \quad \tilde u = U^1(t,x,y)u + U^0(t,x,y),$$

$$\tilde A^0 = \frac{A^0}{T_t+BT_y} - \frac{A^1}{T_t+BT_y}\frac{U^1_x}{U^1} + \frac{A^2}{T_t+BT_y}\left(\left(\frac{U^1_x}{U^1}\right)^2 - \left(\frac{U^1_x}{U^1}\right)_x\right) + \frac{1}{U^1}\frac{U^1_t+BU^1_y}{T_t+BT_y},$$

$$\tilde A^1 = A^1\frac{X_x}{T_t+BT_y} - \frac{X_t+BX_y}{T_t+BT_y} + A^2\frac{X_{xx}-2X_xU^1_x/U^1}{T_t+BT_y},$$

$$\tilde A^2 = A^2\frac{(X_x)^2}{T_t+BT_y}, \quad \tilde B = \frac{Y_t+BY_y}{T_t+BT_y}, \quad \tilde C = \frac{U^1}{T_t+BT_y}\left(C - E\frac{U^0}{U^1}\right),$$

where T, X, Y, U^1 and U^0 are arbitrary smooth functions of their arguments with $(T_tY_y - T_yY_t)X_xU^1 \neq 0$, and $E = \partial_t + B\partial_y - A^2\partial_{xx} - A^1\partial_x - A^0$.

$$u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y) \qquad (2)$$

$(B, A^2, A^1, A^0, C) = (x, 1, 0, 0, 0)$ for (1).

Theorem

The complete point symmetry group G of (1) consists of the transformations

$$\begin{aligned}
&\tilde t = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde x = \frac{\hat x}{\gamma t + \delta} - \frac{3\gamma \hat y}{(\gamma t + \delta)^2}, \quad \tilde y = \frac{\hat y}{(\gamma t + \delta)^3}, \\
&\tilde u = \sigma(\gamma t + \delta)^2 \exp\left(\frac{\gamma \hat x^2}{\gamma t + \delta} - \frac{3\gamma^2 \hat x \hat y}{(\gamma t + \delta)^2} + \frac{3\gamma^3 \hat y^2}{(\gamma t + \delta)^3}\right) \\
&\qquad \times \exp\left(3\lambda_3(y - tx) - \lambda_2 x - (3\lambda_3^2 t^3 + 3\lambda_3\lambda_2 t^2 + \lambda_2^2 t)\right)\left(u + f(t,x,y)\right),
\end{aligned} \qquad (4)$$

where

$$\hat x := x + 3\lambda_3 t^2 + 2\lambda_2 t + \lambda_1, \quad \hat y := y + \lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t^1 + \lambda_0,$$

and $\varepsilon = \pm 1$; α, β, γ and δ are arbitrary constants with $\alpha\delta - \beta\gamma = 1$; $\lambda_0, \ldots, \lambda_3$ and σ are arbitrary constants with $\sigma \neq 0$, and f is an arbitrary solution of (1).

$$\tilde t = \frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x = \frac{\hat x}{\gamma t+\delta} - \frac{3\gamma\hat y}{(\gamma t+\delta)^2},\quad \tilde y = \frac{\hat y}{(\gamma t+\delta)^3},$$
$$\tilde u = \sigma(\gamma t+\delta)^2 \exp\left(\frac{\gamma\hat x^2}{\gamma t+\delta} - \frac{3\gamma^2\hat x\hat y}{(\gamma t+\delta)^2} + \frac{3\gamma^3\hat y^2}{(\gamma t+\delta)^3}\right)$$
$$\times \exp\left(3\lambda_3(y-tx) - \lambda_2 x - (3\lambda_3^2t^3 + 3\lambda_3\lambda_2t^2 + \lambda_2^2t)\right)\left(u + f(t,x,y)\right),$$

Remark

The point transformations of the form $\tilde t = t$, $\tilde x = x$, $\tilde y = y$, $\tilde u = u + f(t,x,y)$, constitute the normal (pseudo)subgroup $G^{\rm lin}$ of the group G.

The group G splits over $G^{\rm lin}$,

$$G = G^{\rm ess} \ltimes G^{\rm lin},$$

where the subgroup $G^{\rm ess}$ of G consists of the transformations with $f = 0$.

The subgroup $G^{\rm ess}$ itself splits over R,

$$G^{\rm ess} = F \ltimes R,$$

where $R(\alpha = \delta = 1, \beta = \gamma = 0)$ is a normal subgroup of $G^{\rm ess}$,
$F(\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0)$ is the subgroup of $G^{\rm ess}$.

$$\tilde t = \frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x = \frac{\hat x}{\gamma t+\delta} - \frac{3\gamma\hat y}{(\gamma t+\delta)^2},\quad \tilde y = \frac{\hat y}{(\gamma t+\delta)^3},$$
$$\tilde u = \sigma(\gamma t+\delta)^2 \exp\left(\frac{\gamma\hat x^2}{\gamma t+\delta} - \frac{3\gamma^2\hat x\hat y}{(\gamma t+\delta)^2} + \frac{3\gamma^3\hat y^2}{(\gamma t+\delta)^3}\right)$$
$$\times \exp\big(3\lambda_3(y-tx) - \lambda_2 x - (3\lambda_3^2 t^3 + 3\lambda_3\lambda_2 t^2 + \lambda_2^2 t)\big)\big(u + f(t,x,y)\big),$$

$R(\alpha=\delta=1, \beta=\gamma=0) \lhd G^{\rm ess}$, $F(\lambda_3=\lambda_2=\lambda_1=\lambda_0=0) < G^{\rm ess}$.

$R \simeq \mathrm{H}(2,\mathbb R)\times\mathbb Z_2$ and $F \simeq \mathrm{SL}(2,\mathbb R)$, and their Lie algebras coincide with $\mathfrak r \simeq \mathrm{h}(2,\mathbb R)$ and $\mathfrak f \simeq \mathrm{sl}(2,\mathbb R)$.

$R \simeq R_c \times R_d$, $\quad R_c \simeq \mathrm{H}(2,\mathbb R)$, $\quad R_d \simeq \mathbb Z_2$.
$R_c(\sigma>0)$ and $R_d(\lambda_3=\dots=\lambda_0=0, \sigma\in\{-1,1\})$.

The isomorphisms of F to $\mathrm{SL}(2,\mathbb R)$ and of $R_{\rm c}$ to $\mathrm{H}(2,\mathbb R)$ are established by the correspondences

$$(\alpha,\beta,\gamma,\delta)_{\alpha\delta-\beta\gamma=1} \mapsto \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\quad (\lambda_3,\lambda_2,\lambda_1,\lambda_0,\sigma)_{,\ \sigma>0} \mapsto \begin{pmatrix}1 & 3\lambda_3 & -\lambda_2 & \ln\sigma\\ 0 & 1 & 0 & \lambda_0\\ 0 & 0 & 1 & \lambda_1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

$$(\alpha,\beta,\gamma,\delta)_{\alpha\delta-\beta\gamma=1}\mapsto\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\quad (\lambda_3,\lambda_2,\lambda_1,\lambda_0,\sigma)_{,\ \sigma>0}\mapsto\begin{pmatrix}1 & 3\lambda_3 & -\lambda_2 & \ln\sigma\\ 0 & 1 & 0 & \lambda_0\\ 0 & 0 & 1 & \lambda_1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

The natural conjugacy action of the group F on the normal subgroup R

$$(\tilde\lambda_3,\tilde\lambda_2,\tilde\lambda_1,\tilde\lambda_0,\tilde\sigma)^{\mathsf T}=A\,(\lambda_3,\lambda_2,\lambda_1,\lambda_0,\sigma)^{\mathsf T},$$

where $A:=\varrho_3(\alpha,\beta,\gamma,\delta)\oplus(1)$,

$$\varrho_3\colon\ (\alpha,\beta,\gamma,\delta)_{\alpha\delta-\beta\gamma=1}\mapsto\begin{pmatrix}\alpha^3 & \alpha^2\gamma & \alpha\gamma^2 & \gamma^3\\ 3\alpha^2\beta & 2\alpha\beta\gamma+\alpha^2\delta & 2\alpha\gamma\delta+\beta\gamma^2 & 3\gamma^2\delta\\ 3\alpha\beta^2 & 2\alpha\beta\delta+\beta^2\gamma & 2\beta\gamma\delta+\alpha\delta^2 & 3\gamma\delta^2\\ \beta^3 & \beta^2\delta & \beta\delta^2 & \delta^3\end{pmatrix}.$$

Consider the natural projection $\pi : \mathbb{R}^4_{t,x,y,u} \to \mathbb{R}_t$ (onto).

Then $\pi_* G^{\mathrm{ess}}$ coincides with the group of linear-fractional transformations of t, which is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

At the same time, the subgroup F of G singled out by the constraints $\lambda_i = 0$, $i = 0,\dots,3$, $\sigma = 1$ and $f = 0$ is isomorphic to the group $\mathrm{SL}(2,\mathbb{R})$.

The Iwasawa decomposition of this subgroup is given by one-parameter sobgroups of G respectively generated by $\mathcal{P}^t + \mathcal{K}$, $\mathcal{D}$, $\mathcal{P}^t$.
The one-parameter subgroup generated by $\mathcal{P}^t + \mathcal{K}$:

$$\tilde t = \frac{t\cos\epsilon - \sin\epsilon}{t\sin\epsilon + \cos\epsilon}, \quad \tilde x = \frac{x}{t\sin\epsilon + \cos\epsilon} - \frac{3y\sin\epsilon}{(t\sin\epsilon + \cos\epsilon)^2}, \quad \tilde y = \frac{y}{(t\sin\epsilon + \cos\epsilon)^3},$$
$$\tilde u = (t\sin\epsilon + \cos\epsilon)^2 \exp\left(\frac{x^2\sin\epsilon}{t\sin\epsilon + \cos\epsilon} - \frac{3xy\sin\epsilon^2}{(t\sin\epsilon + \cos\epsilon)^2} + \frac{3y^2\sin\epsilon^3}{(t\sin\epsilon + \cos\epsilon)^3}\right) u,$$

Remark

$\mathcal{J}\colon (t,x,y,u) \mapsto (t,-x,-y,u)$

The transformation $\mathcal{J}$ belongs to the one-parameter subgroup of G generated by $\mathcal{P}^t + \mathcal{K}$,

$$\tilde t = \frac{t\cos\epsilon - \sin\epsilon}{t\sin\epsilon + \cos\epsilon}, \quad \tilde x = \frac{x}{t\sin\epsilon + \cos\epsilon} - \frac{3y\sin\epsilon}{(t\sin\epsilon + \cos\epsilon)^2}, \quad \tilde y = \frac{y}{(t\sin\epsilon + \cos\epsilon)^3},$$
$$\tilde u = (t\sin\epsilon + \cos\epsilon)^2 \exp\left(\frac{x^2\sin\epsilon}{t\sin\epsilon + \cos\epsilon} - \frac{3xy\sin\epsilon^2}{(t\sin\epsilon + \cos\epsilon)^2} + \frac{3y^2\sin\epsilon^3}{(t\sin\epsilon + \cos\epsilon)^3}\right) u,$$

and $\epsilon = \pi$ for $\mathcal{J}$, $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$. The value $\epsilon = \pi/2$ corresponds to the transformation

$$\mathcal{K}'\colon \quad \tilde t = -\frac{1}{t}, \quad \tilde x = \frac{x}{t} - 3\frac{y}{t^2}, \quad \tilde y = \frac{y}{t^3}, \quad \tilde u = t^2 \mathrm{e}^{\frac{x^2}{t} - \frac{3xy}{t^2} + \frac{3y^2}{t^3}} u.$$

Corollary

A complete list of discrete point symmetry transformations that are independent up to combining with each other and with continuous point symmetry transformation of this equation is exhausted by the single involution $\mathcal{I}'$ alternating the sign of u, $\mathcal{I}'\colon (t,x,y,u) \mapsto (t,x,y,-u)$. Thus, the factor group of the complete point symmetry group G with respect to its identity component is isomorphic to $\mathbb{Z}_2$.

$$F(t,y,x,t',y',x') = \frac{\sqrt{3}H(t-t')}{2\pi(t-t')^2}\exp\left(-\frac{(x-x')^2}{4(t-t')}-\frac{3\big(y-y'-\frac{1}{2}(x+x')(t-t')\big)^2}{(t-t')^3}\right)$$

The formal application of the point symmetry transformation $\Phi := \mathcal{J}(\ln\frac{\sqrt{3}}{2\pi})\circ\mathcal{P}^0(y')\circ\mathcal{P}^1(x')\circ\mathcal{P}^t(t')\circ\mathcal{K}'$ of the equation (1),

$$\Phi:\quad \tilde{t} = -\frac{1}{t}+t',\quad \tilde{x} = \frac{x}{t}-3\frac{y}{t^2}+x',\quad \tilde{y} = \frac{y}{t^3}+y',$$
$$\tilde{u} = \frac{\sqrt{3}}{2\pi}t^2\exp\left(\frac{x^2}{t}-3\frac{xy}{t^2}+3\frac{y^2}{t^3}+3x'x-3x'\frac{y}{t}+3(x')^2t\right)u,$$

maps the function $u(t,x,y) = 1-H(t)$ to the fundamental solution of the equation (1).

Classification of inequivalent subalgebras

We want to classify all inequivalent Lie reductions of $u_t + xu_y = u_{xx}$. Such classification has never been done before.

We use the Levi decomposition of $\mathfrak{g}^{\rm ess}$, $\mathfrak{g}^{\rm ess} = \mathfrak{f} \Subset \mathfrak{r}$ and decomposition $G^{\rm ess} = F \ltimes R$

The radical $\mathfrak{r} = \langle \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I} \rangle \simeq {\rm h}(2, \mathbb{R})$.

The Levi factor $\mathfrak{f} = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K} \rangle \simeq {\rm sl}(2, \mathbb{R})$.

Patera J., Winternitz P. and Zassenhaus H. Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group, *J. Math. Phys.* **16** (1975), 1597–1614.

Patera J. and Winternitz P., Subalgebras of real three and four-dimensional Lie algebras, *J. Math. Phys.* **18** (1977), 1449–1455.

Popovych R.O., Boyko V.M., Nesterenko M.O. and Lutfullin M.W., Realizations of real low-dimensional Lie algebras, arXiv:math-ph/0301029v7 (2005) (extended and revised version of paper *J. Phys. A* **36** (2003), 7337–7360).

$\mathfrak{g}^{\rm ess} = \mathfrak{f} \Subset \mathfrak{r}$, $G^{\rm ess} = F \ltimes R$

$\mathfrak{f} = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K} \rangle$ is isomorphic to $\mathrm{sl}(2, \mathbb{R})$.
$\mathfrak{r} = \langle \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I} \rangle$ is isomorphic to $\mathrm{h}(2, \mathbb{R})$.

Inequivalent subalgebras of $\mathfrak{f}$: $\{0\}$, $\langle \mathcal{P}^t \rangle$, $\langle \mathcal{D} \rangle$, $\langle \mathcal{P}^t + \mathcal{K} \rangle$, $\langle \mathcal{P}^t, \mathcal{D} \rangle$ and $\mathfrak{f}$ itself.

$\pi_\mathfrak{f}$ and $\pi_\mathfrak{r}$ are the natural projections of $\mathfrak{g}^{\rm ess}$ onto $\mathfrak{f}$ and $\mathfrak{r}$, $\mathfrak{g}^{\rm ess} = \mathfrak{f} \dotplus \mathfrak{r}$

The subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of $\mathfrak{g}^{\rm ess}$ are definitely $G^{\rm ess}$-inequivalent if their projections $\pi_\mathfrak{f}\mathfrak{s}_1$ and $\pi_\mathfrak{f}\mathfrak{s}_2$ are F-inequivalent.

Technique:

- fix the dimension d of subalgebras $\mathfrak{s}$ ($d = 1$ or $d = 2$)
- consider $\mathfrak{s}_\mathfrak{f}$ of $\mathfrak{f}$ from the list of inequivalent subalgebras of $\mathfrak{f}$ with $\dim \mathfrak{s}_\mathfrak{f} \leq \dim \mathfrak{s}$.
- take the set subalgebras d-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ with $\mathfrak{s}$ with $\pi_\mathfrak{f}\mathfrak{s} = \mathfrak{s}_\mathfrak{f}$ and construct a complete list of $G^{\rm ess}$-inequivalent subalgebras in this set.

When $\dim \pi_\mathfrak{f}\mathfrak{s} = 0$ we use the classification of real binary cubics.

Olver P.J., *Classical Invariant theory*, Cambridge University Press, Cambridge, 1999.

$$\mathcal{P}^t=\partial_t,\ \ \mathcal{D}=2t\partial_t+x\partial_x+3y\partial_y-2u\partial_u,$$
$$\mathcal{K}=t^2\partial_t+(tx+3y)\partial_x+3ty\partial_y-(x^2+2t)u\partial_u,$$
$$\mathcal{P}^3=3t^2\partial_x+t^3\partial_y+3(y-tx)u\partial_u,\ \ \mathcal{P}^2=2t\partial_x+t^2\partial_y-xu\partial_u,\ \ \mathcal{P}^1=\partial_x+t\partial_y,$$
$$\mathcal{P}^0=\partial_y,\ \ \mathcal{I}=u\partial_u.$$

Lemma 1

A complete list of $G^{\rm ess}$-inequivalent one-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$

$$\mathfrak{s}_{1.1}=\langle\mathcal{P}^t+\mathcal{P}^3\rangle,\ \ \mathfrak{s}^\delta_{1.2}=\langle\mathcal{P}^t+\delta\mathcal{I}\rangle,\ \ \mathfrak{s}^\nu_{1.3}=\langle\mathcal{D}+\nu\mathcal{I}\rangle,\ \ \mathfrak{s}^\mu_{1.4}=\langle\mathcal{P}^t+\mathcal{K}+\mu\mathcal{I}\rangle,$$
$$\mathfrak{s}^\varepsilon_{1.5}=\langle\mathcal{P}^2+\varepsilon\mathcal{P}^0\rangle,\ \ \mathfrak{s}_{1.6}=\langle\mathcal{P}^1\rangle,\ \ \mathfrak{s}_{1.7}=\langle\mathcal{P}^0\rangle,\ \ \mathfrak{s}_{1.8}=\langle\mathcal{I}\rangle,$$

where $\varepsilon\in\{-1,1\}$, $\delta\in\{-1,0,1\}$, and μ and ν are arbitrary real constants with $\nu\geqslant0$.

Lemma 2

A complete list of G-inequivalent one-dimensional subalgebras of $\mathfrak{g}$ consists of the one-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ listed in Lemma 2 and the subalgebras of the form $\langle\mathcal{Z}(f)\rangle$, where the function f belongs to a fixed complete set of $G^{\rm ess}$-inequivalent nonzero solutions of the equation (1).

Lemma 3

The complete list of $G^{\rm ess}$-inequivalent two-dimensional subalgebras of $\mathfrak g^{\rm ess}$ is given by

$$\mathfrak s^\mu_{2.1}=\langle\mathcal P^t,\mathcal D+\mu\mathcal I\rangle,\quad \mathfrak s^\delta_{2.2}=\langle\mathcal P^t+\delta\mathcal I,\mathcal P^0\rangle,\quad \mathfrak s_{2.3}=\langle\mathcal P^t,\mathcal P^0+\mathcal I\rangle,$$
$$\mathfrak s^\mu_{2.4}=\langle\mathcal D+\mu\mathcal I,\mathcal P^1\rangle,\quad \mathfrak s^\mu_{2.5}=\langle\mathcal D+\mu\mathcal I,\mathcal P^0\rangle,$$
$$\mathfrak s_{2.6}=\langle\mathcal P^0,\mathcal P^1\rangle,\quad \mathfrak s_{2.7}=\langle\mathcal P^0,\mathcal P^2\rangle,\quad \mathfrak s^\varepsilon_{2.8}=\langle\mathcal P^1,\mathcal P^3+\varepsilon\mathcal P^0\rangle,$$
$$\mathfrak s_{2.9}=\langle\mathcal P^t+\mathcal P^3,\mathcal I\rangle,\quad \mathfrak s^\delta_{2.10}=\langle\mathcal P^t,\mathcal I\rangle,\quad \mathfrak s_{2.11}=\langle\mathcal D,\mathcal I\rangle,\quad \mathfrak s_{2.12}=\langle\mathcal P^t+\mathcal K,\mathcal I\rangle,$$
$$\mathfrak s^\varepsilon_{2.13}=\langle\mathcal P^2+\varepsilon\mathcal P^0,\mathcal I\rangle,\quad \mathfrak s_{2.14}=\langle\mathcal P^1,\mathcal I\rangle,\quad \mathfrak s_{2.15}=\langle\mathcal P^0,\mathcal I\rangle,$$

where $\varepsilon\in\{-1,1\}$, $\delta\in\{-1,0,1\}$, and μ is an arbitrary real constant.

$$\mathcal P^t=\partial_t,\quad \mathcal D=2t\partial_t+x\partial_x+3y\partial_y-2u\partial_u,$$
$$\mathcal K=t^2\partial_t+(tx+3y)\partial_x+3ty\partial_y-(x^2+2t)u\partial_u,$$
$$\mathcal P^3=3t^2\partial_x+t^3\partial_y+3(y-tx)u\partial_u,\quad \mathcal P^2=2t\partial_x+t^2\partial_y-xu\partial_u,\quad \mathcal P^1=\partial_x+t\partial_y,$$
$$\mathcal P^0=\partial_y,\quad \mathcal I=u\partial_u.$$

Lemma 4

A complete list of G-inequivalent two-dimensional subalgebras of $\mathfrak{g}$ consists of

1. the two-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ listed in Lemma 3,
2. the subalgebras of the form $\langle \hat Q, \mathcal Z(f)\rangle$, where $\hat Q$ is the basis element of one of the one-dimensional subalgebras $\mathfrak{s}_{1.1}$–$\mathfrak{s}_{1.7}$ of $\mathfrak{g}^{\rm ess}$ listed in Lemma 1, and the function f belongs to a fixed complete set of $\mathrm{St}_{G^{\rm ess}}(\langle \hat Q\rangle)$-inequivalent nonzero $\langle \hat Q+\lambda\mathcal I\rangle$-invariant solutions of the equation (1) with $\mathrm{St}_{G^{\rm ess}}(\langle \hat Q\rangle)$ denoting the stabilizer subgroup of $G^{\rm ess}$ with respect to $\langle \hat Q\rangle$ under the action of $G^{\rm ess}$ on $\mathfrak{g}^{\rm ess}$ and with $\lambda\in\{0,1\}$ if $\hat Q\in\{\mathcal P^1,\mathcal P^0\}$, $\lambda\in\{-1,0,1\}$ if $\hat Q=\mathcal P^t$, $\lambda\geqslant 0$ if $\hat Q=\mathcal P^2+\varepsilon\mathcal P^0$ and $\lambda\in\mathbb R$ otherwise,
3. the subalgebras of the form $\langle \mathcal I, \mathcal Z(f)\rangle$, where the function f belongs to a fixed complete set of $G^{\rm ess}$-inequivalent nonzero solutions of the equation (1), and
4. the subalgebras of the form $\langle \mathcal Z(f^1), \mathcal Z(f^2)\rangle$, where the function pair (f^1, f^2) belongs to a fixed complete set of $G^{\rm ess}$-inequivalent (up to linearly recombining of components) pairs of linearly independent solutions of the equation (1).

Thanks

Thank you for your attention!