Finding the "geometry" of the heart and brain with moving frames

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Symmetry, Invariants, and their Applications:
A Celebration Peter Olver's 70th Birthday
3rd August, 2022

My research background

Computational electromagnetics (Ph.D.)
(EM propagation on metamaterials)

Cardiac electrophysiology (Postdoc)
(AP propagation around the PVs)

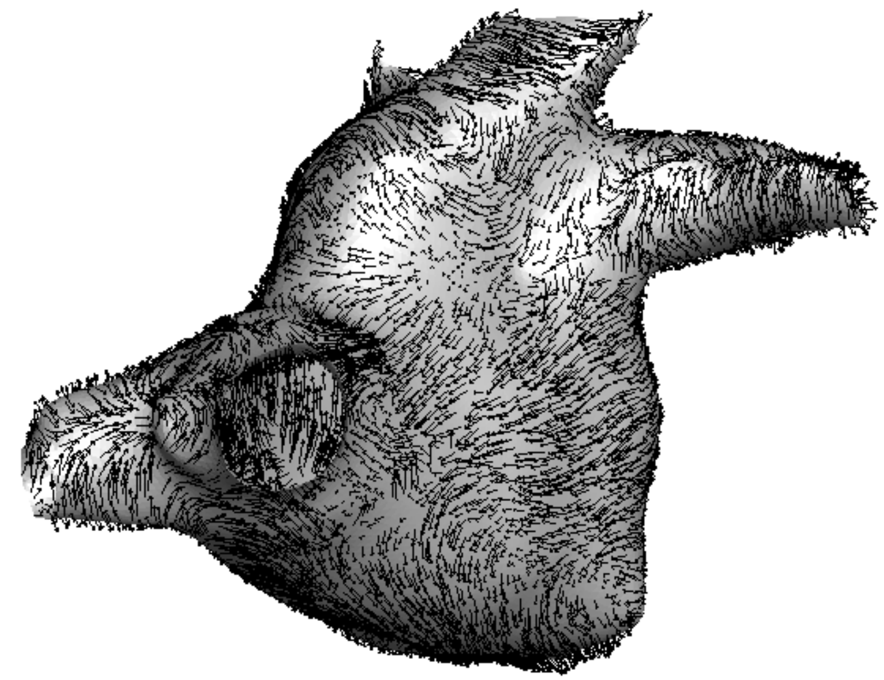
Neuroscience (Recent)
(White matter tractography)

Solely depends → Geometry

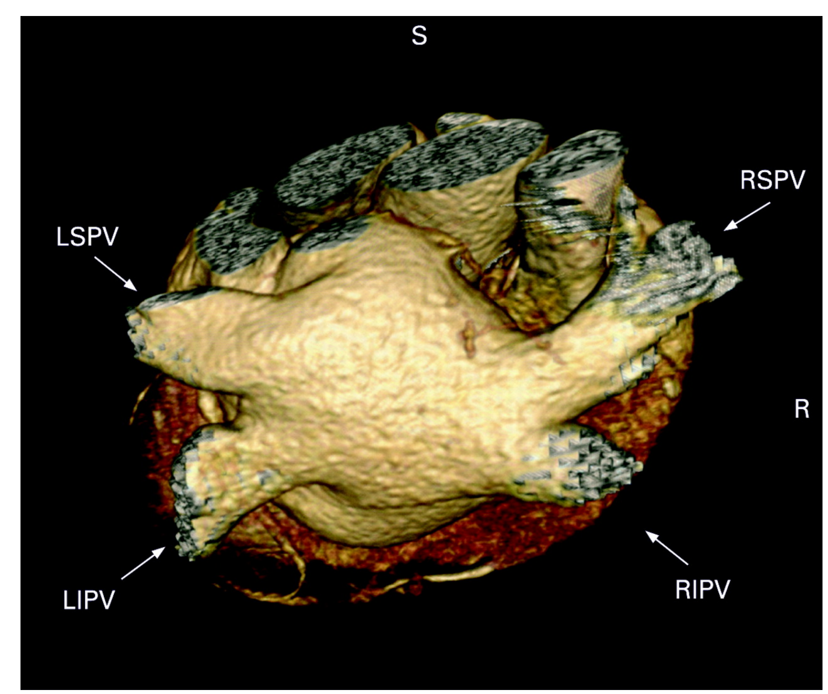
Signal propagation = information propagation
= fluid dynamics with massless particle

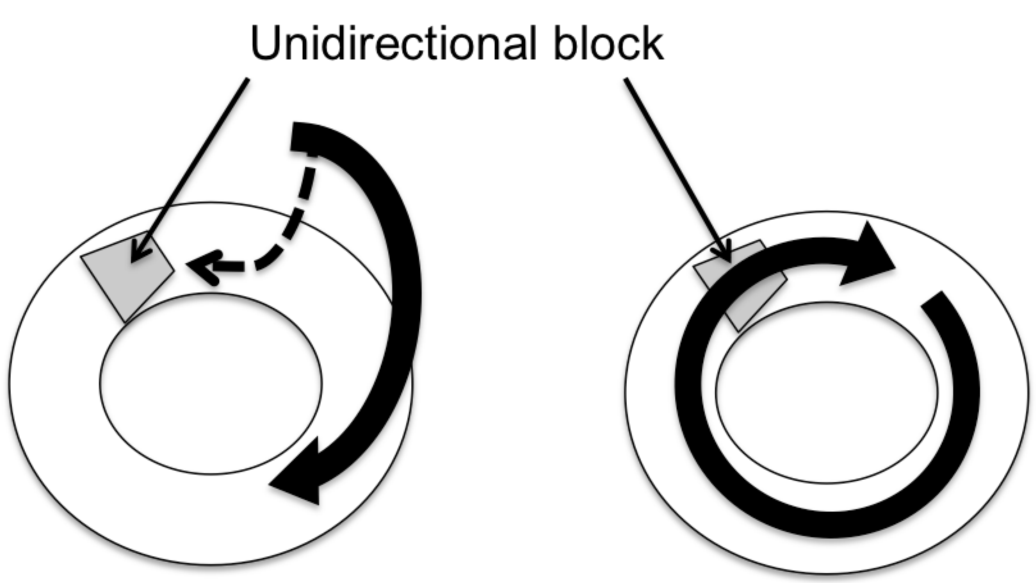
Important facts/questions in cardiac electrophysiology

- Many pathologies are related with conduction block (region where the electric conduction fails).
- Analyzing and prediction the conduction block in the heart is challenging because the heart is filled with complex anisotropic fiber
- Mathematical scheme should be derived to find or predict the conduction block region in the multidimensional complex space



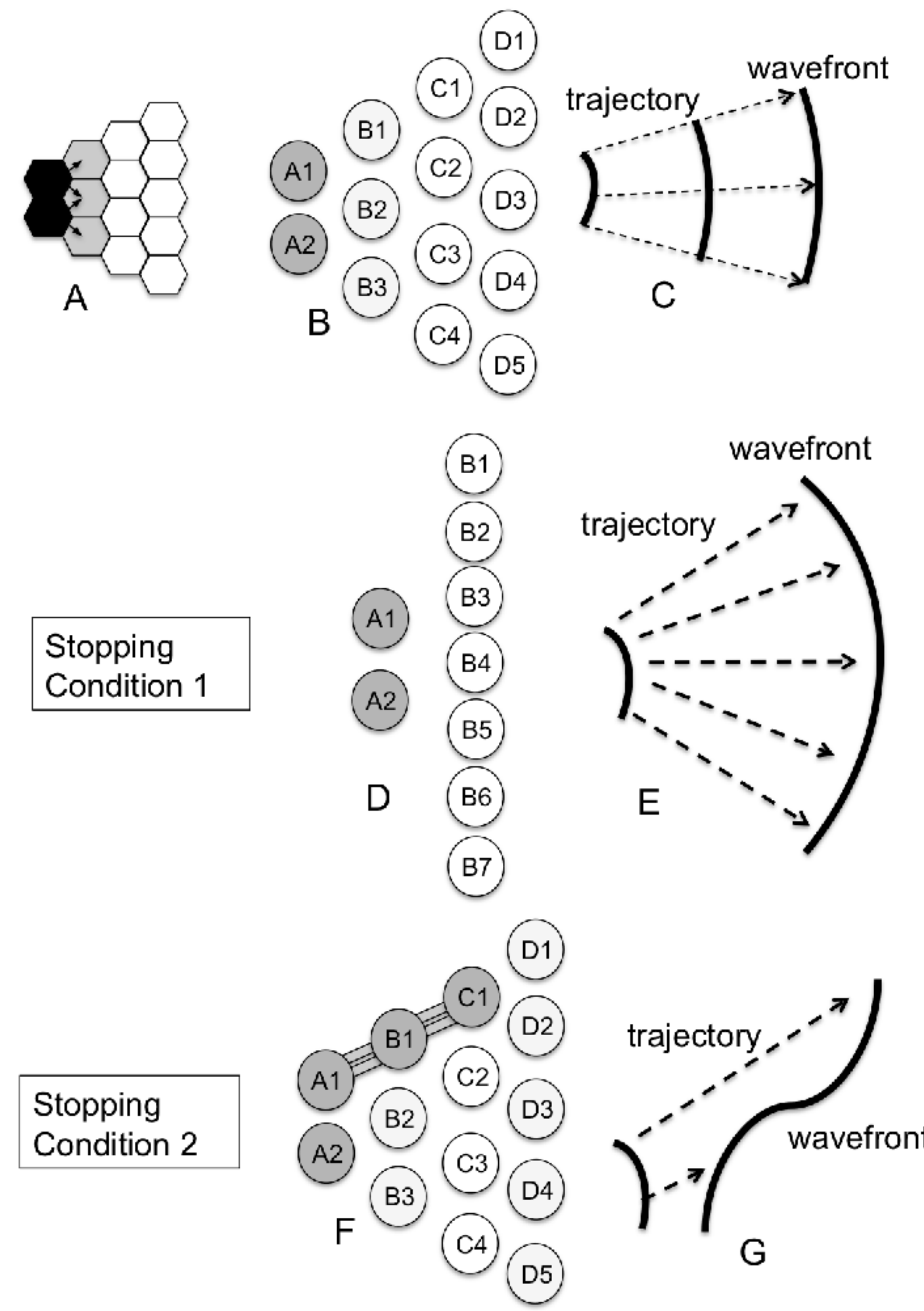
Atrium





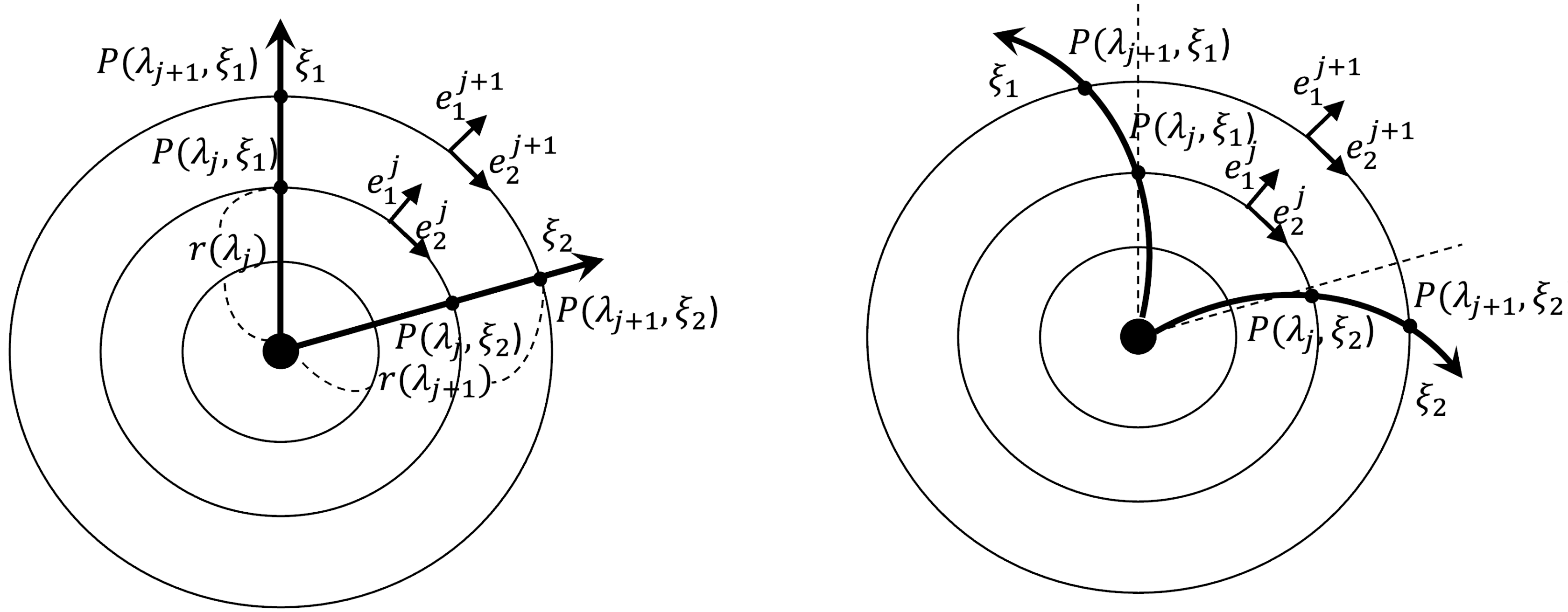
Wave trajectory to represent source/sink ratio

- The relative acceleration is related to the **divergence and convergence** of the trajectory of a wave propagation.
- In cardiac electric signal propagation, a component of this relative acceleration represents the **ratio between the sink and source**(Chun2014a).
- Conduction block represents the status that ions are not sufficient for electric signal propagation



Courtesy from Chun2014a

A large relative acceleration causes conduction block

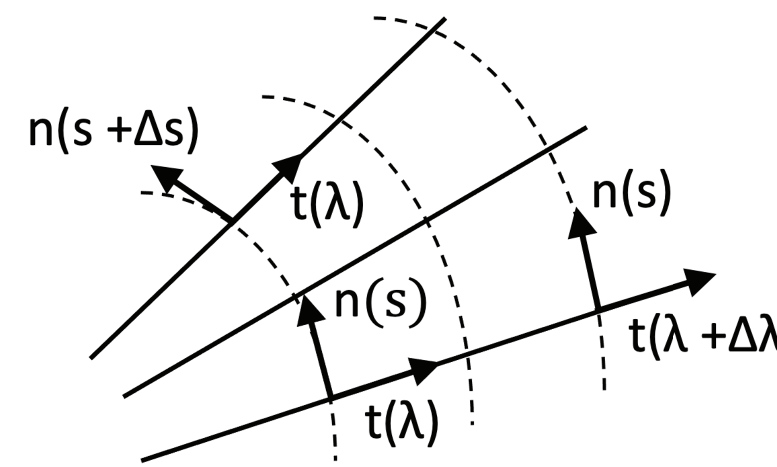


- A relative acceleration is the same as the Riemann curvature tensor.

$$\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\mathbf{n} = \mathcal{R}(\cdots, \mathbf{u}, \mathbf{n}, \mathbf{u})$$

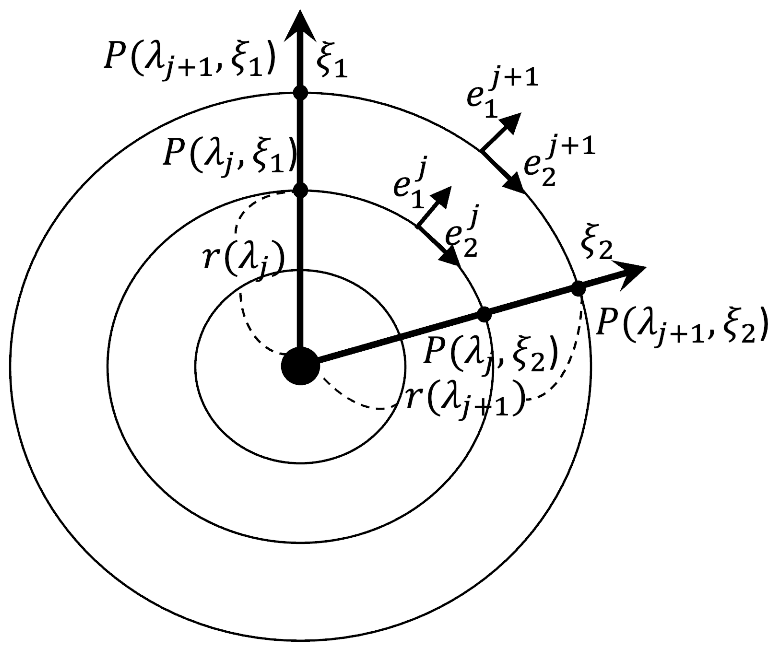
$$\mathbf{u} = \text{tangent vector of the trajectory}$$

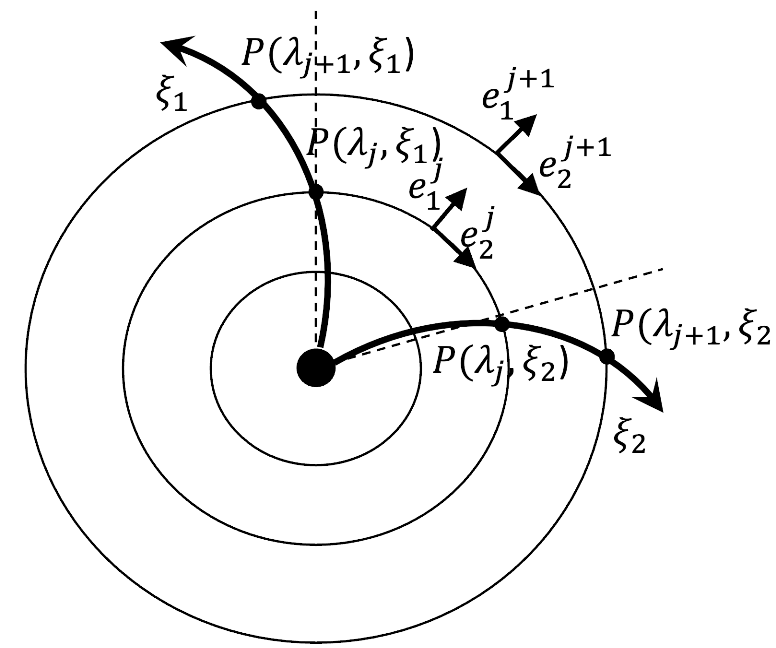
$$\mathbf{n} = \text{separation vector}$$



- Conduction block corresponds to the instantaneous low ion density.
- The cardiac tissue without ions absorbs all the signals and not emits signal, so it is a kind of ‘black hole’.

Relative acceleration in moving frames





Suppose $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be orthonormal basis vectors constructed everywhere

Let $\mathbf{e}_1$ is aligned along $\mathbf{u}$ and $\mathbf{e}_2$ is aligned along $\mathbf{n}$

Then, what can we say about $\nabla_{\mathbf{e}_1} \nabla_{\mathbf{e}_1} \mathbf{e}_2$ in relation to $\nabla_{\mathbf{u}} \nabla_{\mathbf{u}} \mathbf{n}$?

> Suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit tangent vector of the trajectory $\mathbf{u}$ and the separation vector $\mathbf{n}$, respectively. Then, the covariant derivative $\nabla_{\mathbf{n}} \mathbf{u}$ has the same sign as $\nabla_{\mathbf{e}_2} \mathbf{e}_1$.

Relative acceleration for conduction failure of cardiac electric propagation on a 2D general curved surface, in submission, 2022

Relative acceleration of moving frames for conduction block

Consider a sufficiently smooth two-dimensional curved domain. Suppose the moving frames of the first order $\mathbf{e}_i$ are constructed in this domain such that the first tangent vector $\mathbf{e}_1$ is aligned along the propagational direction as the gradient of the action potential. Suppose that $\nabla_{\mathbf{e}_2}\mathbf{e}_1 \cdot \mathbf{e}_2$ is negative and the relative acceleration is positive and sufficiently large for trajectory divergence as follows.

$$\nabla_{\mathbf{e}_2}\mathbf{e}_1 \cdot \mathbf{e}_2 < 0 \quad \text{and} \quad \nabla_{\mathbf{e}_1}\nabla_{\mathbf{e}_1}\mathbf{e}_2 \cdot \mathbf{e}_2 > 0. \tag{1}$$

Then, the propagation stops due to sink and source mismatch and the corresponding region is likely to develop conduction block.

In words, if the trajectory currently diverges and accelerates to diverge furthermore, then conduction block occurs at that region

Numerical algorithm in plain words

1. Construct moving frames lying on a surface without any preference direction on the tangent plane.

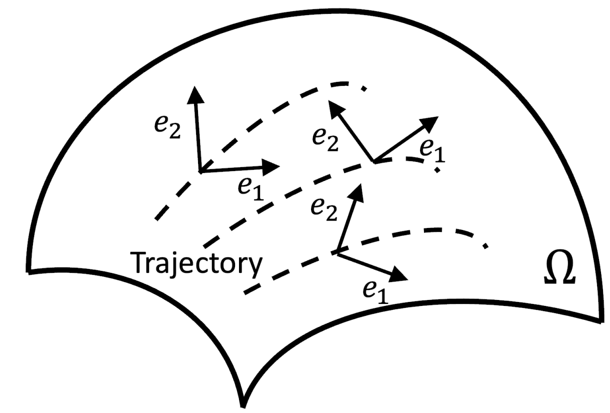
2. Solve diffusion-reaction equations.

3. Derive the propagational direction by computing the gradient of the action potential.

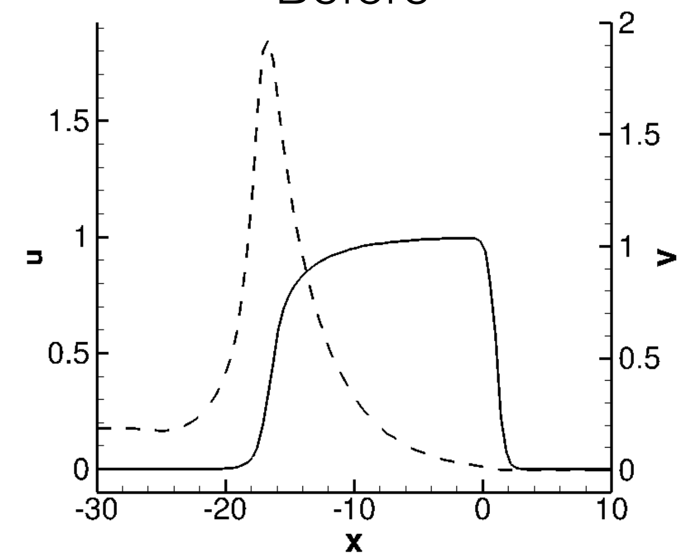
4. Align the first tangent vector of the moving frames along the propagational direction.

5. Compute the connection (geometry of trajectory).

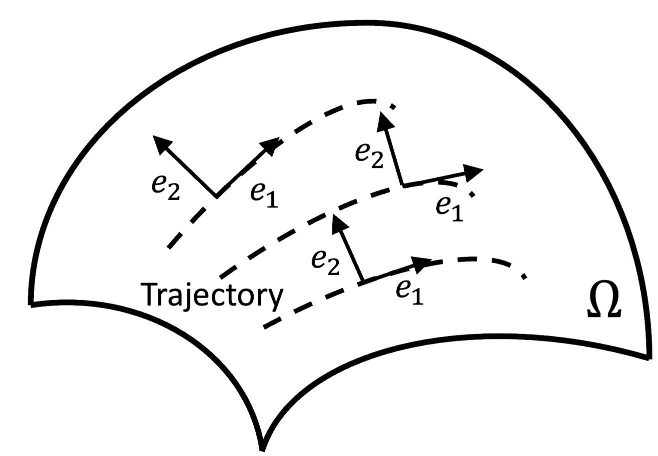
6. Finally, we compute the relative acceleration such that a large relative acceleration corresponds to the conduction region.



Before



action potential



After

Covariant derivatives by moving frames

1) By the tangent vectors

$$\boldsymbol{\nabla}_j \mathbf{u} = \boldsymbol{\nabla}_{x^j} \mathbf{u} = \sum_{i=1}^{2} \left[\frac{\partial \widetilde{u}^i}{\partial x^j} + \sum_{k=1}^{2} \Gamma^i_{kj} \widetilde{u}^k \right] \frac{\partial}{\partial x^i},$$

where

$$\Gamma^k_{ij} = \sum_{m=1}^{2} \frac{g^{km}}{2} \left[\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right]$$

2) By moving frames

$$\boldsymbol{\nabla}_{\mathbf{e}_j} \mathbf{u} = \sum_{i=1}^{2} \left[\nabla u^i \cdot \mathbf{e}_j + \sum_{k=1}^{2} \omega_{ik}(\mathbf{e}_j) u^k \right] \mathbf{e}_i$$

where

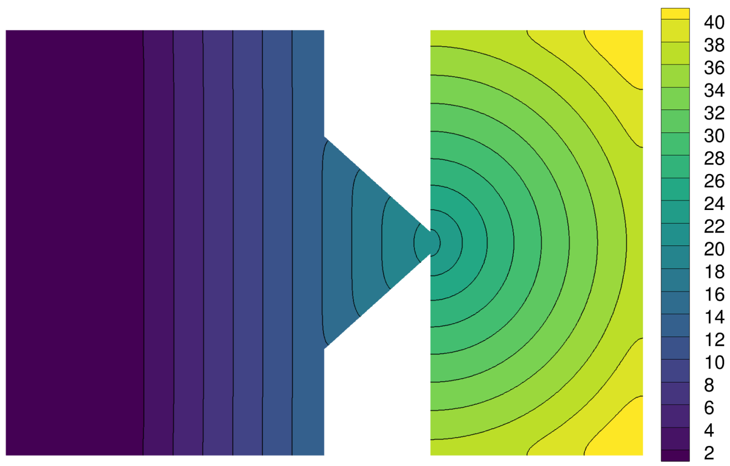
$$\omega_{ij}(\mathbf{e}_k) = [e^x_j \; e^y_j \; e^z_j] \begin{bmatrix} \nabla e^x_i \cdot \mathbf{e}_k \\ \nabla e^y_i \cdot \mathbf{e}_k \\ \nabla e^z_i \cdot \mathbf{e}_k \end{bmatrix}.$$

On the sphere of the unit radius, the covariant derivative of $\mathbf{u}$ along a vector $\mathbf{v}$ by using the Christoffel symbol is the same as that by using the connection 1-form

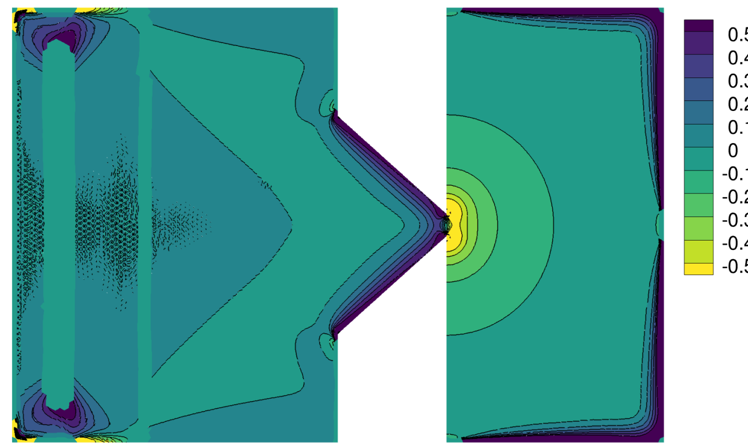
\High-order method with moving frames to compute the covariant derivatives of vectors on general 2D curved surfaces, in revision, 2022

Examples

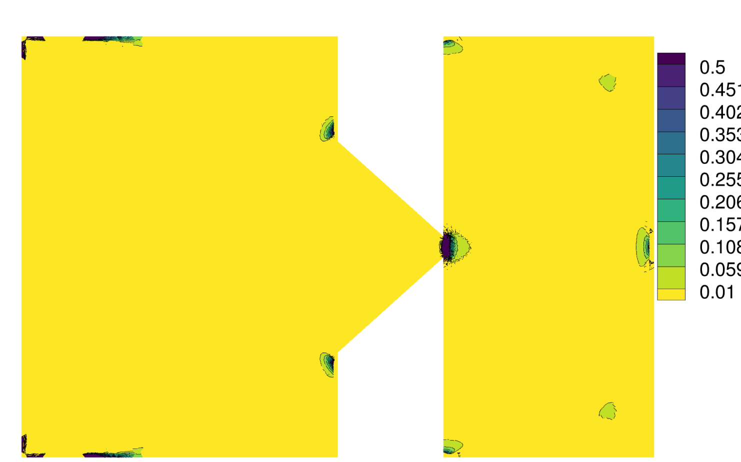
A) Propagation through a small gap



Arrival time map



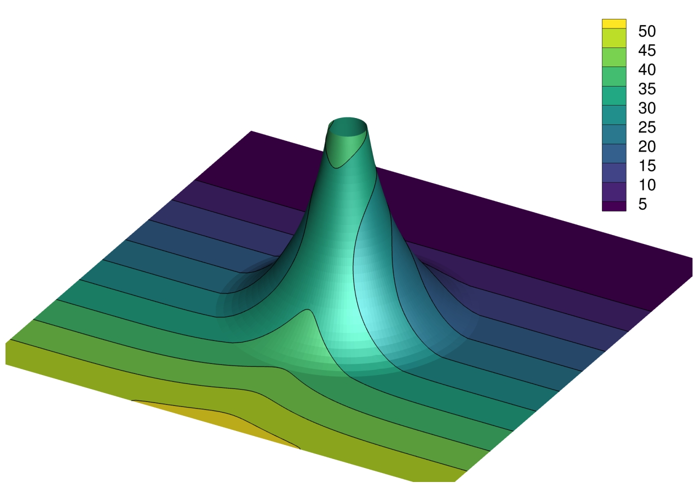
Connection



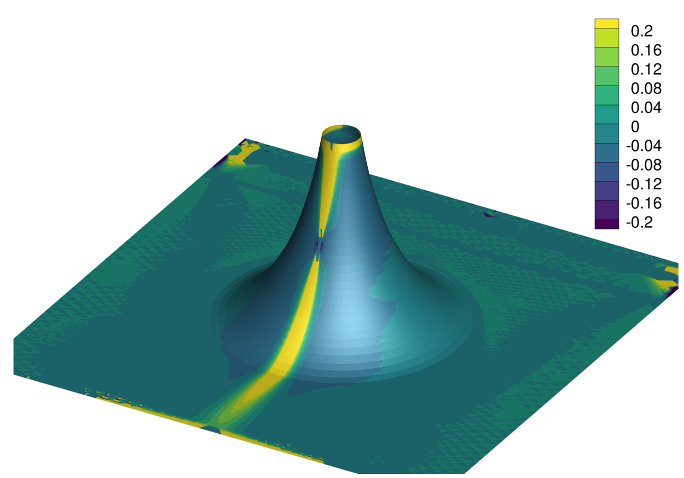
Conduction block

B) Propagation through a pseudosphere

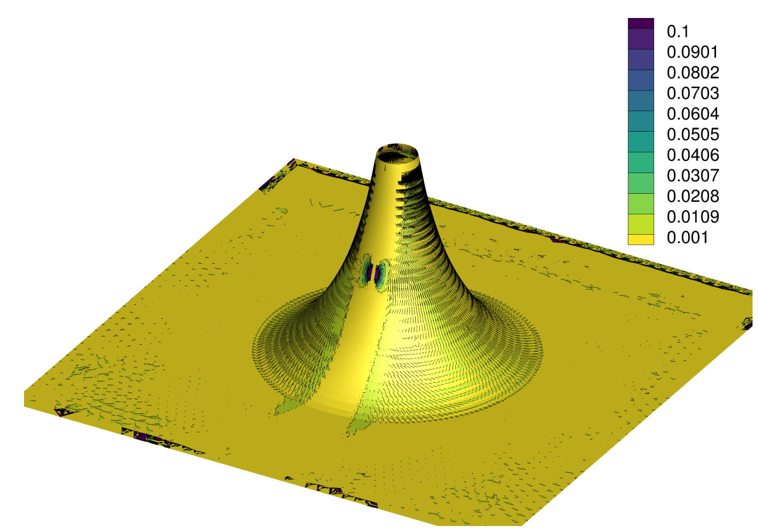
(a model of pulmonary vein)



Arrival time map



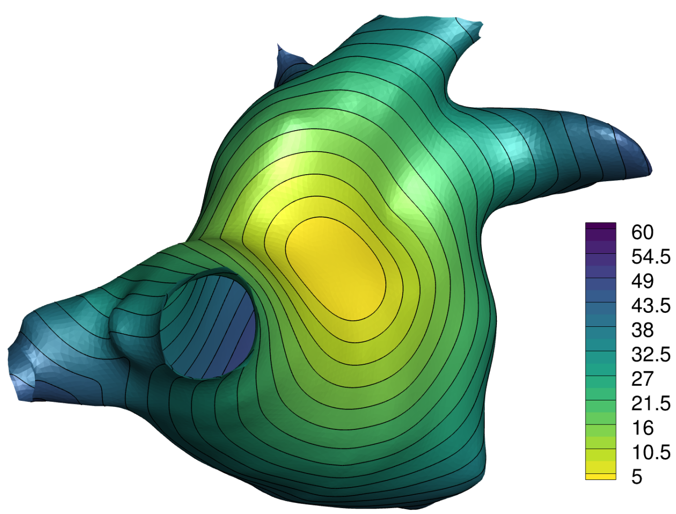
Connection



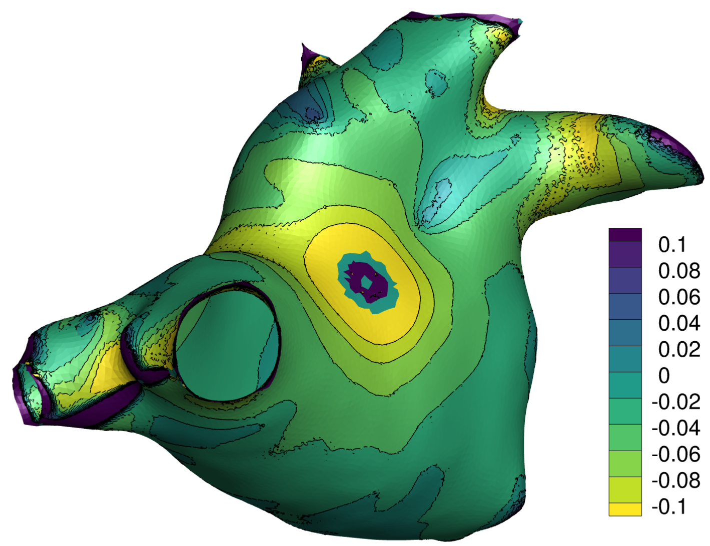
Conduction block

Application to 2D atrium (I)

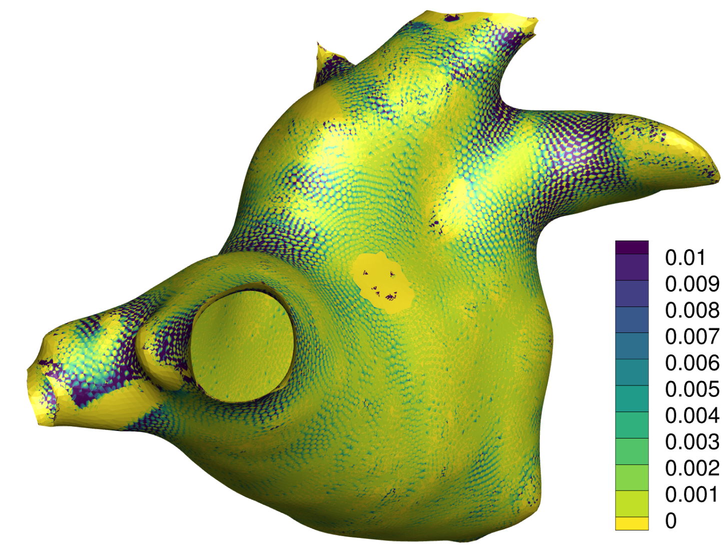
A) Isotropic heart



Arrival time map

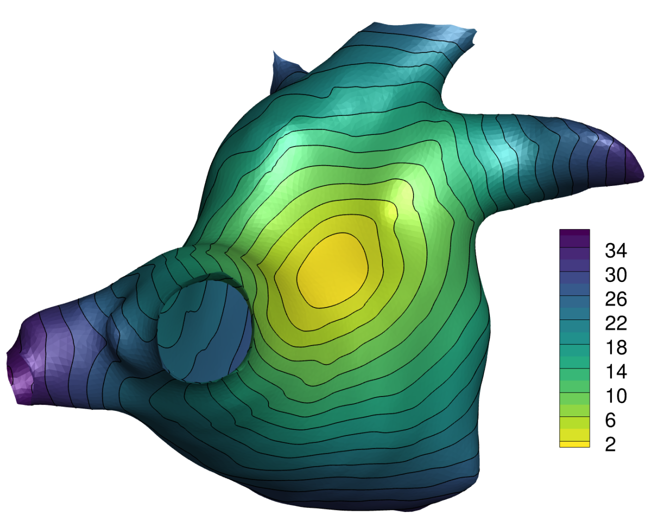


Connection

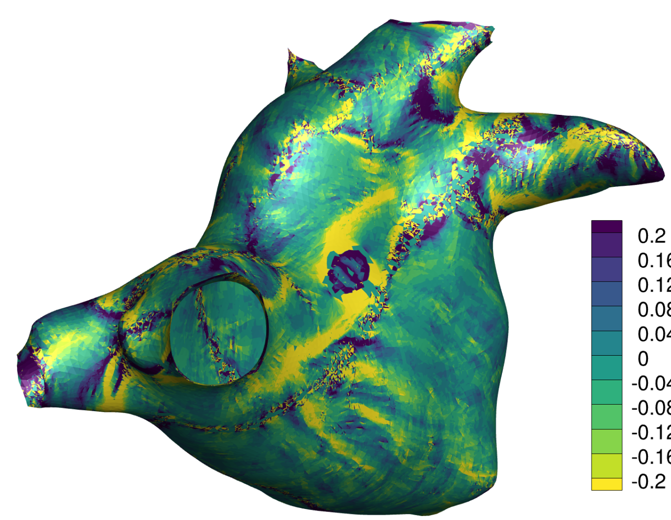


Conduction block

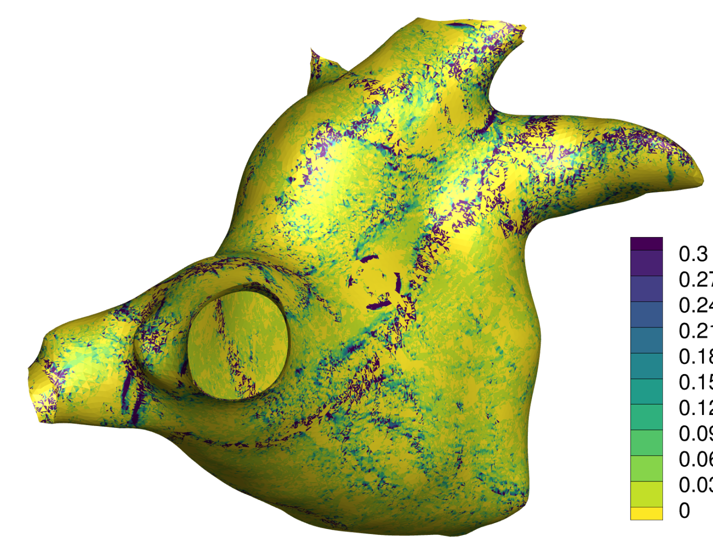
B) Anisotropic heart



Arrival time map



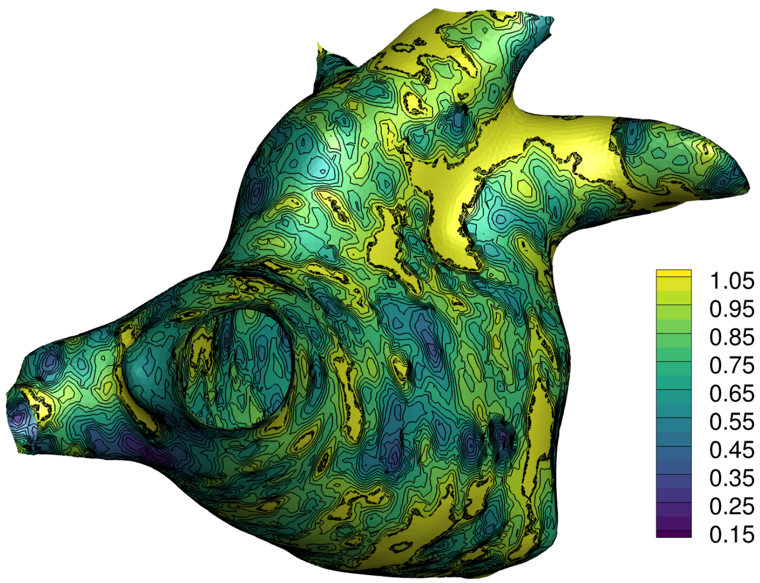
Connection



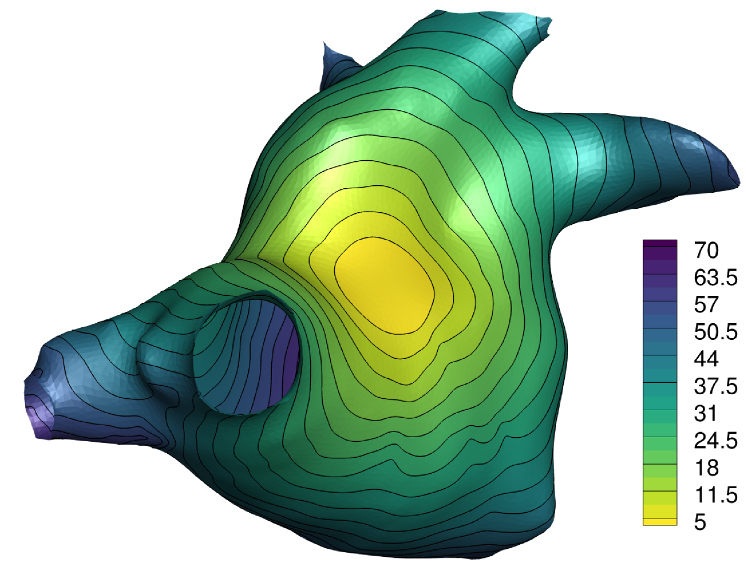
Conduction block

Application to 2D atrium (II)

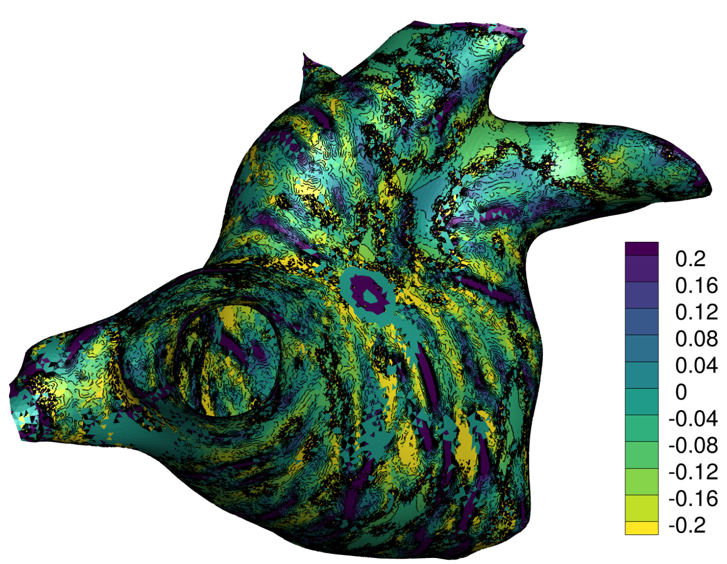
Isotropic hear with various conductivity variations (with fibrosis)



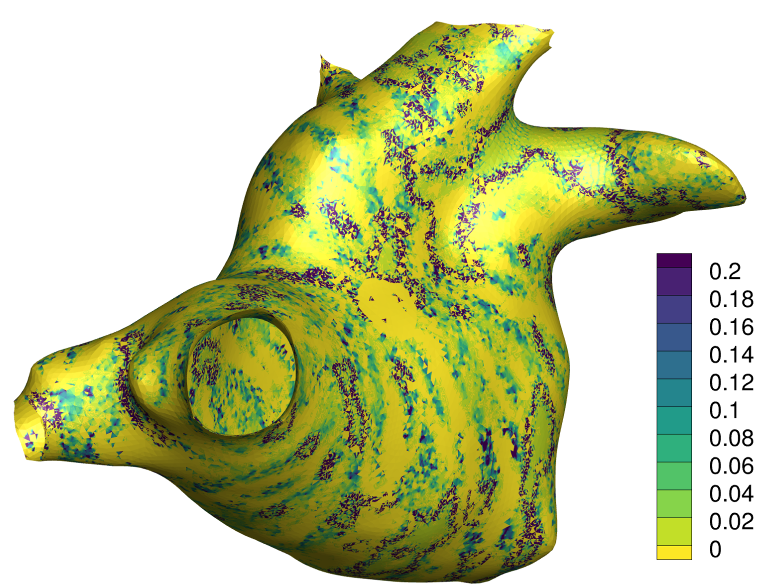
Conductivity map



Arrival time map



Connection

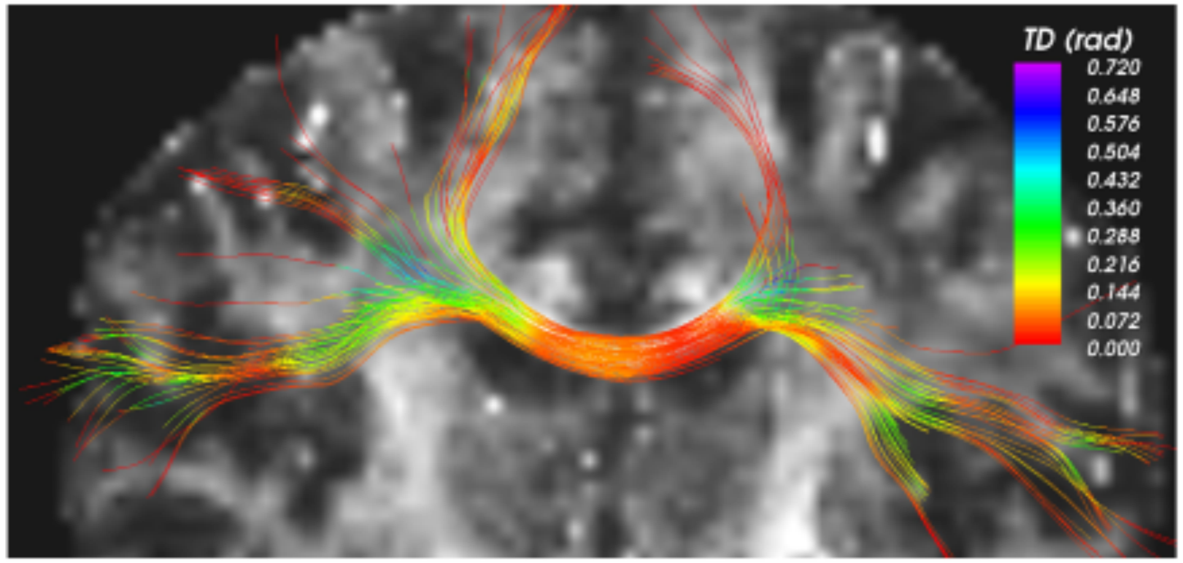


Conduction block

Application to Neuroscience with moving frames

- Contrary to the heart, the brain's connectivity is supposed to be flexible according to experience, learning, and environment.
- However, some indications exist to suggest that there is intrinsic geometric features in the brain.
- Ex) Conscience, Conceptualization, Resonant brain etc…

Gigandet X et. a. 2008

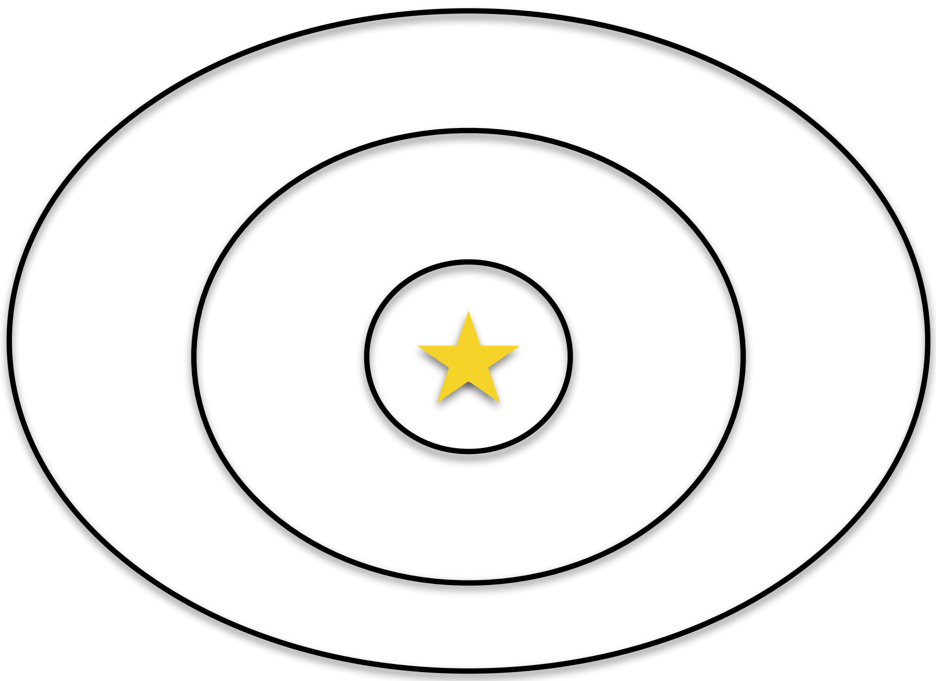


Courtesy from Savadjiev2014

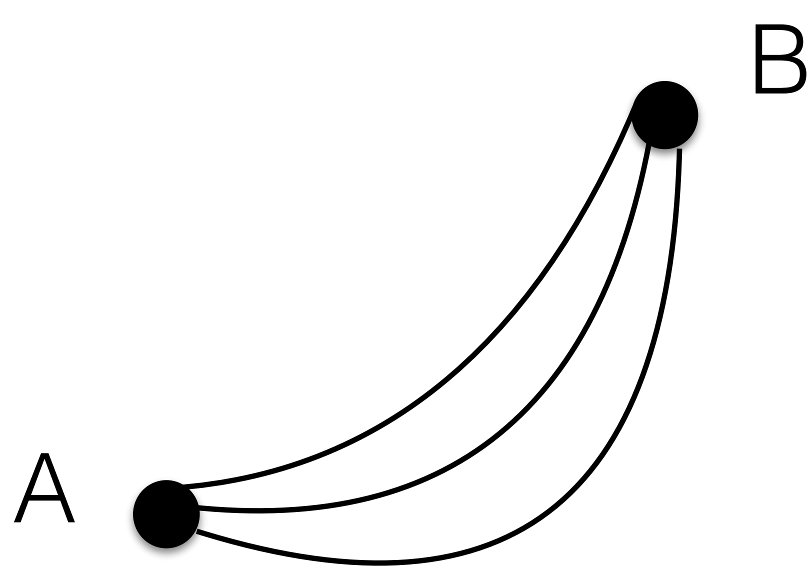
Brain’s 3Ns,
and its solution by moving frames

- **Nonlinear**: Neural function is not linear, so does not add up.
- **Nonlocal**: Neural function requires local and global connectivity at the same time.
- **Nonstationary**: Some neural functions change over time

A. Nonlinear - two pairs of moving frames (I)



- A light propagates with the same intensity all the directions, regardless of the distance it travels.
- However, the energy at a very large radius does not mean that the initial light has an enormous energy.

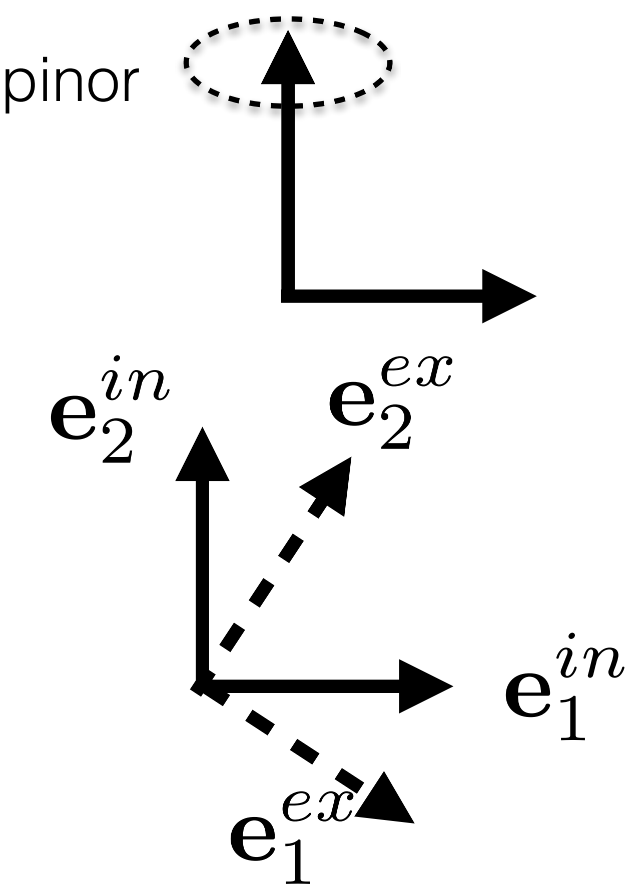


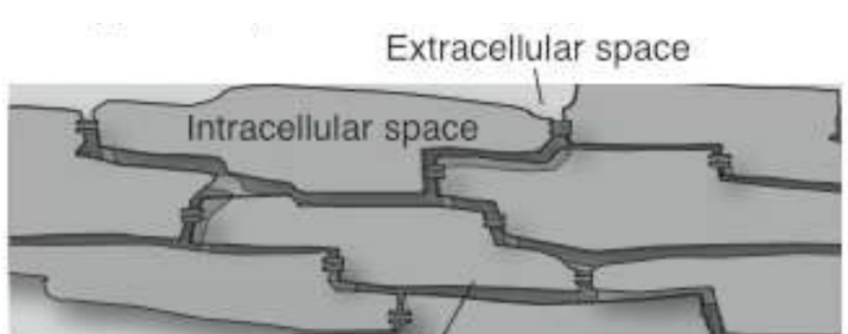
- For three different length of neural fiber connecting the same end points A and B, the initial signal starting at A along each neural fiber reaches B at the same time.

A. Nonlinear - two pairs of moving frames (II)

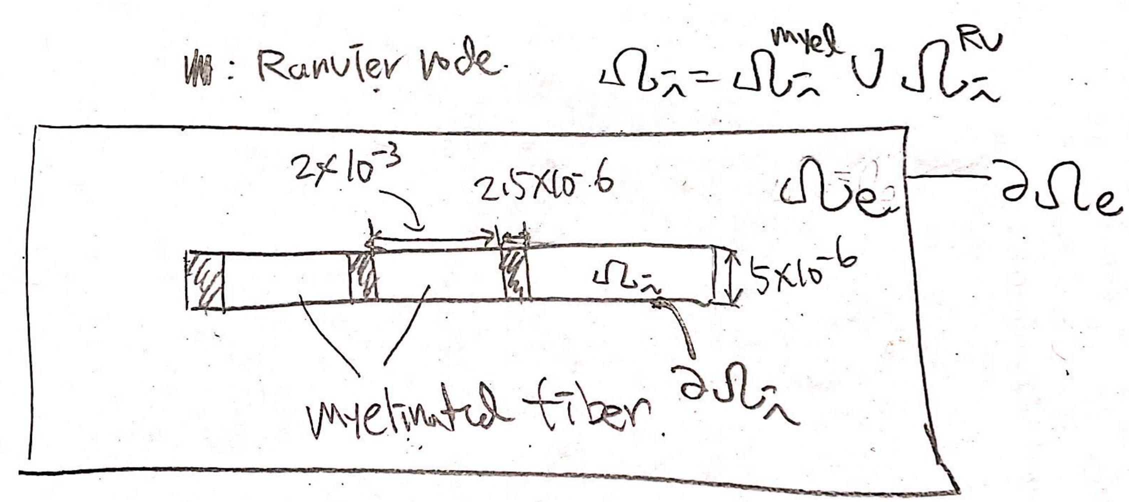
- The energy should be provided by the medium. For example, in the physical space, it is powered by particles' electrons and photons.

- In the biological space, the light is the voltage difference between the two spaces (intercellular domain and extracellular domain). Thus, nonlinear phenomena should be analyzed by two moving frames at each point.





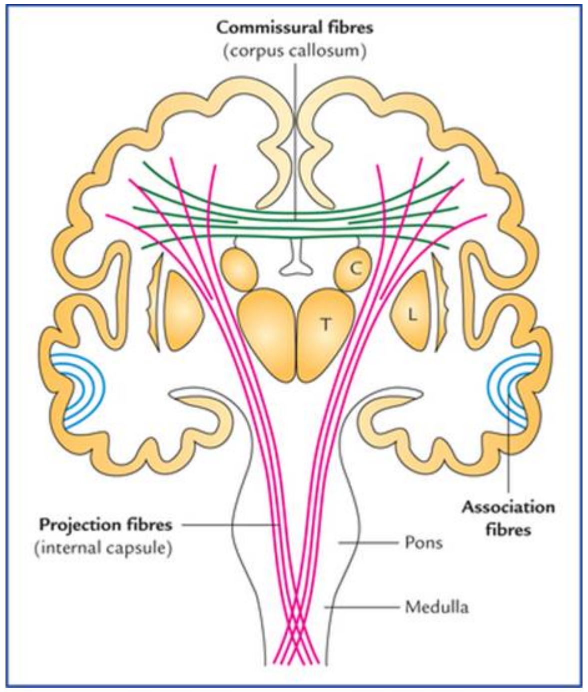
cardiac tissue



neural fiber

B. Nonlocal - a new connection

- White matter fiber provides a long range and fast connectivity between the distanced compartments.
- Different meaning of ‘neighbor’ and consequently different definition of ‘derivatives’.
- How do we define ‘connection’ ?

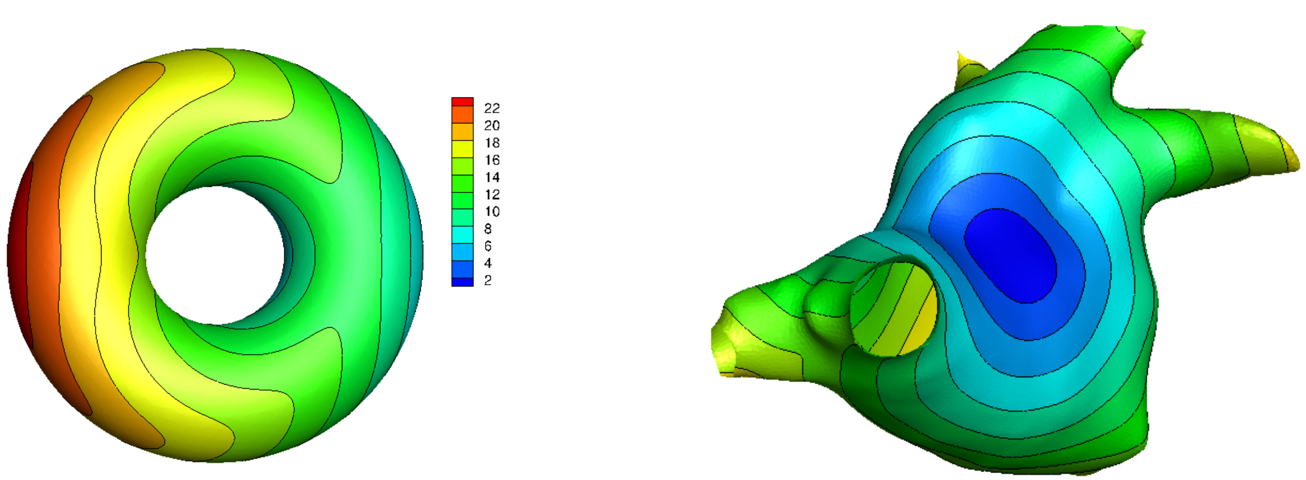


Textbook of Clinical Neuroanatomy

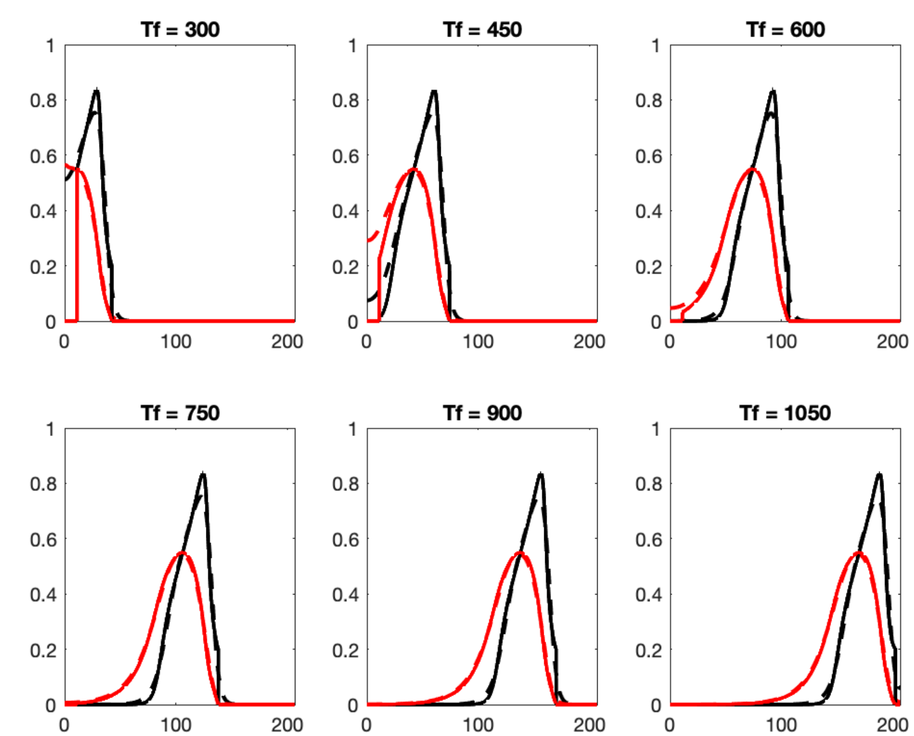
C. Nonstationary - What is time? (I)

- One of the modern concept of time is the 'sequence of events'.
- In the signal propagation, it can be considered the arrival time of the signal. We call it time map.
- The orthogonal direction to time map shows the 'causal relation' between excited and excitable units.
- The diffusion-reaction equations can be boiled down to ODEs such as

PDEs = ODEs + Time Map



Time map of point-initiated torus (left) and atrium (right)



ODE (dashed) vs. PDE (solid)

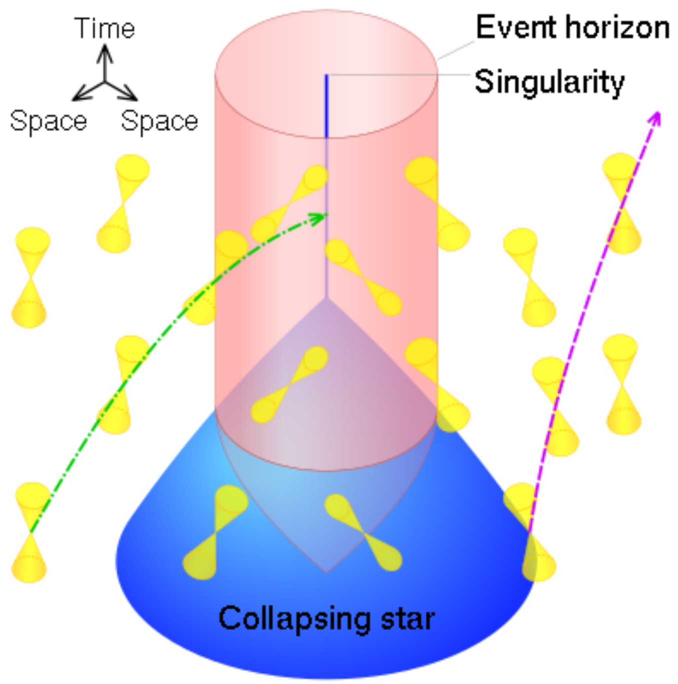
C. Nonstationary - What is time? (II)

- Instead of adding more dimension to space dimension for time, time domain can be embedded in space domain such that

$$\mathbf{e}_t = a\mathbf{e}_1 + b\mathbf{e}_2, \quad a, b \in \mathbf{R}$$

- 1) This representation might express elegantly time-dependent conduction phenomena, such as memory, dynamic conduction block, resonance etc.

- 2) Light cone for global causal relationship



Thank you !