The geometry of Cartan's realization problems

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Symmetry, Invariants and their Applications
A Celebration of Peter Olver's 70th Birthday
Halifax, August, 2022

University of Minnesota c. 1990

Some words from our guru:

Chapter 8

Equivalence of Coframes

By definition, a coframe on a manifold is a "complete" collection of one-forms in the sense that, at each point, it provides a basis for the cotangent space. Two coframes are said to be equivalent if they are mapped to each other by a diffeomorphism. The equivalence problem for coframes is, in fact, the most important of the equivalence problems that we are to treat, because it ultimately includes all the others as special cases. Indeed, the remarkable and powerful Cartan equivalence method, [**39**], [**78**], [**230**], provides an *explicit*, *practical* algorithm for reducing most other equivalence problems to an equivalence problem for a suitable coframe. (The exceptions are those problems admitting an infinite dimensional symmetry group, which will be handled by similar, but more sophisticated methods to be discussed later on.) Therefore, it is crucial that we learn how to deal properly with this apparently special, but, in reality, quite general equivalence problem. In this chapter, we

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Based on joint work with Peter's grand child **Ivan Struchiner** (USP):

- The Classifying Lie Algebroid of a Geometric Structure I: Classes of Coframes. *Transactions of the AMS* **366** (2014), 2419–2462.
- The Classifying Lie Algebroid of a Geometric Structure II: G-structures with connection. *São Paulo J. Math. Sci.* **15** (2021), 524–570.
- The Global Solutions to a Cartan's Realization Problem `arXiv:1907.13614`.

Inspired by:

- R. Bryant, Bochner-Kähler metrics. *J. Amer. Math. Soc.* **14** (2001), 623-715.
- P. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, UK, 1995.

Overview

Starting from the classical correspondence:

$$\text{Geometric structures} \quad \longleftrightarrow \quad G\text{-structures(with connection)}$$

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The main steps of the program:

Classification problem for class of geometric structures	$\longleftrightarrow$	G-structure algebroid (with connection)
Solutions to classification problem	$\longleftrightarrow$	Integrate G-structure algebroid to G-structure groupoid (with connection)

Equivalence of G-structures

Notation:

- M – manifold with $\dim M = n$
- $\pi : \mathrm{F}(M) \to M$ – **frame bundle** of M:

$$\pi^{-1}(x) = \{p : \mathbb{R}^n \to T_x M : \text{ linear isomorphism}\}$$

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Given G-structures $P_1 \subset \mathrm{F}(M_1)$ and $P_2 \subset \mathrm{F}(M_2)$, a G-equivalence is a diffeomorphism $\phi : M_1 \to M_2$ such that:

$$\phi_*(P_1) = P_2.$$

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$$\theta_p(\xi) = p^{-1}(\mathrm{d}_p\pi(\xi)) \quad (p \in \mathrm{F}(M))$$

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Theorem

A principal bundle isomorphism

$$\begin{tikzcd} P_1 \arrow[r, "\Phi"] \arrow[d] & P_2 \arrow[d] \\ M_1 \arrow[r, "\varphi"'] & M_2 \end{tikzcd}$$

is an equivalence of G-structures if and only if $\Phi^*\theta_2 = \theta_1$.

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1. *Strongly horizontal:* $\theta_p(v) = 0$ iff $v = \widetilde{\alpha}|_p$, for some $\alpha \in \mathfrak{g}$
2. *G-equivariant:* $g^*\theta = g^{-1} \cdot \theta$, for all $g \in G$
3. *Pointwise surjective:* $\theta_p : T_p\mathrm{F}_G(M) \to \mathbb{R}^n$

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Theorem

If $\pi : P \to M$ is a G-principal bundle with $G \subset \mathrm{GL}_n(\mathbb{R})$ and $\theta_P \in \Omega^1(P, \mathbb{R}^n)$ satisfies 1-3, then there exists a unique embedding of principal bundles

$$i : P \hookrightarrow \mathrm{F}(M), \qquad i^*\theta = \theta_P.$$

A form θ_P satisfying 1-3 will also be call a **tautological form**.

Examples of G-structures

- Coframes $\Longleftrightarrow$ $\{e\}$-structures;
- Riemannian structures $\Longleftrightarrow$ O_n-structures;
- Almost symplectic structures $\Longleftrightarrow$ Sp_n-structures;
- Almost complex structures $\Longleftrightarrow$ $\mathrm{GL}_n(\mathbb{C})$-structures;
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- Integrable O_n-structures $\Longleftrightarrow$ flat Riemannian structures;
- Integrable Sp_n-structures $\Longleftrightarrow$ symplectic structures ;
- Integrable $\mathrm{GL}_n(\mathbb{C})$-structures $\Longleftrightarrow$ complex structures;
- Integrable U_n-structures $\Longleftrightarrow$ flat Kähler structures.

Connections on G-structures

Definition

A **connection** on $P \to M$ is a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying:

1. *Vertical:* $\omega(\widetilde{\alpha}) = \alpha$, for all $\alpha \in \mathfrak{g}$;
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If (P, θ, ω) is a G-structure with connection:

$$(\theta, \omega)_p : T_pP \xrightarrow{\simeq} \mathbb{R}^n \oplus \mathfrak{g}$$

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$\Longrightarrow$ (θ, ω) is a **coframe** which satisfies the structure equations:

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where:

- $T : P \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathbb{R}^n)$ – **torsion**
- $R : P \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathfrak{g})$ – **curvature**

Equivalence of G-structures with connection

$(P_i, \theta_i, \omega_i)$ – G-structures with connection

Definition

A (local) equivalence is a (local) G-bundle isomorphism $\phi : P_1 \to P_2$ which preserves the coframes:

$$\phi^*\theta_2 = \theta_1, \quad \phi^*\omega_2 = \omega_1.$$

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Equivalence problem:

- When are two G-structures with connection (locally) equivalent?

Some more words from our guru:

to each other by a diffeomorphism $\Phi\colon M \to \overline{M}$, so that

$$\Phi^* \overline{\theta}^i = \theta^i, \qquad i = 1, \ldots, m. \tag{8.7}$$

In our treatment, we shall only look at *local equivalence*, meaning that the diffeomorphism Φ is only required to be defined on a suitable open subset of M — see the end of Chapter 14 for some remarks on the global equivalence problem. Cartan made the fundamental observation that the invariance of the exterior derivative operator d under smooth maps, cf. Theorem 1.36, is the key to the solution of the coframe equivalence problem. Thus, if (8.7) holds, we also necessarily have

$$\Phi^* \, d\overline{\theta}^i = d\theta^i, \qquad i = 1, \ldots, m. \tag{8.8}$$

The solution to the equivalence problem for coframes lies in the detailed analysis of the differentiated conditions (8.8).

Invariant forms

(P, θ, ω) – G-structure with connection

Definition

A form $\alpha \in \Omega^k(P)$ is called invariant if for every local equivalence $\phi : P \to P$:

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We denote by $\Omega^\bullet(P, \theta, \omega)$ the space of invariant forms.

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Basic Remark: $\Omega^\bullet(P, \theta, \omega)$ is preserved under exterior differentiation.

$$\alpha \in \Omega^\bullet(P, \theta, \omega) \quad \Longrightarrow \quad \mathrm{d}\alpha \in \Omega^\bullet(P, \theta, \omega)$$

Fully regular G-structures

Definition

(P, θ, ω) is called fully regular when the spaces

$$\left\{ \mathrm{d}_p I : I \in \Omega^0(P, \theta, \omega) \right\} \subset T_p^* P \Big\},$$

have constant dimension (independent of $p \in P$).

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If (P, θ, ω) is fully regular, $\exists$ space $X_{(\theta,\omega)}$ & submersion $h : P \to X_{(\theta,\omega)}$ so that:

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In fact, more is true:

Proposition

For a fully regular G-structure with connection (P, θ, ω):

$$\Omega^\bullet(P, \theta, \omega) \simeq \Omega^\bullet(A_{(\theta,\omega)}) := \Gamma(\wedge^\bullet A^*_{(\theta,\omega)}),$$

where $A_{(\theta,\omega)} = X_{(\theta,\omega)} \times (\mathbb{R}^n \oplus \mathfrak{g}) \to X_{(\theta,\omega)}$.

Lie algebroids and G-structures

It follows that for a fully regular (P, θ, ω) we have:

- A vector bundle $A \to X$;
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- A Lie bracket $[\cdot,\cdot]_A; \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;
- A bundle map $\rho_A : A \to TM$;

satisfying the Leibniz identity:

$$[s_1, f\, s_2]_A = f[s_1, s_2]_A + \rho(s_1)(f)\, s_2.$$

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Main idea: Think of $(A, [\cdot,\cdot]_A, \rho_A)$ as a *generalized tangent bundle*.

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The A-forms $\Omega^\bullet(A) := \Gamma(\wedge^\bullet A^*)$ inherit a differential $\mathrm{d}_A : \Omega^k(A) \to \Omega^{k+1}(A)$:

$$\mathrm{d}_A\alpha(s_0,\dots,s_k) = \sum_i (-1)^i \rho_A(s_i)(\alpha(s_0,\dots,\widehat{s}_i,\dots,s_k)) + \sum_{i<j}(-1)^{i+j}\alpha([s_i,s_j]_A, s_0,\dots,\widehat{s}_i,\dots,\widehat{s}_j,\dots,s_k)$$

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One defines the anchor $\rho_A : A \to TM$ and Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ by:

$$\begin{aligned} \rho_A(s)(f) &:= \mathrm{d}_A f(s), \\ \langle [s_1, s_2]_A, \alpha\rangle &:= \mathrm{d}_A(\alpha(s_2))(s_1) - \mathrm{d}_A(\alpha(s_1))(s_2) - \mathrm{d}_A\alpha(s_1, s_2). \end{aligned}$$

satisfying:

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Summary

Given (P,θ,ω) a fully regular G-structure with connection, we have a classifying Lie algebroid $A_{(\theta,\omega)} \to X_{(\theta,\omega)}$ and a classifying bundle map:

$$\begin{array}{ccc} TP & \xrightarrow{H} & A_{(\theta,\omega)} \\ \downarrow & & \downarrow \\ P & \xrightarrow[h]{} & X_{(\theta,\omega)} \end{array}$$

such that:

$$\Omega^\bullet(P,\theta,\omega) = H^*\Omega^\bullet(A_{(\theta,\omega)}).$$

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Moreover:

- (H,h) is a Lie algebroid map: $H^*(\mathrm{d}_A\alpha) = \mathrm{d}(H^*\alpha)$, $\forall \alpha \in \Omega^\bullet(A_{(\theta,\omega)})$;
- TP and $A_{(\theta,\omega)}$ carry G-actions by Lie algebroid automorphisms for which (H,h) is G-equivariant;
- $A_{(\theta,\omega)}$ is a transitive Lie algebroid: $\mathrm{Im}\,\rho = TX_{(\theta,\omega)}$.

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One seeks all G-structures $P \to M$, satisfying structure eqs:

$$\mathrm{d}\theta = T(\theta \wedge \theta) - \omega \wedge \theta, \quad \mathrm{d}\omega = R(\theta \wedge \theta) - \omega \wedge \omega \tag{1}$$

where possible values of torsion and curvature are constrained:

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- There is G-equivariant map $h : P \to X$ into G-manifold X such that torsion and curvature factor through X:

$$R = R(h) \quad T = T(h),$$

for G-equivariant maps $R : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathfrak{g})$ and $T : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathbb{R}^n)$.

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- Since (θ, ω) is a coframe, $h : P \to X$ also satisfies structure equations:

$$\mathrm{d}h = F(h, \theta) + \psi(h, \omega), \tag{2}$$

  where $F : X \times \mathbb{R}^n \to TX$ is G-equivariant and $\psi : X \times \mathfrak{g} \to TX$ is $\mathfrak{g}$-action.

Cartan's Realization Problem

One is given **Cartan Data**:

- a closed Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$;
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Again, these can be interpreted in terms of a Lie algebroid!

Cartan's Realization Problem: algebroid picture

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Proposition

There exists a solution through every $x \in X$ if and only if $A \to X$ is a Lie algebroid.

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Definition

A Lie algebroid of this form is called a *G*-structure Lie algebroid with connection.

An example: Extremal Kähler surfaces

Definition

A Kähler surface (M^2, g, σ, J) is called extremal if the Hamiltonian vector field X_K associated with the Gaussian curvature K is an infinitesimal isometry.

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Unitary frame bundle:

$$\mathrm{F}_{U(1)}(M) := \left\{ u : \mathbb{C} \to (T_x M, J_x) \mid \text{complex linear isomorphism} \right\}$$

Tautological form: $\theta \in \Omega^1(\mathrm{F}_{U(1)}(M), \mathbb{C})$

Levi-Civita connection: $\omega \in \Omega^1(\mathrm{F}_{U(1)}(M), \mathfrak{u}(1))$

Structure Equations: Identifying $\mathfrak{u}(1) \simeq i\mathbb{R}$

$$\begin{cases} \mathrm{d}\theta = -\omega \wedge \theta \\ \mathrm{d}\omega = \frac{K}{2}\, \theta \wedge \bar{\theta} \end{cases}$$

Differential analysis

Pullback of the symplectic form σ under $\pi : \mathrm{F}_{U(1)}(M) \to M$:

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– Since X_K is infinitesimal isometry, $\mathcal{L}_{\widetilde{X}_K}\omega = 0$, 2nd structure equation yields:

$$\mathrm{d}U = \tfrac{i}{2}\,\mathrm{d}\,\imath_{\widetilde{X}_K}\omega = -\tfrac{i}{2}\,\imath_{\widetilde{X}_K}\mathrm{d}\omega = -\tfrac{i}{4}K\,\imath_{\widetilde{X}_K}(\theta\wedge\bar{\theta}) \quad\Longrightarrow\quad \mathrm{d}U = -\tfrac{K}{2}(\bar{T}\theta + T\bar{\theta})$$

Algebroid of extremal Kähler surfaces

Conclusion: An extremal Kähler surface amounts to:

- $U(1)$-structure $P \to M$ w/ tautological form $\theta \in \Omega^1(P, \mathbb{C})$, connection form $\omega \in \Omega^1(P, i\mathbb{R})$, and
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It comes with a right $U(1)$-action:

$$(K,T,U)g = (K, g^{-1}T, U), \qquad (z,\alpha)g = (g^{-1}z,\alpha).$$

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Remark. Each $\mathbf{t}^{-1}(x)$ is a G-principal bundle over $M = \mathbf{t}^{-1}(x)/G$. Hence, a G-principal groupoid is a *family* of G-principal bundles parameterized by X. .

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Global Objects

Given a closed subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$, the relevant global objects for the equivalence problem are G-structure groupoids with connection:

$$\Gamma \rightrightarrows X, \quad \Theta \in \Omega^1_L(\Gamma, \mathbb{R}^n), \quad \Omega \in \Omega^1_L(\Gamma, \mathfrak{g}).$$

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Theorem

If $(\Gamma \rightrightarrows X, \Theta, \Omega)$ is a G-structure groupoid with connection, then its Lie algebroid $A = Lie(\Gamma) \to X$ is a G-structure Lie algebroid with connection.

Solving Cartan realization problem

$$\text{Classification problem for geometric structures} \xrightarrow{\text{differential analysis}} G\text{-structure algebroid with connection}$$

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Classification problem for geometric structures	$\xrightarrow{\text{differential analysis}}$	G-structure algebroid with connection
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G-structure groupoid with connection	$\Longrightarrow$	Solutions: $\mathbf{t}^{-1}(x)/G$

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Important addenda:

- It is enough to G-integrate the restriction of $A \to X$ to the orbit containing $x \in X$.
- There is a (computable) obstruction theory for G-integrability.

Finding the extremal Kähler surfaces

The algebroid $A \to X \simeq \mathbb{R}^4$

$$A = (\mathbb{R} \times \mathbb{C} \times \mathbb{R}) \times (\mathbb{C} \oplus i\mathbb{R}) \longrightarrow X = \mathbb{R} \times \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^4$$

(with global coordinates (K, T, U))

$$[(z,\alpha),(w,\beta)]|_{(K,T,U)} := (\alpha w - \beta z, -\tfrac{K}{2}(z\bar{w} - \bar{z}w))$$
$$\rho(z,\alpha)|_{(K,T,U)} := \left(-T\bar{z} - \bar{T}z, Uz - \alpha T, -\tfrac{K}{2}T\bar{z} - \tfrac{K}{2}\bar{T}z\right),$$

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- the 2-dimensional submanifolds of $\mathbb{R}^4$ given by $U(1)$-rotation of the curves in $\mathbb{R}^3$:

$$U = \tfrac{1}{4}K^2 - c_1, \quad |T|^2 = -\tfrac{1}{12}K^3 + c_1 K + c_2.$$

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The values of c_1 and c_2 (hence $\Delta = \frac{1}{48}(16c_1^3 - 9c_2^2)$) determine if an orbit $\mathcal{O}$ has topology, and hence determines the G-integrability of $A|_{\mathcal{O}}$.

The 1-connected extremal Kähler surfaces

Conditions	$U(1)$**-frame bundle**: $\mathbf{s}^{-1}(x)$	**Solutions**: $\mathbf{s}^{-1}(x)/U(1)$	**complete solutions**
$K=0$	$SO(2) \ltimes \mathbb{R}^2$	$\mathbb{R}^2$	Yes
$K=c>0$	$\mathbb{S}^3$	$\mathbb{S}^2$	Yes
$K=c<0$	$SO(2,1)$	$\mathbb{H}^2$	Yes
$\Delta=0$, $c_1=c_2=0$	$(\mathbb{R}^2\times\mathbb{R})/\mathbb{Z}$	$\mathbb{R}^2$	No
$\Delta=0$, $c_2<0$	$\mathbb{R}^2\times\mathbb{S}^1$	$\mathbb{R}^2$	No
$\Delta=0$, $c_2>0$	$(\mathbb{R}^2\times\mathbb{R})/\mathbb{Z}$ $(\mathbb{R}^2\times\mathbb{S}^1)$	$\mathbb{R}^2$	Yes No
$\Delta<0$	$\mathbb{R}^2\times\mathbb{S}^1$	$\mathbb{R}^2$	No
$\Delta>0$ (if $\frac{4c_1-r_2^2}{r_3^2-4c_1}=\frac{p}{q}$)	$\mathbb{R}^2\times\mathbb{S}^1$ $\mathbb{S}^3$	$\mathbb{R}^2$ $\mathbb{CP}^1_{p,q}$	No Yes

Final comments

Other byproducts of the theory:

- Give existence and unique of solutions for any Cartan realization problem.
- Allows to determine which solutions are complete.
- Yields infinitesimal and global symmetries of solutions.
- Describes deformations and moduli space of solutions.

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- Yields infinitesimal and global symmetries of solutions.
- Describes deformations and moduli space of solutions.

Still many things to be worked out, e.g (in progress):

- Extend theory to G-structure of finite type $k > 1$;
- Extend theory to infinite dimensions (profinite algebroids);
- Work out more examples.

Thank you!