Bowley's Law Revisited

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Symmetry, Invariants, and theirApplications: A Celebration PeterOlver's 70th Birthday

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Abstract

Bowley's Law is a stylized fact of economics stating that the share of national income paid out to the employees as compensation for their work (i.e., the wage or labor share) remains unchanged over time. The economic data collected in different countries from the end of the 19th century until about 1980 gave rise to and strongly supported this law, which was widely accepted by the economics community at the time. This law is now subject to doubt, however, as recent data patterns appear to deviate from it.

We present a mathematical model demonstrating that the wage share can be treated as a time-independent invariant under certain conditions, thus defining the limitations of Bowley's Law.

Joint work with **Kunpeng Wang** (Sichuan University-Pittsburgh Institute)

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Outline

Introduction

Bowley's Law, also known as the law of the constant wage (labor) share, is a stylized fact of economics which states that the wage share of a national economy, i.e., the share of a country's economic output that is given to employees as compensation for their work (usually in the form of wages), remains constant over time. It is named after an English economist, mathematician, and statistician Sir Arthur Bowley[1].

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[1] A. L. Bowley, *Wages in the United Kingdom in the Nineteenth Century: Notes for the Use of Students of Social and Economic Questions*, Cambridge, UK: Cambridge University Press, 1900; A. L. Bowley, *Wages and Income in the United Kingdom Since 1860*, Cambridge: Cambridge University Press, 1937.

"As we see on the basis of statistical data the relative share in gross income shows only small changes both in the long and short period. We shall try to explain this 'law' and establish under which conditions it is valid." (Kalecki, 1938).

"The share of wages and salaries in national income has edged up very slightly over the long run." (Samuelson and Nordhous, 1992)

"... the result remains a bit of a miracle" (Keynes, 1939)

"... a mystery" (Schumpeter, 1939)

"... a parallel to Newton's gravitational constant g ... " (Weintroub, 1959)

"... the mystery of the constant relative shares remains as a reproach to theoretical economics" (Robinson, 1966)

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Bowley's Law $\Rightarrow$ important political and economic ramifications (e.g., why to study how a national income is divided between wages, profits, and rents?)

It must be noted that in the works of classical economists like Ricardo, Smith and Marx, income shares of the socio-economic classes are variable in the long-run according to the level of economic development.

According to the general consensus in the economic community, Bowley's Law could be observed in the past, but it is no longer valid for the post-1980s data.

For example, it is reported by Krämer[2] that the wage share of income in the G7 economies in percent (weighted average) was in decline during the period 1970-2010.

[2] Krämer, M. H.: Bowley's Law: The diffusion of an empirical supposition into economic theory. Papers in Political Economy, 2011, No. 61, pp. 19–49.

A similar result is reported by Elsby *et al*[3] with regards to the US wage share.

[3] Eslby, M. W., Hobijn, B., Sahin, A.: The Decline of the US labor share. regarding the US labor share. BPEA article. 2013.

"Nature imitates mathematics" (Gian-Carlo Rota)

When is the wage share constant? Formulating the problem

Consider the configuration space defined by the following variables:

Y - production,

K - capital,

L - labor.

The corresponding unconstrained optimization problem

$$\Pi = pY - wL - rK \to \max,$$

where Π is the profit with nominal wage (w), nominal rent (r), nominal price (p). We assume $Y = f(K, L)$ is such that there is an interior solution for $K, L, Y \geq 0$ and so, in particular, we have

$$\frac{\partial Y}{\partial L} = \frac{w}{p}.$$

Then, in view of the above, the wage (labor) share is given by

$$s_L = \frac{wL}{pY} = \frac{\partial Y}{\partial L}\frac{L}{Y}. \tag{1}$$

Problem: Build a mathematical model *rooted in reality* that assures s_L is a time-independent invariant.

Solving the problem

Definition

In economics, the *Cobb-Douglas production function* is a particular functional form of the production function, widely used to represent the technological relationship between the amounts of two (or more) inputs. In its most standard form the Cobb-Douglas function is given by

$$Y = AL^{\beta}K^{\alpha}, \tag{2}$$

where Y is the total production, L - labor, K - capital, A - total factor productivity, α and β – the output elasticities.

Of particular importantce is the case when $\alpha + \beta = 1$ (i.e., the function (2) is homogeneous.

A historical overview

"... For it was in 1927 that I computed the index numbers of the total number of manual workers (L) employed in American manufacturing by years from 1899 to 1922, did the same for fixed capital (C), expressed these in logarithmic terms on a chart, and then added the index for physical production (P) in manufacturing. I found the curve for product to lie, in general, approximately one-quarter of the distance between the curve for labor, which had increased the least (to 162), and the curve of capital, which had increased the most (to 431)."

(P. H. Douglas, *The Cobb-Douglas production function once again: Its history, its testing, and some new empirical values*, Journal of Political Economy, 84(51), 1976, 903–915.)

Consider now the data studied by Cobb and Douglas in 1927 and compare it to the following system

$$\dot{K} = aK,\ \dot{L} = bL,\ \dot{Y} = cY,$$

where $a, b, c > 0$, or, the corresponding one-parameter transformation group

$$\overline{K} = K_0 e^{at},\ \overline{L} = L_0 e^{bt},\ \overline{Y} = Y_0 e^{ct}.$$

Using R programming language, we obtain the following estimates of the parameters with excellent accuracy:

$$\begin{aligned} a &= 0.06472564, \quad K_0 = 4.61213588 \quad \text{(capital)}, \\ b &= 0.02549605, \quad L_0 = 4.66953290 \quad \text{(labor)}, \\ c &= 0.03592651, \quad Y_0 = 4.66415363 \quad \text{(production)}. \end{aligned} \tag{3}$$

Therefore, the dynamical system above is a *data-driven dynamical system* indeed.

The corresponding infinitesimal generator is given by

$$\mathbf{u}_2 = aK\frac{\partial}{\partial K} + bL\frac{\partial}{\partial L} + cY\frac{\partial}{\partial Y}. \tag{4}$$

Let $(\mathbb{R}^3, \pi, \mathbb{R}^2)$ be a trivial bundle that $\pi = pr_1$ and (K, L, Y) be adapted coordinates. Then, the corresponding jet bundles are $(J^1\pi, \pi_1, \mathbb{R}^2)$ and $(J^1\pi, \pi_{1,0}, \mathbb{R}^3)$, where

$$J^1\pi = \left\{ j^1_p\phi : p \in \mathbb{R}^2, \phi \in \Gamma_p(\pi) \right\} \tag{5}$$

is the first jet manifold with adapted coordinates (K, L, Y, Y_K, Y_L) and the source projection $\pi_1 = \pi \circ \pi_{1,0}$.

The first prolongation of $\mathbf{u}_2$ on $\mathbb{R}^3$ is $\mathbf{u}_2^{(1)}$. It is a symmetry of the Cartan distribution on $J^1\pi$, that is,

$$\begin{aligned} \mathbf{u}_2^{(1)} = \quad & aK\frac{\partial}{\partial K} + bL\frac{\partial}{\partial L} + cY\frac{\partial}{\partial Y} + \xi_1(K, L, Y, Y_K, Y_L)\frac{\partial}{\partial Y_K} \\ & + \xi_2(K, L, Y, Y_K, Y_L)\frac{\partial}{\partial Y_L}. \end{aligned} \tag{6}$$

The components $\xi_1(K, L, Y, Y_K, Y_L)$ and $\xi_2(K, L, Y, Y_K, Y_L)$ are abbreviated respectively as ξ_1 and ξ_2 in what follows.

Let us consider a basic contact form $\omega = dY - Y_K dK - Y_L dL$. We require that the one-form $\mathcal{L}_{\mathbf{u}_2^{(1)}}(\omega)$ be a contact form, namely,

$$
\begin{aligned}
\mathcal{L}_{\mathbf{u}_2^{(1)}}(\omega) &= \mathcal{L}_{\mathbf{u}_2^{(1)}}(dY - Y_K dK - Y_L dL) \\
&= \mathcal{L}_{\mathbf{u}_2^{(1)}}(dY) - (\mathcal{L}_{\mathbf{u}_2^{(1)}} Y_K) dK - Y_K(\mathcal{L}_{\mathbf{u}_2^{(1)}}(dK)) \\
&\quad -(\mathcal{L}_{\mathbf{u}_2^{(1)}} Y_L) dL - Y_L(\mathcal{L}_{\mathbf{u}_2^{(1)}}(dL)) \\
&= d(\mathbf{u}_2^{(1)}(Y)) - (\mathbf{u}_2^{(1)}(Y_K)) dK - Y_K d(\mathbf{u}_2^{(1)}(K)) \\
&\quad -(\mathbf{u}_2^{(1)}(Y_L)) dL - Y_L d(\mathbf{u}_2^{(1)}(L)) \\
&= c dY - \xi_1 dK - a Y_K dK - \xi_2 dL - b Y_L dL \\
&= c(\omega + Y_K dK + Y_L dL) - \xi_1 dK - a Y_K dK - \xi_2 dL \\
&\quad - b Y_L dL \\
&= c\omega + (cY_K - \xi_1 - aY_K) dK + (cY_L - \xi_2 - bY_L) dL,
\end{aligned}
\tag{7}
$$

the last line of the equation (7) implies that the expressions in the parentheses above vanish, which entails

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$$\xi_1 = \xi_1(K, L, Y, Y_K, Y_L) = (c - a)Y_K \tag{8}$$

and

$$\xi_2 = \xi_2(K, L, Y, Y_K, Y_L) = (c - b)Y_L. \tag{9}$$

Hence, the first prolongation $\mathbf{u}_2^{(1)}$ of $\mathbf{u}_2$ is found to be

$$\begin{aligned}\mathbf{u}_2^{(1)} &= aK\frac{\partial}{\partial K} + bL\frac{\partial}{\partial L} + cY\frac{\partial}{\partial Y} + (c-a)Y_K\frac{\partial}{\partial Y_K} \\ &+(c-b)Y_L\frac{\partial}{\partial Y_L},\end{aligned} \tag{10}$$

Next, a set of four (5-1 = 4) corresponding fundamental invariants is found to be

$$
\begin{aligned}
I_1 &= LK^{-\frac{b}{a}}, \\
I_2 &= YK^{-\frac{c}{a}}, \\
I_3 &= Y_K K^{\frac{a-c}{a}}, \\
I_4 &= Y_L K^{\frac{b-c}{a}}.
\end{aligned}
\tag{11}
$$

Combaning the fundamental invariants in such a way that the parameters of the exponential growth a, b, and c vanish, we arrive at

$$s_L = \mathcal{I}_1(I_1, I_2, I_4) = \frac{I_1 \cdot I_4}{I_2} = \frac{Y_L L}{Y}, \tag{12}$$

as expected.

Conclusions

We have built a mathematical model showing that the exponential growth in Y (production), L (labor), and K (capital) implies that Bowley's Law holds true in this case, ergo when Bowley's Law is not observed the economy is not growing fast enough – as per the growth in Y, L, and K.

"Anyone who believes that exponential growth can go on forever in a finite world is either a madman or an economist."

(Kenneth E. Boulding)

Theorem (Extreme value theorem)

If K is a compact set and $f : K \to \mathbb{R}$ is a continuous function, then f is bounded and there exist $p, q \in K$ such that $f(p) = \sup_{x \in K} f(x)$ and $f(q) = \inf_{x \in K} f(x)$.

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- Wikipedia.

Thank You!

Happy anniversary, Peter!