# Symmetries and exact solutions of the diffusive Lotka-Volterra system

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**... And this devotion [to biology]
is achieved only by deep understanding
of beauty, infinity, symmetry,
and harmony in nature.**

Taras Shevchenko (1814–1861), the great Ukrainian poet.

# Outline

1. Introduction: references, motivation, examples
2. Traveling fronts (TFs) and other exact solutions of two-component Lotka-Volterra system and their biological interpretation
3. Definitions of Q-conditional symmetry
4. Exact solutions of the two- and three-component diffusive Lotka-Volterra systems (DLVS) and their interpretation
5. Construction of exact solutions of the DLVS in 'no-go case'
6. 'The end is where we start from'

The talk is based on the recent paper
R.Cherniha and V. Davydovych: Construction and application of exact solutions of the diffusive Lotka–Volterra system: A review and new results. *Commun. in Nonlin. Sci. and Num. Simul.* **113** (2022) 106579
The relevant older works are

- Cherniha R.2010 Conditional symmetries for systems of PDEs: new definition and their application for reaction-diffusion systems. *J Phys A Math Theor.* **43** 405207.
- Cherniha R and Davydovych V 2011 Conditional symmetries and exact solutions of the diffusive Lotka-Volterra system *Math. Comput. Modelling.* **54** 1238–51
- Cherniha R and Davydovych V 2013 Lie and conditional symmetries of the three-component diffusive Lotka-Volterra system. *J Phys A: Math and Theor* ; vol.46, 185204 (18pp)
- Cherniha R., Davydovych V. Nonlinear reaction-diffusion systems — conditional symmetry, exact solutions and their applications in biology. *Springer* 2017.

About 100 years ago, Alfred Lotka (1920) and Vito Volterra(1926) independently developed a mathematical model, which nowadays serves as a mathematical background for population dynamics, chemical reactions, ecology, etc. The model is based on a system of ordinary differential equations (ODEs) involving quadratic nonlinearities (typically two equations). Following some earlier papers, in which linear models were used chemical reactions [Hirniak-1908,Hirniak-1911], Lotka has shown that the densities in periodic chemical reactions can be adequately described by a model involving ODEs with quadratic nonlinearities. In contrast to Lotka, Volterra, as a mathematician, was inspired by the information that the amount of predatory fish caught in Italy varied periodically and suggested a prey–predator model for the interaction of two populations of fishes.
The classical Lotka–Volterra system consists of two nonlinear ODEs of the form

$$\begin{aligned} \frac{du}{dt} &= u(a - bv), \\ \frac{dv}{dt} &= v(-c + du), \end{aligned} \tag{1}$$

where the functions $u(t)$ and $v(t)$ represent the numbers of prey and predators, while $a,\ b,\ c$ and $d$ are positive parameters.

Later it was shown that systems of differential equations with quadratic nonlinearities can describe many other types of interaction between species, chemicals, cells, etc. Moreover, their diffusion in space can also be taken into account. Thus, the $m$-component system is obtained

$$u^i_t = d_i u^i_{xx} + u^i\left(a_i + \sum_{j=1}^{m} b_{ij} u^j\right), \quad i = 1, \ldots, m, \tag{2}$$

which is called the diffusive Lotka–Volterra system (DLV system).

Hereinafter $u^1(t,x), u^2(t,x), \ldots, u^m(t,x)$ are unknown functions, $d_i \geq 0,\ a_i$ and $b_{ij}$ are arbitrary parameters. However, these parameters should guarantee that the system is nonlinear and all equations cannot be autonomous (otherwise its applicability is questionable).

Depending on the signs of these coefficients, different types of interaction between $m$ types of species, chemicals, cells, etc. can be modeled.

In the scalar case, the DLV system is reducible to the famous Fisher equation

$$u_t = u_{xx} + u(1-u) \tag{3}$$

is $u(t,x)$ suggested by RA Fisher in 1937 in order to describe the spread in space of a favoured gene in a population. The Fisher equation cannot be exactly solved taking into account any reasonable initial and boundary conditions in a bounded domain (e.g. a finite interval). There is only known the exact solution in the form of the traveling front (TF), which satisfies the natural conditions at infinity $x = \pm\infty$. The solution was found in [Ablowitz M & Zeppetella A, 1979]:

$$u = \frac{1}{4}\Big(1 - \tanh\Big(\frac{1}{2\sqrt{6}}(x - \frac{5}{\sqrt{6}}t + x_0)\Big)\Big)^2, \tag{4}$$

It should be also stressed that the Fisher eq. admits only a trivial Lie symmetry and does not possesses non-classical symmetry (conditional symmetry). Thus, construction its exact solutions still is an highly non-trivial problem.
There are some studies devoted to search for exact solutions of the Fisher equation claiming that new those have been found. However, to the best of my knowledge, they either can be reduced to (4), or are not smooth, or don't satisfy any reasonable boundary conditions.

The problem which occurs in finding exact solutions of the Fisher equation is very natural, because one is a nonlinear PDE. Typically, the mathematical models used for description of biomedical processes are based on nonlinear PDEs and usually they are non-linearizable (in contrast to many problems occurring in physics !).

The well known principle of linear superposition cannot be applied to generate new exact solutions to nonlinear PDEs. Thus, the classical methods (the Fourier method, the methods of the Laplace transformations and the Green function, etc) are not applicable for solving such PDEs.

Thus, construction of particular exact solutions for these equations is a non-trivial problem. Finding exact solutions that have an appropriate physical, chemical or biological interpretation is of fundamental importance.

Modern symmetry based (group-theoretical) methods, which are based on the classical Lie method (CLM), are the most powerful methods to construct exact solutions of nonlinear PDEs. CLM was created by Sophus Lie in the end of XIX century. There are several excellent books devoted to this method and its applications to PDEs (Ovsiannikov, Olver, Bluman et al, Fushchych et al,...).

In 1969, Bluman & Cole introduced an essential generalization of CLM, which is often called the non-classical symmetry method. Notably their paper was widely ignored until 1986 when Olver & Rosenau brought attention to the Bluman & Cole work.
Both methods allow us to construct Lie symmetries and Q-conditional (non-classical) symmetries in the form of linear 1st order differential operators.
Having the known Lie or conditional symmetry for a nonlinear PDE in question and using a standard algorithm, one constructs ansatz, which allows to reduce the nonlinear PDE to an equation of lower dimensionality. A typical form of the ansatz for a (1+1)-dimensional PDE is

$$U = g_0(t,x) + \varphi(\omega)g_1(t,x) \qquad (5)$$

where $\varphi(\omega)$ is new unknown function, $\omega = \omega(t,x)$ is the invariant variable, and $g_0(t,x)$ and $g_1(t,x)$ are the known functions. Solving ODE for $\varphi(\omega)$, one constructs exact solutions for PDE in question. The most common particular case is $g_0 = 0,\ g_1 = 1$ and $\omega = x - Ct$ leading to the traveling fronts (TF).

Consider a system of $m$ evolution equations ($m \geq 2$) with $2$ independent $(t, x)$ and $m$ dependent $u = (u_1, u_2, \ldots, u_m)$ variables. Let us assume that the $k$-th order ($k \geq 2$) equations of evolution type

$$u_t^i = F^i\left(t, x, u, u_x, \ldots, u_x^{(k_i)}\right), \ i = 1, 2, \ldots, m, \tag{6}$$

are defined on a domain $\Omega \subset \mathbf{R}^2$ of independent variables $t$ and $x$.
It is well-known that to find Lie invariance operators, i.e. Lie symmetry, one needs to consider system (6) as the manifold

$$\mathcal{M} = \{S_1 = 0, S_2 = 0, \ldots, S_m = 0\}$$

where

$$S_i \equiv u_t^i - F^i\left(t, x, u, u_x, \ldots, u_x^{(k_i)}\right) = 0, \ i = 1, 2, \ldots, m, \tag{7}$$

in the prolonged space of the variables:

$$t, x, u, \underset{1}{u}, \ldots, \underset{k}{u},$$

where $k = \max\{k_i, \ i = 1, \ldots, m\}$

## Definition

**1.** System (6) is invariant ( in Lie sense ) under the transformations generated by the infinitesimal operator, i.e. Lie symmetry

$$Q = \xi^0(t,x,u)\partial_t + \xi^1(t,x,u)\partial_x + \\ +\eta^1(t,x,u)\partial_{u_1} + \ldots + \eta^m(t,x,u)\partial_{u_m}, \tag{8}$$

if the following invariance conditions are satisfied:

$$\underset{k}{Q} S_i \equiv \underset{k}{Q}\left(u_t^i - F^i\left(t,x,u,u_x,\ldots,u_x^{(k_i)}\right)\right)\Big|_{\mathcal{M}} = 0,\ i = 1,2,\ldots,m \tag{9}$$

Here the operator

$$\underset{k}{Q} = \underset{k-1}{Q} + \sigma^{k1}\partial_{u^{(k)}_{1,x}} + \ldots + \sigma^{km}\partial_{u^{(k)}_{m,x}},\ \underset{0}{Q} = Q$$

is the k-th order prolongation of the operator $Q$ and its coefficients are expressed via the functions $\xi^0, \xi^1, \eta^1, \ldots, \eta^m$ and their derivatives by the well-known formulae.

The crucial idea, which Bluman & Cole used for introducing the notion of $Q$-conditional symmetry, is to change the manifold $\mathcal{M}$, namely: the operator $Q$ is used to realize this. In [R.Ch.2010] it was noted that there are a few possibilities to realize this idea in the case of PDE systems. As a result, a new definition and the relevant algorithm how to simplify solving the problem described above were proposed.

## Definition

**2.**[R.Ch. 2010: J. Phys. A Math.Theor.] Operator (8) with $\xi^0 \neq 0$ is called the $Q$-conditional symmetry of the first type for an evolution system of the form (6) if the following invariance conditions are satisfied:

$$\underset{k}{Q} S_i \equiv \underset{k}{Q}\left(u_t^i - F^i\left(t, x, u, u_x, \ldots, u_x^{(k_i)}\right)\right)\Big|_{\mathcal{M}_1} = 0 \qquad (10)$$

for all $i = 1, 2, \ldots, m$, where the manifold

$$\mathcal{M}_1 = \{S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u_{i_1}) = 0\}$$

with a fixed number $i_1\,(1 \leq i_1 \leq m)$.

## Definition

**3.** Operator (8) is called the $Q$-conditional symmetry of the $p$-th type for an evolution system of the form (6) if the following invariance conditions are satisfied:

$$\underset{k}{Q} S_i \equiv \underset{k}{Q} \left( u_t^i - F^i \left( t, x, u, u_x, \ldots, u_x^{(k_i)} \right) \right) \Big|_{\mathcal{M}_p} = 0 \qquad (11)$$

for all $i = 1, 2, \ldots, m$, where the manifold

$$\mathcal{M}_p = \{ S_1 = 0, S_2 = 0, \ldots, S_m = 0,$$

$$Q(u_{i_1}) = 0, \ldots, Q(u_{i_p}) = 0 \}$$

with a set of the given numbers $i_1, \ldots, i_p \, (1 \leq p \leq i_p \leq m)$.

## Definition

**4.** Any $Q$-conditional symmetry of the $m$-th type is called the $Q$-conditional symmetry (non-classical symmetry) for an evolution system of the form (6).

Remark. All three definitions coincide in the case of $m = 1$, i.e. a single evolution equation.

If $m > 1$ then one obtains a hierarchy of conditional symmetry operators. It is easily seen that

$$\mathcal{M}_m \subset \mathcal{M}_p \subset \mathcal{M}_1 \subset \mathcal{M}$$

It means that each Lie symmetry is automatically a $Q$-conditional symmetry of the first, while $Q$-conditional symmetry of the first type is that of the $m$-th type (non-classical symmetry).

From the formal point of view is enough to find all the $Q$-conditional symmetry (non-classical symmetry) operators. On the other hand, to construct a complete list of $Q$-conditional symmetries for a system of PDEs, one needs to solve another nonlinear system, which usually is much more complicated and cumbersome. This problem arises even in the case of single linear PDE and it was the reason why G.Bluman and J.Cole in their pioneering work [ J. Math. Mech.,1969] were unable to describe all $Q$-conditional symmetries in explicit form even for the linear heat equation (it was done much later by other authors).

Thus, all three definitions are important from theoretical and practical point of view.

In the case $m = 2$, the DLV system (2) produces the two-component that

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(a_1 + b_1 u + c_1 v), \\ \lambda_2 v_t &= v_{xx} + v(a_2 + b_2 u + c_2 v), \end{aligned} \tag{12}$$

which is the most common particular case and was intensively studied starting from 1970s [Conway & Smoller,1977; Hastings, 1978] by different techniques. In particular, Rodrigo & Mimura [Hiroshima Math.J.2000] were able to construct the first examples of TFs for (12). Later new TFs were constructed in [R. Ch.& V.Dutka, 2004],[LC Hung, 2011, 2012],[CC Chen et al, 2012], [CC Chen & LC Hung, 2016]. An example from [R. Ch.& V.Dutka, 2004]:

$$\begin{aligned} u &= \tfrac{a}{4b}\left[1 - \tanh\left(\tfrac{\sqrt{a}}{2\sqrt{6}}\left(x - \tfrac{5\sqrt{a}}{\sqrt{6}}t + x_0\right)\right)\right]^2 \\ v &= \beta_0 + \tfrac{\beta_1 a}{4b}\left[1 - \tanh\left(\tfrac{\sqrt{a}}{2\sqrt{6}}\left(x - \tfrac{5\sqrt{a}}{\sqrt{6}}t + x_0\right)\right)\right]^2, \end{aligned} \tag{13}$$

where all the parameters are defined by the parameters $a_k, b_k, c_k$ from (12). Here we are looking for exact solutions of the DLV system with more complicated structure.

Lie symmetry of the DLV system (12) is poor. As it follows from [R.Ch.& J.R.King,2000,2003] as a particular case, in order to admit a non-trivial Lie symmetry, several parameters among $a_k, b_k, c_k$ must vanish. As a result, (12) reduces to an unrealistic model.
Thus, one needs to look for conditional symmetries of the DLV system in order to find new exact solutions.

**Theorem**

*In the case $\lambda_1 \neq \lambda_2$, DLV system (12) is $Q$-conditionally invariant under operator*

$$Q = \xi^0(t,x,u,v)\partial_t + \xi^1(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v, \quad \xi^0 \neq 0 \tag{14}$$

*if and only if $b_1 = b_2 = b, c_1 = c_2 = c.$*
*In the case $\lambda_1 = \lambda_2$, DLV system (12) admits only such operators of the form (14), which are equivalent to the Lie symmetry operators.*

## Theorem

*In the case $\lambda_1 \neq \lambda_2$, DLV system (12) is invariant under Q-conditional operators of the first type only in two cases. The corresponding systems and Q-conditional symmetries (up to local transformations $u \to bu,\ v \to \exp(\frac{a_2}{\lambda_2}t)v,\ b \neq 0$ and $u \to \exp(\frac{a_1}{\lambda_1}t)v,\ cv \to u,\ c \neq 0$ ) have the forms*

$$(i) \qquad \begin{array}{l} \lambda_1 u_t = u_{xx} + u(a_1 + u + v), \\ \lambda_2 v_t = v_{xx} + v(a_2 + u + v), \quad a_1 \neq a_2, \end{array} \tag{15}$$

$$Q_1 = (\lambda_1 - \lambda_2)\partial_t + (a_1 - a_2)u(\partial_u - \partial_v), \tag{16}$$

$$Q_2 = (\lambda_1 - \lambda_2)\partial_t - (a_1 - a_2)v(\partial_u - \partial_v). \tag{17}$$

$$(ii) \qquad \begin{array}{l} \lambda_1 u_t = u_{xx} + u(a_1 + u), \\ \lambda_2 v_t = v_{xx} + vu, \end{array} \tag{18}$$

$$\begin{array}{l} Q = \partial_t + \frac{2\alpha_1}{\lambda_1 - \lambda_2}\,\partial_x + \Big(\exp(\alpha_1 x + \frac{\alpha_1^2}{\lambda_2}t)\times \\ \big((\alpha_3 + \alpha_4 \exp(-\frac{a_1}{\lambda_2}t))u + \alpha_3 a_1\big) + \alpha_2 v\Big)\partial_v, \end{array} \tag{19}$$

*where $\alpha_k,\ k = 1, \dots, 4$ are arbitrary constants with the restriction $\alpha_3^2 + \alpha_4^2 \neq 0$. There are no any other Q-conditional operators of the first type.*

Let's apply the conditional symmetries obtained for constructing exact solutions. According to the theorem presented above, system (15)

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(a_1 + u + v), \\ \lambda_2 v_t &= v_{xx} + v(a_2 + u + v), \quad a_1 \neq a_2, \end{aligned}$$

admits two operators of conditional symmetry. Consider, say, the second one:

$$Q_2 = (\lambda_1 - \lambda_2)\partial_t - (a_1 - a_2)v(\partial_u - \partial_v).$$

Using the standard procedure one obtains the relevant ansatz

$$\begin{aligned} u(t,x) &= \varphi_1(x) - \varphi_2(x)\exp\left(\tfrac{a_1 - a_2}{\lambda_1 - \lambda_2}t\right), \\ v(t,x) &= \varphi_2(x)\exp\left(\tfrac{a_1 - a_2}{\lambda_1 - \lambda_2}t\right) \end{aligned}$$

reducing (15) to the nonlinear ODE system

$$\begin{aligned} &\varphi_1'' + \varphi_1^2 + a_1\varphi_1 = 0, \\ &\varphi_2'' + \tfrac{a_2\lambda_1 - a_1\lambda_2}{\lambda_1 - \lambda_2}\varphi_2 + \varphi_1\varphi_2 = 0 \end{aligned}$$

Because the first equation can be solved separately we were able to construct several particular solutions of this reduced system and each of them produces exact solution of the two-component DLV system (15). Let me present one of them only (see [R.Ch.& V.Davydovych, 2011] for more solutions).

Obviously, system (15) can be rewritten in the form ( $u \rightarrow bU,\, v \rightarrow cV$)

$$\begin{array}{l} \lambda_1 U_t = U_{xx} + U(a_1 - bU - cV), \\ \lambda_2 V_t = V_{xx} + V(a_2 - bU - cV), \end{array} \qquad (20)$$

System (20) is used to describe the competition (e.g., for food) of two species.

**Theorem**

*The classical solution of boundary-value problem for the competition system (20) and the initial profile*

$$\begin{array}{l} U(0,x) = \frac{a_1}{b} + \frac{1}{(a_1 - a_2)b}\, C_2 \sin(\sqrt{-\beta\lambda_1}x), \\ V(0,x) = \frac{1}{(a_2 - a_1)c}\, C_2 \sin(\sqrt{-\beta\lambda_1}x), \end{array} \qquad (21)$$

*and boundary conditions*

$$\begin{array}{l} x = 0: \; U = \frac{a_1}{b}, \; V = 0, \\ x = \frac{\pi}{\sqrt{-\beta\lambda_1}}: \; U = \frac{a_1}{b}, \; V = 0, \end{array} \qquad (22)$$

*in domain* $\Omega = \{(t,x) \in (0,+\infty) \times \left(0, \frac{\pi}{\sqrt{-\beta\lambda_1}}\right)\}$ *is given by formulae (23).*

$$\begin{aligned} U(t,x) &= \tfrac{a_1}{b} + \tfrac{1}{(a_1-a_2)b}\ C_2 \sin(\sqrt{\beta\lambda_1}x)e^{-\beta t}, \\ V(t,x) &= \tfrac{1}{(a_2-a_1)c}\ C_2 \sin(\sqrt{\beta\lambda_1}x)e^{-\beta t}, \end{aligned} \tag{23}$$

where $\beta = \frac{a_2-a_1}{\lambda_1-\lambda_2} > 0,\ a_1 > 0,\ a_2 > 0.$
The solution (23) with $\beta < 0$ has the time asymptotic

$$(U,\ V) \to (\frac{a_1}{b},\ 0), \quad t \to +\infty. \tag{24}$$

Thus, this solution describes the competition between the two species when the species $U$ eventually dominate while the species $V$ die.

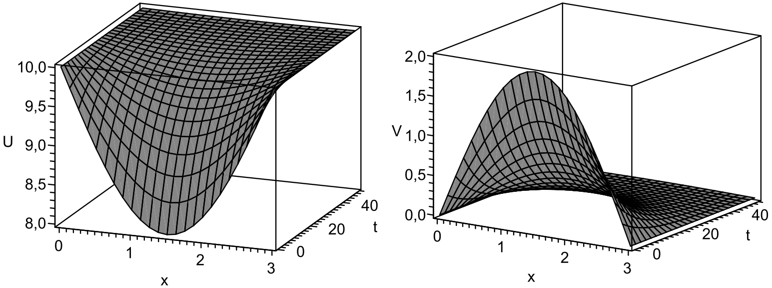


Figure: Solution (23) of system (20) with
$a_1 = 1,\ a_2 = 2,\ \lambda_1 = 11,\ \lambda_2 = 1,\ b = 0.1,\ c = 0.1,\ \ C_2 = 0.2,\ \beta = \ -0.1.$

The DLV system consists of evolution equations, hence the so-called no-go case arises when one constructs Q-conditional (non-classical) symmetries. So, one should look for the operators of the form (8) with $\xi^0 = 0$, i.e.

$$Q = \xi^1(t,x,u)\partial_x + +\eta^1(t,x,u)\partial_{u_1} + \ldots + \eta^m(t,x,u)\partial_{u_m}, \qquad (25)$$

Very recently this case was studied in [R.Ch. & V. Davydovych, 2021, 2022]. It is well-known [Zhdanov & Lahno, Physica D, 1998] that application of the standard definition of Q-conditional symmetry leads to a complicated system and its solving is equivalent to solving the evolution PDE in question.

## Definition

Operator (25) is called $Q$-conditional symmetry of the first type for the $m$-component DLV system (2) if the following invariance criterion is satisfied:

$$\underset{2}{Q}\,(S_i)\Big|_{\mathcal{M}_1^j} = 0, \ \ i = 1, 2, \ldots, m, \qquad (26)$$

where $j$ is a fixed number and

$$\mathcal{M}_1^j = \{S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u^j) = 0, \tfrac{\partial}{\partial t}\, Q(u^j) = 0, \tfrac{\partial}{\partial x}\, Q(u^j) = 0\},$$
$$S_i = u_t^i - d_i u_{xx}^i + u^i\left(a_i + \sum_{j=1}^{m} b_{ij}u^j\right). \qquad (27)$$

**Theorem**

*The DLV system (12)*

$$\begin{aligned}\lambda_1 u_t &= u_{xx} + u(a_1 + b_1 u + c_1 v),\\ \lambda_2 v_t &= v_{xx} + v(a_2 + b_2 u + c_2 v),\end{aligned}$$

*is invariant under $Q$-conditional symmetry operator(s) of the first type in the no-go case*

$$Q = \ \xi(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v, \ \xi \neq 0,$$

*if and only if the system and the relevant operator(s) are as specified in Table 1. Any other DLV system (12) admitting a $Q$-conditional symmetry of the first type and the corresponding operator(s) are reducible to those listed in Table 1 by an appropriate transformation from the set*

$$t^* = t + t_0, \ x^* = e^{\gamma_0}(x + x_0), \ u^* = \beta_{11}\, e^{\gamma_1 t} u + \beta_{12}\, v, \ v^* = \beta_{22}\, e^{\gamma_2 t} v + \beta_{21}\, u,$$

*where $t_0, \ x_0, \ \beta_{ij}$ and $\gamma_j$ are correctly-specified constants.*

Table: $Q$-conditional symmetries of the first type of the DLV system (12)

| | Reaction terms in (12) | Restrictions | Operators |
|---|---|---|---|
| 1 | $u(a_1+u+v)$<br>$v(a_2+\frac{\lambda_2}{\lambda_1}u+\frac{\lambda_2}{\lambda_1}v)$ | $\lambda_1 \neq \lambda_2$ | $Q_1^u=\partial_x+\frac{g_x^1}{g^1}u(\partial_u-\partial_v),$<br>$Q_1^v=\partial_x+\frac{g_x^2}{g^2}v(\partial_v-\partial_u)$ |
| 2 | $u(a+u+2v)$<br>$v(a+v)$ | $\lambda_1=1$ | $Q_2^v=G(x,v)\left(\partial_x+F(x,v)(\partial_u-\partial_v)\right)$ |
| 3 | $uv$<br>$v(a_2+c_2v)$ | $a_2c_2\neq 0$<br>$\lambda_1=1$ | $Q_3^u=\partial_x+r(t,x)\,u\,\partial_u,$<br>$Q_3^v=\left(h^1(\omega)-2th^2(\omega)\right)\partial_x$<br>$+\left((h^2(\omega)x+h^3(\omega))u+p(t,x,v)\right)\partial_u$ |
| 4 | $uv$<br>$c_2v^2$ | $c_2\neq 0$<br>$\lambda_1=1$ | $Q_3^u,\ Q_4^v=\left(h^1(\theta)-2th^2(\theta)\right)\partial_x$<br>$+\left((h^2(\theta)x+h^3(\theta))u+p(t,x,v)\right)\partial_u$ |
| | | | |

Table: Continuation: $Q$-conditional symmetries of the first type of the DLV system (12) with $\lambda_1 = \lambda_2 = 1$

| | Reaction terms in (12) | Restrictions | Operators |
|---|---|---|---|
| 5 | $uv$<br>$v\left(a_2 + \frac{v}{2}\right)$ | $a_2 \neq 0$ | $Q_3^u, \quad Q_3^v, \quad Q_5^v = \partial_x + e^{a_2 t} u \partial_v$<br>$+ \left(\alpha\, u - \frac{e^{-a_2 t}}{2}\, v^2 - a_2 e^{-a_2 t} v\right) \partial_u$ |
| 6 | $uv$<br>$\frac{1}{2}\, v^2$ | | $Q_3^u, \ Q_4^v,$<br>$Q_6^v = (\alpha_1 t + \alpha_0)\partial_x + (\alpha_1 t + \alpha_0) u \partial_v$<br>$+ \left(\left(\alpha_2 - \frac{\alpha_1}{2}\, x\right) u - \frac{\alpha_1 t + \alpha_0}{2}\, v^2 - \alpha_1 v\right) \partial_u$ |
| 7 | $uv$<br>$v(a_2 + v)$ | $a_2 \neq 0$ | $\alpha_1^2 + \alpha_2^2 \neq 0, \ Q_3^u, \ Q_3^v,$<br>$Q_7^u = \partial_x + \left(-\frac{x}{2t}\, u + \frac{\alpha_1}{t}\right.$<br>$\left. + \left(\frac{\alpha_2 e^{-a_2 t}}{t} + \frac{\alpha_1}{a_2 t}\right) v\right) \partial_u$ |
| 8 | $uv$<br>$v^2$ | $\alpha_1^2 + \alpha_2^2 \neq 0$ | $Q_3^u, \ Q_4^v,$<br>$Q_8^u = \partial_x + \left(-\frac{x}{2t}\, u + \frac{\alpha_1}{t} + \left(\frac{\alpha_2}{t} + \alpha_1\right) v\right) \partial_u$ |

In Table:
$\omega = \frac{a_2+c_2v}{\lambda v}\, e^{\frac{a_2}{\lambda}\,t},\ \theta = t + \frac{\lambda}{c_2 v};$
$h^1,\ h^2$ and $h^3$ are arbitrary smooth functions,
function $p(t,x,v)$ is the general solution of the linear ODE

$$p_t = p_{xx} - \frac{v(a_2+c_2v)}{\lambda}p_v + vp,$$

the functions $F$ and $G$ form the general solution of the system

$$FF_v - F_x + av + v^2 = 0,\ G_x = FG_v,$$

the function $r(t,x)$ is the general solution of the Burgers equation

$$r_t = r_{xx} + 2rr_x,$$

$$g^i(t,x) = \begin{cases} \alpha_0 \exp\left(\frac{\kappa^2 t}{\lambda_i}\right) + \alpha_1 \sin(\kappa\, x) + \alpha_2 \cos(\kappa\, x), & \text{if} \quad \frac{\lambda_1 a_2 - \lambda_2 a_1}{\lambda_1 - \lambda_2} > 0, \\ \alpha_0 \exp\left(-\frac{\kappa^2 t}{\lambda_i}\right) + \alpha_1 e^{\kappa x} + \alpha_2 e^{-\kappa x}, & \text{if} \quad \frac{\lambda_1 a_2 - \lambda_2 a_1}{\lambda_1 - \lambda_2} < 0, \\ \alpha_0 + \alpha_1 x + \alpha_2 \lambda_i x^2 + 2\alpha_2 t, & \text{if} \quad \lambda_1 a_2 = \lambda_2 a_1, \end{cases} \tag{28}$$

where $i = 1, 2,\ \kappa = \sqrt{\left|\frac{\lambda_1 a_2 - \lambda_2 a_1}{\lambda_1 - \lambda_2}\right|},\ \alpha_0,\ \alpha_1$ and $\alpha_2$ are arbitrary constants.

The DLV system with positive parameters

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(a_1 - b\,u - c\,v), \\ \lambda_2 v_t &= v_{xx} + v(a_2 - \tfrac{\lambda_2 b}{\lambda_1} u - \tfrac{\lambda_2 c}{\lambda_1} v). \end{aligned} \tag{29}$$

is applicable for describing the competition of two population of species. A four-parameter family of exact solutions obtained via Q-conditional symmetry reads as

$$u(t,x) = \frac{a_1 \exp(\frac{a_1}{\lambda_1}\,t)}{C_1 + \alpha_0 b \exp(\frac{a_1}{\lambda_1}\,t) + C_2 \lambda_2 \exp(\frac{a_2}{\lambda_2}\,t)}$$

$$\times\left(\alpha_0 + \alpha_1 \exp(\tfrac{\lambda_2 a_1 - \lambda_1 a_2}{\lambda_1(\lambda_1 - \lambda_2)}\,t)\sin\left(\sqrt{\tfrac{\lambda_1 a_2 - \lambda_2 a_1}{\lambda_1 - \lambda_2}}\,x\right)\right), \tag{30}$$

$$v(t,x) = \frac{1}{c}\,\frac{\alpha_0 a_1 b \exp(\frac{a_1}{\lambda_1}\,t) + C_2 a_2 \lambda_1 \exp(\frac{a_2}{\lambda_2}\,t)}{C_1 + \alpha_0 b \exp(\frac{a_1}{\lambda_1}\,t) + C_2 \lambda_2 \exp(\frac{a_2}{\lambda_2}\,t)} - \frac{b}{c}\,u(t,x).$$


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