Conservation laws and symmetries that depend on arbitrary functions

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Happy 70th Birthday, Peter!

Athena SWAN Silver Award
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LONDON MATHEMATICAL SOCIETY
GOOD PRACTICE SCHEME

Generalized symmetries of a system of PDEs

Consider an involutive system of PDEs, $\mathcal{A} = 0$, with components

$$\mathcal{A}_\ell(\mathbf{x}, [\mathbf{u}]) = 0, \quad \ell = 1, \dots, L;$$

- independent variables are $\mathbf{x} = (x^1, \dots, x^p)$,
- dependent variables are $\mathbf{u} = (u^1, \dots, u^q)$,
- $[\mathbf{u}]$ denotes $\mathbf{u}$ and finitely many derivatives $u^\alpha_{\mathbf{J}}$.

Assumption: All functions of $(\mathbf{x}, [\mathbf{u}])$ are analytic (locally).

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The total derivative with respect to x^i is

$$D_i=\frac{\partial}{\partial x^i}+u_i^\beta\,\frac{\partial}{\partial u^\beta}+u_{ji}^\beta\,\frac{\partial}{\partial u_j^\beta}+\cdots.$$

Multi-index notation: for $\mathbf{J}=(j^1,\dots,j^p)$, let $D_{\mathbf{J}}=D_1^{j^1}\cdots D_p^{j^p}$.

The (infinitely prolonged) vector field

$$X = Q^{\alpha}\frac{\partial}{\partial u^{\alpha}} + (D_i Q^{\alpha})\frac{\partial}{\partial u_i^{\alpha}} + \cdots = (\mathbf{D_J} Q^{\alpha})\frac{\partial}{\partial u_{\mathbf{J}}^{\alpha}}$$

generates generalized symmetries if the LSC holds:

$$X\mathcal{A}_{\ell} = 0 \quad \text{when} \quad [\mathcal{A} = 0], \qquad \ell = 1, \ldots, L.$$

Square brackets denote the enclosed expression and its derivatives. The (symmetry) characteristic is $\mathbf{Q} = (Q^1(\mathbf{x}, [\mathbf{u}]), \ldots, Q^q(\mathbf{x}, [\mathbf{u}]))$.

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The LSC is linear homogeneous in $\mathbf{Q}$; for some PDEs, it has solutions that are linear homogeneous in one or more arbitrary functions $\mathrm{g}^r(\mathbf{x})$, perhaps subject to linear differential constraints,

$$K_{sr}(\mathrm{g}^r) = 0, \qquad s = 1, \ldots, S.$$

Running example

Various applications generate a pseudoparabolic conservation law of the following form, for given functions M and Ψ:

$$\mathcal{A} := u_t - D_x\{M(u)D_xD_t(\Psi(u))\} = 0. \qquad (1)$$

This admits symmetries with characteristic $Q = \mathrm{g}(t)u_t$; here, g is arbitrary subject to $D_x\mathrm{g} = 0$.

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Reduction of order: Write (8) in terms of the differential invariants $x, u, p = u_x$ and simplify to get

$$D_x\big\{M(u)D_u(p\Psi'(u))\big\} + D_u\big\{pM(u)D_u(p\Psi'(u)) - u\big\} = 0.$$

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Reduction of order for PDEs **may** occur for Lie pseudogroups; the details of the group foliation are critical (Ovsiannikov, Thompson & Valiquette, Kruglikov, Schneider).

Conservation laws

A conservation law (CLaw) for $\mathcal{A} = 0$ is a divergence expression, $\mathcal{C} = D_i F^i$, that vanishes on solutions of the PDE:

$$\mathcal{C} = 0 \quad \text{when} \quad [\mathcal{A} = 0]. \tag{2}$$

A CLaw is *trivial* whenever it is the sum of:

1. a CLaw whose components F^i vanish when $\mathcal{A} = 0$,
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Each CLaw, $\mathcal{C}$, has an equivalent *characteristic form*:

$$\widetilde{\mathcal{C}} := \mathcal{Q}^{\ell} \mathcal{A}_{\ell} = \mathrm{Div}(\widetilde{\mathbf{F}}).$$

The function $\mathcal{Q}$ is the *characteristic* (or multiplier) for $\widetilde{\mathcal{C}}$.

Lie pseudogroups of variational symmetries are reflected in conservation laws of Euler–Lagrange (E–L) equations,

$$\mathcal{A}_\alpha = \mathbf{E}_{u^\alpha}(L(\mathbf{x},[\mathbf{u}]) := (-\mathbf{D})_\mathbf{J} \frac{\partial L}{\partial u^\alpha_\mathbf{J}} = 0.$$

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Integrate by parts to get a conservation law in characteristic form:

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The correspondence between (3) and (4) gives

Noether's (First) Theorem

$\mathbf{Q}$ is a characteristic of a variational symmetry generator if and only if it is a characteristic of a CLaw for the E–L equations.

"Noether 1.5" (H. and Mansfield)

Theorem Suppose that a characteristic of variational symmetry generators is linear homogeneous in R independent functions $\mathrm{g}^r(\mathbf{x})$ that are subject to a complete set of linear differential constraints,

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Then the E–L equations $\mathcal{A}_\alpha := \mathbf{E}_{u^\alpha}(L) = 0$ satisfy the identities

$$\mathbf{E}_{\mathrm{g}^r}\left\{ Q^\alpha(\mathbf{x}, [\mathbf{u}; \mathrm{g}])\mathcal{A}_\alpha \right\} = (K_{sr})^\dagger(\mu^s), \qquad r = 1, \ldots, R. \tag{5}$$

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If there are differential relations (syzygies) between the E–L equations, these are obtained by eliminating μ^s from (5).

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The identities (5) give families of conservation laws corresponding to Noether's (First) Theorem:

$$\widetilde{\mathcal{C}} = \mu^s K_{sr}(\mathrm{g}^r) - \mathrm{g}^r (K_{sr})^\dagger(\mu^s).$$

Example For $M = \Psi'$, set $u = w_x$ in (8) to get an E–L equation,

$$\mathcal{A} := w_{xt} - D_x\{\Psi'(w_x)D_xD_t(\Psi(w_x))\} = 0. \qquad (6)$$

This has variational symmetries with $Q = \mathrm{g}^1(t)w_t + g^2(t)$; again, regard each g^r as arbitrary subject to $D_x\mathrm{g}^r = 0$.

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The Noether 1.5 identities (5) are $w_t \mathcal{A} = -D_x \mu^1, \ \mathcal{A} = -D_x \mu^2$, whose solutions are the first integrals

$$\mu^1 = \tfrac{1}{2}\{D_t(\Psi(w_x))\}^2 - \tfrac{1}{2} w_t^2, \qquad \mu^2 = \Psi(w_x) D_x D_t(\Psi(w_x)) - w_t.$$

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Eliminate w_t and use the symmetries to get a reduced PDE for u:

$$\{D_t(\Psi(u))\}^2 - \{\Psi(u) D_x D_t(\Psi(u))\}^2 = \mathrm{sgn}(c), \qquad c \in \mathbb{R}.$$

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For $c = 0$, there is a symmetry reduction to an ODE for $p(x, u)$:

$$D_u\{p\Psi'(u) \pm \ln|\Psi(u)|\} = 0.$$

Example: Area-preserving variational symmetries

If $\mathbf{Q}$ depends on $\mathrm{g}^1(x,y),\ \mathrm{g}^2(x,y)$, subject to $\mathrm{g}^1_x + \mathrm{g}^2_y = 0$, the conservation laws and differential relations come from

$$\mathbf{E}_{\mathrm{g}^1}\{Q^\alpha \mathcal{A}_\alpha\} = -D_x\mu, \qquad \mathbf{E}_{\mathrm{g}^2}\{Q^\alpha \mathcal{A}_\alpha\} = -D_y\mu.$$

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Notes

1. Noether 1.5 bridges the gap between Noether's Theorems.
2. Generally, the conservation laws from Noether 1.5 are not in characteristic form.
3. If g is arbitrary in $p-1$ independent variables, the Noether 1.5 conservation laws are first integrals (cf Popovych & Bihlo).

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$$\mathcal{A} := u_t - D_x\{\exp(-\Psi(u))D_xD_t(\Psi(u))\} = 0, \tag{8}$$

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has a CLaw characteristic $\mathcal{Q} = \mathrm{g}\exp(\Psi(u))$, subject to $D_t\mathrm{g} = 0$. The identity (7) amounts to

$$\exp(\Psi(u))\mathcal{A} = -D_t\mu,$$

so μ is a first integral. It yields the reduced equation

$$D_x^2(\Psi(u)) - \tfrac{1}{2}\{D_x(\Psi(u))\}^2 - \int \Psi(u)\,\mathrm{d}u = f(x)$$

(a parametrized ODE).

Example (2-D Euler equations)

The 2-D Euler equations (with unit density) are $\mathcal{A}_\alpha = 0$, where

$$\mathcal{A}_1 = u_t + uu_x + vu_y + p_x, \quad \mathcal{A}_2 = v_t + uv_x + vv_y + p_y, \quad \mathcal{A}_3 = u_x + u_y.$$

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Applying the theorem gives something more meaningful:

$$\mathcal{A}_1 + (u + xD_t)\mathcal{A}_3 = -D_x\mu^1 - D_y\mu^2 \quad \text{(and similarly for } \mathcal{A}_2\text{)}$$

$$\rightarrow \qquad \widetilde{\mathcal{C}} = D_x\{xu_t + u^2 + p\} + D_y\{xv_t + uv\} \qquad \text{(force balance).}$$

Questions?