Ramified local isometric embeddings of singular Riemannian metrics

Niky Kamran

McGill University, Montréal, Canada

Symmetry, Invariants and their Applications, a celebration of Peter Olver's 70th birthday
Halifax, August 4, 2022

Joint with Alberto Enciso (ICMAT, Madrid)

1. Local isometric embeddings - the regular case (Cartan-Janet)

Section 1 is textbook material. Main references include :

- Han, Q. and Hong, J., *Isometric embedding of Riemannian manifolds in Euclidean spaces*, Mathematical Surveys and Monographs, **130**, American Mathematical Society, Providence, RI, 2006.
- Jacobowitz, H., *Local isometric embeddings*, Seminar on Differential Geometry, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.
- M. Spivak, *A comprehensive introduction to differential geometry*, Vol. 5, Chapter 11, Publish or Perish, Houston, 1979.

Local isometric embeddings

The regular case

Let $U \ni 0$ be an open neighbourhood in $\mathbb{R}^n$, with local coordinates $x = (x_\alpha)_{1 \leq \alpha \leq n}$. We write $x = (x', x_n)$ where $x' = (x_k)_{1 \leq k \leq n-1}$. On U, we consider a C^ω Riemannian metric, written with no loss of generality as

$$g = g_{nn}(x)\, dx_n^2 + g_{kl}(x)\, dx_k\, dx_l \,, \tag{1}$$

where $1 \leq k, l \leq n-1$ and where the coefficients g_{nn} and g_{kl} are functions of all the local coordinates $x = (x_\alpha), 1 \leq \alpha \leq n$.

Remark :
We may choose coordinates in which $g_{nn} = 1$, but we won't do it here, in anticipation of the singular case that will be treated below.

Local isometric embedding (LIE) problem

Basic question : Can one locally isometrically embed (U, g) into a Euclidean space $\mathbb{E}^N$ for some N?

The PDE system governing LIEs

The LIE problem is a PDE problem :

Recall, that by definition, (U, g) can be locally isometrically embedded in $\mathbb{E}^N$ if $\forall x \in U, \exists$ an open neighborhood $W \subset U$ of x and a smooth map $\mathbf{u} : W \to \mathbb{E}^N$ of rank n satisfying

$$g = \mathbf{u}^* g_{\mathbb{E}^N} \, .$$

This is equivalent to the system of $n(n+1)/2$ first-order PDEs given by

$$\|\partial_n \mathbf{u}\|^2 = g_{nn} \, , \tag{2}$$

$$\partial_k \mathbf{u} \cdot \partial_n \mathbf{u} = 0 \, , \tag{3}$$

$$\partial_k \mathbf{u} \cdot \partial_l \mathbf{u} = g_{kl} \, , \tag{4}$$

where the dot to denotes the Euclidean inner product in the ambient Euclidean space $\mathbb{E}^N$.

Remark : We won't distinguish between U and W from now on since the problem is local.

The Cartan-Janet Theorem

If $N = n(n+1)/2$, we have as many equations as unknowns. The Cartan-Janet Theorem says that in this case the system always admits a solution in the C^ω category :

Theorem (E. Cartan, M. Janet, 1928)

(U, g) can be locally isometrically embedded in $\mathbb{E}^{n(n+1)/2}$ by a C^ω map.

Remark : The much harder case of global isometric embeddings of Riemannian manifolds of class C^k was solved much later in the breakthrough work of Nash (1954), who took a very different approach based on the development of a powerful open mapping theorem. Important subsequent improvements are due to Gromov, Rokhlin and Günther, with improved codimensions. For the Lorentzian case, we refer to the important results of Müller and Sánchez (2011), where the global isometric embedding problem is solved under the assumptions of stable causality and the existence of a steep temporal function.

The proof of the Cartan-Janet Theorem is obtained by successive application of the Cauchy-Kovalevskaia Theorem, working one dimension at a time. We will recall the main steps of the proof, in preparation for the singular case.

We thus consider Cauchy data for the system (2), (3), (4) along the hypersurface $x_n = 0$, given by C^ω maps $\mathbf{u}_0, \mathbf{u}_1$,

$$\mathbf{u}|_{x_n=0} = \mathbf{u}_0 \,, \quad \partial_n \mathbf{u}|_{x_n=0} = \mathbf{u}_1 \,. \tag{5}$$

Observe that the data are constrained in view of (2), (3) and (4) by

$$\|\mathbf{u}_1\|^2 = g_{nn}(\cdot, 0) \,, \tag{6}$$

$$\partial_k \mathbf{u}_0 \cdot \mathbf{u}_1 = 0 \,, \tag{7}$$

$$\partial_k \mathbf{u}_0 \cdot \partial_l \mathbf{u}_0 = g_{kl}(\cdot, 0) \,. \tag{8}$$

The system (2), (3), (4) is not in Cauchy-Kovalevskaia form. We therefore differentiate the equations (2), (3), (4) with respect to x_n to obtain the system of $n(n+1)/2$ second-order PDEs given by

$$\partial_n \mathbf{u} \cdot \partial_{nn} \mathbf{u} = \frac{1}{2} \partial_n g_{nn} \,, \tag{9}$$

$$\partial_k \mathbf{u} \cdot \partial_{nn} \mathbf{u} = -\frac{1}{2} \partial_k g_{nn} \,, \tag{10}$$

$$\partial_{kl} \mathbf{u} \cdot \partial_{nn} \mathbf{u} = \partial_{kn} \mathbf{u} \cdot \partial_{ln} \mathbf{u} - \frac{1}{2} \partial_{nn} g_{kl} - \frac{1}{2} \partial_{kl} g_{nn} \,. \tag{11}$$

The Cauchy data $\mathbf{u}_0 \, , \mathbf{u}_1$ must satisfy besides (6), (7), (8), the additional constraint

$$\partial_{kl} \mathbf{u}_0 \cdot \mathbf{u}_1 = -\frac{1}{2} \partial_n g_{kl}(\cdot, 0) \,. \tag{12}$$

This process can be reversed to show that the Cauchy problems for the first and second-order systems are equivalent. We have :

Proposition

Consider a C^ω Riemannian metric (1). The system of first-order PDEs (2), (3), (4) governing local isometric embeddings of (U, g) into $\mathbb{E}^N$, with initial data (5) constrained by (6) to (8), is equivalent to the system of second-order PDEs (9), (10), (11) with initial data (5) constrained by (6) to (8) and (12).

Remark : The above proposition is true even in the C^∞ case since it rests on an ODE argument.

Remark : Closely related to the constraints on the Cauchy data is the fact that a LIE does not always admit an isometric extension. In other words, if $(H, g\,|_H) \subset (U, g)$ is a $(n-1)$-submanifold and $\mathbf{v} : (H, g\,|_H) \to \mathbb{E}^N$ is a LIE, then there doesn't always exist a LIE $\mathbf{u} : U \to \mathbb{E}^N$ such that $\mathbf{u}\,|_H = \mathbf{v}$.

Indeed, let γ be a minimizing geodesic curve between a pair of sufficiently close points $x, y \in U \subset \mathbb{R}^2$ and let ρ be a non-geodesic curve from x to y. Let $\mathbf{v} : \rho \to \mathbf{v}(\rho) \subset \mathbb{E}^3$ be a LIE where $\mathbf{v}(\rho)$ is chosen to be a straight line segment. Let d be the arc length distance from x to y measured along ρ. Since $\mathbf{v}$ is an isometry from ρ onto its image, we have

$$d = \text{length}\,(\rho) = \text{length}\,(\mathbf{v}(\rho))\,,$$

Suppose now that $\mathbf{v}$ admits a local isometric extension $\mathbf{u} : U \to \mathbb{E}^3$. Since $\mathbf{v}(\rho) \subset \mathbb{E}^3$ is a straight line segment, we have

$$\text{length}\,(\mathbf{v}(\rho)) = \text{length}\,(\mathbf{u}(\rho)) = \text{dist}_{\mathbb{E}^3}(\mathbf{u}(x), \mathbf{u}(y))\,.$$

Since γ is a minimizing geodesic and ρ is a non-geodesic, so we have

$$\text{length}\,(\gamma) < \text{length}\,(\rho) = d\,.$$

But

$$\text{length}\,(\gamma) = \text{length}\,(\mathbf{u}(\gamma)) = \text{length}\,(\mathbf{v}(\gamma)) = d\,,$$

which is a contradiction. This extends to all higher dimensions (Jacobowitz).

The proof of the Cartan-Janet Theorem thus amounts to constructing the local isometric embedding by induction on n, applying the Cauchy-Kovalevskaya Theorem at each step to the system (9), (10), (11) with suitably chosen initial data satisfying (6) to (8) and (12).

We summarize the iterative step as a proposition, which follows directly from the Cauchy-Kovalevskaya Theorem by putting the system of PDEs (9), (10), (11) in Cauchy-Kovalevskaya form. This is done by solving for $\partial_{nn}\mathbf{u}$, assuming a rank hypothesis on the initial data :

Proposition

Consider a C^ω Riemannian metric (1). The equivalent system of second-order PDEs (9), (10), (11) governing local isometric embeddings of (U, g) into $\mathbb{E}^N$, with initial data $\mathbf{u}_0, \mathbf{u}_1$ constrained by (6) to (8) and (12) , admits a unique local analytic solution $\mathbf{u}$ if the set $\{\partial_k \mathbf{u}_0, \partial_{kl}\mathbf{u}_0, \mathbf{u}_1,\ 1 \leq k, l \leq n-1\}$ is linearly independent at every point of the initial hypersurface $x_n = 0$.

Proof of Cartan-Janet by induction on n

Since g is regular, we may choose coordinates so that $g_{nn} = 1$,

$$g = dx_n^2 + g_{kl}(x)\, dx_k\, dx_l\,.$$

Start with $n = 2$:

$$g = dx_2^2 + g_{11}(x_1, x_2) dx_1^2\,.$$

We further assume with no loss of generality that

$$g_{11}(x_1, 0) = 1\,, \quad \partial_2 g_{11}(x_1, 0) = 0\,,$$

and choose the Cauchy data to be given by

$$\mathbf{u}_0(x_1) = (\cos x_1, \sin x_1, 0)\,, \quad \mathbf{u}_1(x_1) = (0, 0, 1)\,.$$

The constraints (6) to (8) and (12) are satisfied. Furthermore, $\partial_1 \mathbf{u}_0, \partial_{11} \mathbf{u}_0$ and $\mathbf{u}_1$ are linearly independent. So by the above proposition, we get a C^ω LIE of g in $\mathbb{E}^3$.

We now proceed with the inductive step. Write for $n \geq 3$

$$g = dx_n^2 + g_{kl}(x)\, dx_k\, dx_l\,,$$

where we assume with no loss of generality that

$$g_{kl}(0) = \delta_{kl}\,, \quad \partial_n g_{kl}(0) = 0\,.$$

By the induction hypothesis, $\exists$ a C^ω map $\mathbf{v} : V \to \mathbb{E}^{(n-1)n/2}$ such that

$$g_{kl}(x', 0) = \mathbf{v}^* g_{\mathbb{E}^{n-1}}\,.$$

We take as a candidate for the Cauchy data

$$\mathbf{u}_0 = (\mathbf{v}, \mathbf{0})\,, \quad \tilde{\mathbf{u}}_1 = (\mathbf{0}, 1)\,.$$

By construction, we have

$$\|\tilde{\mathbf{u}}_1\|^2 = 1\,, \quad \partial_k \mathbf{u}_0 \cdot \tilde{\mathbf{u}}_1 = 0\,,$$

and

$$\partial_k \mathbf{u}_0 \cdot \partial_l \mathbf{u}_0 = g_{kl}(\cdot, 0)\,,$$

as required.

However we have

$$\partial_{kl} \mathbf{u}_0 \cdot \tilde{\mathbf{u}}_1 = 0\,,$$

instead of

$$\partial_{kl} \mathbf{u}_0 \cdot \tilde{\mathbf{u}}_1 = -\frac{1}{2} \partial_n g_{kl}(\cdot, 0)\,.$$

The idea is to perturb $\tilde{\mathbf{u}}_1$ by adding a suitable term to ensure that the constraints on the Cauchy data are satisfied. Thus we write

$$\mathbf{u}_1 = \tilde{\mathbf{u}}_1 + \mathbf{p}\,,$$

where $\mathbf{p} : V \to \mathbb{E}^{n(n+1)/2}$ is a C^ω map defined in a neighborhood of the origin $x' = 0$ in the initial hypersurface, such that the constraints are satisfied.

The necessary and sufficient conditions on $\mathbf{p}$ for the Cauchy data $\mathbf{u}_0, \mathbf{u}_1$ to satisfy the constraints are given by

$$\partial_k \mathbf{u}_0 \cdot \mathbf{p} = 0\,, \quad \partial_{kl} \mathbf{u}_0 \cdot \mathbf{p} = -\frac{1}{2}\partial_n g_{kl}(\cdot, 0)\,,$$

and

$$2\,\mathbf{p} \cdot \tilde{\mathbf{u}}_1 + \|\mathbf{p}\|^2 = 0\,.$$

By using the condition $\partial_n g_{kl}(0) = 0$ and applying the Implicit Function Theorem, we obtain the existence of a solution $\mathbf{p}$ near the origin $x' = 0$ in the initial hypersurface, as required.

This completes the proof of the Cartan-Janet Theorem.

2. Local isometric embeddings - the case of admissible singularities

Admissible singularities

Admissible singularities

We say that a Riemannian metric g defined on a domain $U \ni 0$ of $\mathbb{R}^n$ has an admissible singularity at the origin if it is of the form

$$g = (\|x'\|^2 + x_n^{2l})\, F_0(x)\, dx_n^2 + g_{kl}\, dx_k\, dx_l\,, \tag{13}$$

where $l \geq 1$ is an integer, where F_0 and g_{jk} are C^ω with $F_0(0) > 0$, where the quadratic form defined by (g_{kl}) is positive definite, and where

$$\partial_n g_{jk}(x', 0) = O(\|x'\|^2)\,. \tag{14}$$

The main result on ramified LIEs

Theorem (Alberto Enciso, N.K.)

Let g be a C^ω Riemannian metric defined on a domain $U \subset \mathbb{R}^n$ with an admissible singularity at the origin. Then there exists a local C^ω isometric embedding $\mathbf{u} : (U', \Pi^* g) \to \mathbb{E}^{(n^2+3n-4)/2}$, where $\Pi : U' \to U \backslash \{0\}$ is a finite Riemannian branched cover of $(U \backslash \{0\}, g)$.

The proof uses a ramified version of the Cauchy-Kovalevskaia Theorem, due to Leray and Gårding-Kotake-Leray for linear systems, and extended to the non-linear case by Choquet-Bruhat. We now summarize these results.

3. Leray's ramified Cauchy-Kovalevskaia Theorem

The main references for Section 3 are :

- Leray, J., *Problème de Cauchy I. Uniformisation de la solution du problème linéaire analytique de Cauchy près de la variété qui porte les données de Cauchy*, Bull. Soc. Math. France, **85** (1957), pp. 389-429.
- Gårding, L., Kotake. T., and Leray, J., *Uniformisation et développement asymptotique de la solution du problème de Cauchy linéaire, à données holomorphes ; analogie avec la théorie des ondes asymptotiques et approchées (Problème de Cauchy, I bis et VI).*, Bull. Soc. Math. France, **92** (1964), pp. 263-361.
- Gårding, L., *Partial differential equations : Problems and uniformization in Cauchy's problem*, in Lectures on Modern Mathematics, Vol. II, pp. 129-150, Wiley, New York, 1964.
- Choquet-Bruhat, Y., *Uniformisation de la solution d'un problème de Cauchy non linéaire à données holomorphes*, Bull. Soc. Math. France, **94** (1966), pp. 25-48.

We begin with the scalar linear case, and consider on an open subset V of $\mathbb{R}^n$ with local coordinates $x = (x_1 \ldots, x_n)$ an m-th order linear differential operator

$$A = a(x, \partial_x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha \,, \tag{15}$$

with C^ω coefficients.

We are interested in the Cauchy problem for the PDE

$$a(x, \partial_x) u(x) = v(x) \,, \tag{16}$$

where v is C^ω in V, where we prescribe the values of a C^ω function w and its derivatives of order $0 \leq k \leq m-1$ on a C^ω hypersurface $S \subset V$ given as the zero set $s(x) = 0$ of a C^ω function s,

$$u(x) - w(x) = O(s(x)^m) \,. \tag{17}$$

The principal part G of A, defined by

$$G = g(x, \partial_x) := \sum_{|\alpha|=m} a_\alpha(x)\partial_x^\alpha\,, \tag{18}$$

plays an important role in the ramified Cauchy problem. It defines on T^*V a C^ω real-valued function g given in bundle coordinates $(x, p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ by

$$g(x, p) = \sum_{|\alpha|=m} a_\alpha(x)p^\alpha\,. \tag{19}$$

The function $g(x, p)$ is thus homogeneous of degree m in the fiber coordinates $p = (p_1, \ldots, p_n)$, and is invariant under lifts of local diffeomorphisms of V.

The condition for a point $x \in S$ to be characteristic for the Cauchy problem involves the hypersurface S supporting the data and the principal part G of A :

Definition

We say that a point $x \in S$ is characteristic for the Cauchy problem (16),(17) if

$$g(x, \partial_x s(x)) = 0\,. \tag{20}$$

The subset of characteristic points $x \in S$ will be denoted by C.

The classical Cauchy-Kovalevskaya Theorem guarantees the existence of a unique C^ω solution to the Cauchy problem (16),(17) for data that are non-characteristic.

Leray's extension of the Cauchy-Kovalevskaya Theorem is precisely concerned with this case where the data are allowed to be characteristic on a non-empty subset of the initial hypersurface S. In a nutshell :

- The Cauchy problem (16),(17) will have a C^ω solution that is ramified around a characteristic subvariety tangent to the initial hypersurface S. The ramification locus can be described geometrically in terms of the flow of a Hamiltonian vector field on T^*V associated to g and the initial hypersurface S.
- A uniformizing map for the solution will be constructed explicitly through the solution of an auxiliary Cauchy problem for a Hamilton-Jacobi equation associated to g and the initial hypersurface S.

The above results will be formulated precisely below.

We remark that for the application of Leray's results to the LIE problem of the class of metrics with an admissible singularity, we shall be concerned with the case for which the data are characteristic at a single point $x \in S$, with an additional non-degeneracy condition that will be specified below.

The first step is to extend the Cauchy problem (16),(17) by the addition of an auxiliary variable ξ. Consider on $(-\eta,\eta)\times V\subset\mathbb{R}^{l+1}$ with coordinates (ξ,x) the modified Cauchy problem given by

$$a(x,\partial_x)u(\xi,x)=v(\xi,x)\,, \tag{21}$$

where the initial hypersurface S is now replaced by the hypersurface S_ξ defined as the level set $s(x)=\xi$, where v is assumed to be C^ω in $(-\eta,\eta)\times V$, and where the Cauchy problem is now given by prescribing the values of an analytic function $w(\xi,x)$ and its derivatives of order $0\leq k\leq m-1$ on S_ξ,

$$u(\xi,x)-w(\xi,x)=O(s(\xi,x)^m)\,, \tag{22}$$

with $s(\xi,x):=s(x)-\xi$.

The set of characteristic points for the Cauchy problem (21), (22) will be denoted by C_ξ.

The key idea behind the uniformization of the solution of the Cauchy problem is to pass to the modified Cauchy problem (21), (22), and to construct the uniformizing map by means of the solution $\xi(t,x)$ of the Cauchy problem for the auxiliary Hamilton-Jacobi equation associated to the Hamiltonian g defined by (19) :

$$\partial_t \xi + g(x, \partial_x \xi) = 0\,, \quad \xi(0,x) = s(x)\,, \tag{23}$$

This Cauchy problem has a unique C^ω solution $\xi(t,x)$ defined for $|t| < \epsilon$ sufficiently small. Now by construction, the map $f : (-\epsilon, \epsilon) \times V \to (-\eta, \eta) \times V$ defined by

$$(t,x) \mapsto f(t,x) = (\xi(t,x), x)\,, \tag{24}$$

maps the hypersurface $t = 0$ to the hypersurface S_ξ.

Basic observation :

Since the Jacobian determinant of this map equals $\partial_t \xi$, the zero set

$$Z_\xi := \{(t,x) \in (-\epsilon, \epsilon) \times V \mid \partial_t \xi(t,x) = 0\}, \tag{25}$$

corresponds precisely by (23) to the analytic subvariety of $(-\epsilon, \epsilon) \times V$ on which the characteristic condition $g(x, \partial_x \xi(t,x)) = 0$ is satisfied.

This leads one naturally to define the characteristic conoid $K_\xi \subset (-\eta, \eta) \times V$ as the image under f of $Z_\xi \subset (-\epsilon, \epsilon) \times V$,

$$K_\xi = f(Z_\xi)\,. \tag{26}$$

We thus see that for any C^ω function $u(\xi, x)$, the map

$$(u \circ f)(t,x) = u(\xi(t,x), x)\,,$$

obtained by composition of f with u will be in general multivalued, and ramified precisely along the characteristic conoid K_ξ.

Remark :The characteristic conoid K_ξ is directly related to the characteristic strips for the Hamilton-Jacobi equation (23) :

Indeed, given the Hamiltonian vector field X_g on T^*V corresponding to g, the characteristic strips are the integral curves $\gamma(t)$, $-\epsilon < t < \epsilon$, of X_g whose initial points $\gamma(0)$ are located on the submanifold of T^*V obtained as the Lagrangian lift of S. The projections onto V of these integral curves are by definition the bi-characteristic curves $x(t), \epsilon < t < \epsilon$, of the original Cauchy problem (16), (17). Leray shows :

Lemma

The characteristic conoid K_ξ is the union of the projections of the bi-characteristic curves $x(t), \epsilon < t < \epsilon$, whose initial points $x(0)$ are elements of the subset C_ξ of characteristic points of S_ξ.

If we fix a point $x \in C_\xi$, the subset of K_ξ consisting of the images of the bi-characteristics $x(t)$, $-\epsilon < t < \epsilon$ such that $x(0) = x$ will be denoted by K_x and referred to as the conoid with vertex at x. Our analyticity hypotheses on the coefficients a_α of A imply immediately that $K_x \setminus \{x\}$ is a C^ω submanifold of V.

Finally, we need to define what is meant by an exceptional characteristic point for the Cauchy problem (21), (22) :

Definition

A characteristic point $x \in C_\xi$ is said to be exceptional if either the initial hypersurface S_ξ and the conoid K_x are tangent to each other at infinitely many points in a neighbourhood of x, or the characteristic strip $\gamma(t)$, $-\epsilon < t < \epsilon$, with initial point $\gamma(0) = (x, \partial_x s(x))$ consists of a single point.

Remark : In our application of Leray's theory to the local isometric embeddings for the Riemannian metrics admitting an admissible singularity, the characteristic subset will consist of a single non-exceptional characteristic point for the differential system governing local isometric embeddings.

We are now ready to state Leray's uniformization theorem :

Theorem

Let $\xi = \xi(t,x)$ be the solution of the Cauchy problem (23) for the Hamilton-Jacobi equation. In a neighborhood of a non-exceptional characteristic point, the map $(t,x) \mapsto (\xi(t,x),x)$ uniformizes the solution $u(\xi,x)$ of the Cauchy problem (21), (22), in the sense that the composition

$$u(\xi(t,x),x) := u \circ \xi\,,$$

and its derivatives of order $1 \leq j \leq m-1$,

$$\partial_\xi^j u(\xi(t,x),x)\,, \quad 1 \leq j \leq m-1\,,$$

are C^ω for $(t,x) \in (-\epsilon,\epsilon) \times V$. Furthermore, the support of the ramification locus of the multi-valued function $u(\xi,x)$ solving the Cauchy problem (21), (22) lies in the set of points $(\xi,x) \in (-\eta,\eta) \times V$ for which the hypersurface S_ξ is tangent to the conoid K_x. Likewise, the restriction of $u(\xi,x)$ to the locus $\xi(t,x) = 0$ uniformizes the solution $u(0,x)$ of the original Cauchy problem (16), (17). Finally, the singularities of u in the neighbourhood of any non-exceptional characteristic point are algebroid.

Here is a very simple example taken from Gårding's paper : Let

$$a(x, \partial_x) := \partial_{x_1} ,$$

with initial data given on the hypersurface S defined by

$$x_2 - x_1^p = 0 ,$$

where $p > 0$ is a positive integer. The solution of this Cauchy problem is given by

$$u(x) = w(x_2^{1/p}, x_2 \ldots, x_n) + \int_{x_2^{1/p}}^{x_1} v(s, x_2 \ldots, x_n)\, ds .$$

This solution is not analytic, but it is ramified along the hyperplane $x_2 = 0$, with p branches. The ramification locus $x_2 = 0$ is tangent to S (to order p) along the characteristic submanifold of S given by $x_1 = x_2 = 0$, and the uniformization map is simply given by

$$(x_1, x_2, \ldots, x_l) \mapsto (x_1, t^p, \ldots, x_l) .$$

For a lot more examples, see :
Johnsson, G., *The Cauchy problem in $\mathbb{C}^N$ for linear second order partial differential equations with data on a quadric surface*, Trans. Amer. Math. Soc. **344** (1994), no. 1, pp. 1-48

In Choquet-Bruhat's generalization of Leray's uniformization theorem, the scalar linear PDE (21) is replaced a non-linear system of N PDEs for N unknowns, of the form

$$F[u] := \big(F_j(x, \xi, D^m u)\big)_{j=1}^{N} = 0\,, \tag{27}$$

where $u(\xi, x) = (u_1(\xi, x), \ldots, u_N(\xi, x))$, and where F is C^ω in all its arguments. There is no loss of generality in expressing the system (27) in the form

$$F[u] := \big(F_j(x, \xi, D^{m_k - n_j} u)\big)_{j=1}^{N} = 0\,, \tag{28}$$

in which m_k, n_j (with $1 \leq k, j \leq N$) are non-negative integers. The Cauchy problem (22) is then replaced by the prescription of N functions $w_k(\xi, x)$, $1 \leq k \leq N$, of class C^ω and derivatives of order $0 \leq k \leq m_k - 1$ on S_ξ, that is

$$u_k(\xi, x) - w_k(\xi, x) = O(s(\xi, x)^{m_k})\,. \tag{29}$$

One puts the system (27) in quasi-linear form by differentiation. The vanishing condition (20) that defines characteristic points in the scalar linear case is then replaced by the vanishing condition of the determinant of a matrix $\mathcal{A}$ governing the linear dependence of the highest-order derivatives appearing in each of the differentiated equations.

Define the matrix $\mathcal{A}(\xi, x, p) = \mathcal{A}_{jk}(\xi, x, p)$ by

$$\mathcal{A}_{jk}(\xi, x, p) := \sum_{|\alpha| = m_k - n_j} \frac{\partial F_j}{\partial(\partial^\alpha u_k)} \, p^\alpha \, .$$

Definition

We say that $x \in S$ is characteristic for the Cauchy problem (28), (29) if $\mathcal{A}_*(x) = 0$, where

$$\mathcal{A}_*(x) := \det \left(\mathcal{A}(s(x), x, \partial_x s(x)) \big|_{u_k(x) = w_k(x)} \right) = 0 \, . \tag{30}$$

The rest of the analysis is essentially similar to the scalar linear case, with the function induced by $\det \mathcal{A}(s(x), x, p)$ on the cotangent bundle T^*V playing the role of the Hamiltonian $g(x, p)$. Leray's Theorem carries over with the modification that it is now each of the components $u_k(\xi, x)$ of u with its derivatives of order $1 \leq j \leq m_k - 1$ which gets uniformized by the map $(t, x) \mapsto (\xi(t, x)), x)$:

Theorem

In a neighborhood of a non-exceptional characteristic point, the compositions

$$u_k(\xi(t, x), x) := u_k \circ \xi\,, \quad 1 \leq k \leq N\,, \tag{31}$$

and their derivatives of order $1 \leq j \leq m_k - 1$*,*

$$\partial_\xi^j u_k(\xi(t, x), x)\,, \quad 1 \leq k \leq N\,, 1 \leq j \leq m_k - 1\,.$$

are C^ω *for* $(t, x) \in (-\epsilon, \epsilon) \times V$*. The support of the ramification locus admits the same description as above. In particular, the restriction of* $u(\xi, x)$ *to the locus* $\xi(t, x) = 0$ *uniformizes the solution* $u(0, x)$ *of the Cauchy problem given by (28) and (29). Likewise the singularities of* u *in the neighbourhood of any non-exceptional characteristic point will be algebroid.*

4. Proof of the theorem on ramified LIE

We begin by remarking that the normal form (1), in which the components g_{nj} of the metric tensor are identically zero, can be achieved by a suitable C^ω local diffeomorphism for a class singular metrics which include as a special case the metrics with an admissible singularity at the origin.

Proposition

Consider on a domain $U \subset \mathbb{R}^n$ a singular C^ω metric

$$g = g_{\alpha\beta}(x)\, dx_\alpha\, dx_\beta = g_{nn}(x)\, dx_n^2 + 2b_j(x)\, dx_j\, dx_n + g_{jk}(x)\, dx_j\, dx_k\,, \qquad (32)$$

where by singular we mean that g_{nn} has an isolated zero at the origin $0 \in U$ and $\det(g_{kl}) \neq 0$ in U. Then there exists a C^ω local diffeomorphism f of the form

$$x_n = \bar{x}_n\,, \quad x_j = \bar{x}_j + f_j(\bar{x})\,, \qquad (33)$$

such that the components $\bar{b}_j$ of the transformed metric $\bar{g} := f^*g$ are identically zero, that is $\bar{g}$ takes the form

$$\bar{g} = \bar{g}_{nn}(\bar{x})\, d\bar{x}_n^2 + \bar{g}_{kl}(\bar{x})\, d\bar{x}_k\, d\bar{x}_l\,. \qquad (34)$$

The first step in the proof is to show that the coefficients b_j in (32) must vanish at the origin as a consequence of our hypothesis that g_{nn} has an isolated zero at the origin. Let

$$V = V'\partial_{x'} + V_n\partial_n \,,$$

denote a non-zero tangent vector at a point $x \in U$. We have,

$$0 \leq g(V,V) = \|V'\|^2_{g'} + 2V_nV' \cdot b + g_{nn}(V_n)^2 \,,$$

where $g' := (g_{kl})$ is the $(n-1)\times(n-1)$ sub-matrix of g corresponding to the range $1 \leq k,l \leq n-1$. Defining $\mathfrak{b} \in \mathbb{R}^{n-1}$ by $b =: g'\mathfrak{b}$, the above inequality then reads

$$0 \leq g(V,V) = \|V'\|^2_{g'} + 2V_n\, g'(V',\mathfrak{b}) + g_{nn}(V_n)^2 \,.$$

The worst possible case for this inequality occurs when $V' = \lambda\, \mathfrak{b}$,where $\lambda \in \mathbb{R}$, in which case the condition $0 < g(V, V)$ reduces to

$$0 \leq \lambda^2 \|\mathfrak{b}\|^2_{g'} + 2\lambda V_n \|\mathfrak{b}\|^2_{g'} + g_{nn}(V_n)^2 \,.$$

This inequality will hold if an only if

$$\|\mathfrak{b}\|^2_{g'} g_{nn}(V_n)^2 - \|\mathfrak{b}\|^4_{g'}(V_n)^2 \geq 0 \,,$$

for all $V_n \neq 0$, or equivalently

$$\|\mathfrak{b}\|^2_{g'} \leq g_{nn} \,,$$

which establishes the first step.

Next we apply a local diffeomorphism of the form (33) to the metric (32) and determine the conditions that the functions $f_j(\bar{x})$ must satisfy in order for transformed metric f^*g to take the form (34) in which the coefficients $\bar{b}_j$ of the cross terms in the metric

$$f^*g = \bar{g}_{\alpha\beta}(\bar{x})\, d\bar{x}_\alpha\, d\bar{x}_\beta = \bar{g}_{nn}(\bar{x})\, d\bar{x}_n^2 + 2\bar{b}_j(\bar{x})\, d\bar{x}_j\, d\bar{x}_n + \bar{g}_{jk}(x)\, d\bar{x}_j\, d\bar{x}_k\,,$$

are identically zero. A straightforward calculation gives that $\bar{b}_j = 0$ if and only if

$$b_k \partial_n f_k + g_{jk} \partial_l f_j \partial_n f_k = -b_l\,, \tag{35}$$

where all the partial derivatives are taken with respect to the barred coordinates $(\bar{x}', \bar{x}_n)$. We now choose an invertible matrix $A = (A_{jk})$ of small norm, say $\|A\| < \varepsilon$, and define $n-1$ linear functions $\tilde{f}_j(\bar{x})$ by

$$\tilde{f}_j(\bar{x}) = (g^{-1})_{jm} A_{lm} \bar{x}_l.$$

Define next $G(\partial' f) := M^{-1}(\partial' f)$, where

$$M_{lk}(\partial' f) := b_k + \partial_l f_j g_{jk}\,.$$

We claim that the matrix-valued function $G(\partial' f)$ is well defined for f_j in a C^1-small neighbourhood of $\tilde{f}_j$ and x in a small neighbourhood of the origin. Indeed, by definition of $\tilde{f}$ and using the fact that $b_j(0) = 0$, we have

$$\begin{aligned} M_{lk}(\partial' f)(\bar{x}) &= O(\|\bar{x}\|) + (g^{-1})_{jm} A_{lm} g_{jk} + O(\|f - \tilde{f}\|_{C^1}) && (36)\\ &= A_{lk} + O(\|\bar{x}\| + \|f - \tilde{f}\|_{C^1})\,. && (37)\\ & && (38) \end{aligned}$$

Therefore we can write the system of equations (35) to be solved as

$$\partial_n f_k = -G_{kl}(\partial' f) b_l\,, \tag{39}$$

and take as initial data

$$f_k|_{\bar{x}_n=0} = \tilde{f}_k = (g^{-1})_{km} A_{lm} \bar{x}_l\,. \tag{40}$$

We now apply the Cauchy-Kovalevskaya Theorem to (39) with initial data given by (40). By choosing ε small enough, we can ensure that the map (33) obtained by solving the system (39) is a local diffeomorphism, which proves our claim.

Now that the normal form

$$g = g_{nn}(x)\, dx_n^2 + g_{kl}(x)\, dx_k\, dx_l\,. \tag{41}$$

has been established for singular metrics such that g_{nn} has an isolated zero at the origin $0 \in U$, we need to make further assumptions about the leading order behaviour of g_{nn} and of the normal derivative of g_{ij} at the origin in order to be able to apply the results of Leray.

First, we require that both the partial Hessian $(\partial_{jk} g_{nn}(0))_{1\leq j,k\leq n-1}$ and the matrix $(g_{jk}(0))_{1\leq j,k\leq n-1}$ be positive definite. Transforming the partial Hessian at 0 into the identity matrix with a local change of coordinates and employing the division property of analytic functions, this almost leads to the starting point

$$g = (\|x'\|^2 + x_n^{2l})\, F_0(x)\, dx_n^2 + g_{kl}\, dx_k\, dx_l\,,$$

except that we still have to impose the condition

$$\partial_n g_{jk}(x', 0) = O(\|x'\|^2)\,.$$

In analogy with the construction of Cauchy data for the iterated sequence of Cauchy-Kovalevskaya problems needed for the proof of the Cartan-Janet Theorem, our next task is to show that there exist C^{ω} Cauchy data $\mathbf{u}_0$, $\mathbf{u}_1$ for the local isometric embedding problem of the class of singular metrics with an admissible singularity at the origin, which are such that the Cauchy problem for the system (9) to (11) admits an isolated non-exceptional characteristic point at the origin $0 \in U$. This is where the condition

$$\partial_n g_{jk}(x', 0) = O(\|x'\|^2)\,,$$

will enter the picture.

The construction of the Cauchy data turns out to be somewhat more delicate than what we had to do for the proof of the classical Cartan-Janet Theorem. It is the main technical step in the proof.

We begin by noting that letting

$$F(x') := F_0(x', 0), \quad \bar{g}_{ij}(x') := g_{ij}(x', 0), \quad h_{ij}(x') := -\frac{1}{2}\partial_n g_{ij}(x', 0)\,,$$

the constraints (6) to (12) read

$$\|\mathbf{u}_1\|^2 = \|x'\|^2 F\,. \tag{42}$$

$$\partial_i \mathbf{u}_0 \cdot \mathbf{u}_1 = 0\,, \tag{43}$$

$$\partial_i \mathbf{u}_0 \cdot \partial_j \mathbf{u}_0 = \bar{g}_{ij}\,, \tag{44}$$

$$\partial_{ij} \mathbf{u}_0 \cdot \mathbf{u}_1 = h_{ij}\,, \tag{45}$$

and the assumption that the origin should be an isolated characteristic point for the system of PDEs (9),(10), (11) implies that the vectors $\partial_i \mathbf{u}_0, \mathbf{u}_1, \partial_{ij} \mathbf{u}_0$ must be linearly independent at every $x' \neq 0$ in the domain of definition of $\mathbf{u}_0$ and $\mathbf{u}_1$ in the initial hypersurface $x_n = 0$.

We have :

Proposition

Consider a C^ω metric g on a neighbourhood $U \subset \mathbb{R}^n$ admitting an admissible singularity at the origin, that is a metric of the form (13) satisfying (14). Then there exist C^ω initial data $\mathbf{u}_0, \mathbf{u}_1$ for the system (9) to (11), taking values in $\mathbb{E}^{N+n-2}$, which satisfy the constraints (42) to (45) and which are such that $\partial_i \mathbf{u}_0, \mathbf{u}_1, \partial_{ij}\mathbf{u}_0$ are linearly independent on the complement of the origin in the initial hypersurface $x_n = 0$. Furthermore, the function $\Delta : V \to \mathbb{R}$ defined as

$$\Delta(x') := \det(\partial_j \mathbf{u}_0(x'), \mathbf{u}_1(x'), \partial_{jk}\mathbf{u}_0(x'), \mathbf{e}_a)_{1\leq j,k\leq n-1,\ 2\leq a\leq n-1} \qquad (46)$$

has a nondegenerate zero at 0. More precisely, in a neighbourhood of the origin in the initial hypersurface $x_n = 0$, the function Δ is of the form

$$\Delta(x') = x_1\,\Delta_0(x')$$

with $\Delta_0(0) \neq 0$.

Recall first that for any C^ω non-degenerate metric $\hat{g}_{ij}(x')$ on the hypersurface $x_n = 0$, we know from the Cartan-Janet Theorem that there exits a C^ω local isometric embedding from $\Sigma \subset \{x_n = 0\} \to \mathbb{E}^{N'}$, where $N' = n(n-1)/2$, meaning that there exists a C^ω map $\mathbf{v}$ which satisfies

$$\partial_i \mathbf{v} \cdot \partial_j \mathbf{v} = \hat{g}_{ij}\,, \tag{47}$$

where we may assume with no loss of generality that the vectors $\partial_a \mathbf{v}, \partial_{n-1}\mathbf{v}, \partial_{ab}\mathbf{v}$, $1 \leq a, b \leq n-2$, are linearly independent at every $x' \in \Sigma$. Consider now the embedding $\mathbf{w} : \Sigma \to \mathbb{E}^{N-1} = \mathbb{E}^{N'} \times \mathbb{E}^{n-1}$ defined by

$$\mathbf{w} := (\mathbf{v}, \mathbf{V})\,,$$

with $\mathbf{V} = (V_1, \ldots, V_{n-1})$ defined by

$$V_a := \epsilon^5 \sin \frac{x_{n-1}}{\epsilon^4} \sin \frac{x_a}{\epsilon^2}\,, \quad V_{n-1} := -\epsilon^5 \cos \frac{x_{n-1}}{\epsilon^4}\,,$$

where $1 \leq a \leq n-2$.

It is straightforward to verify that

$$\begin{aligned}
\partial_a \mathbf{w} &= \partial_a \mathbf{v} + O(\epsilon^3)\,, \\
\partial_{n-1} \mathbf{w} &= \partial_{n-1} \mathbf{v} + O(\epsilon) \\
\partial_{ab} \mathbf{w} &= \partial_{ab} \mathbf{v} + O(\epsilon) \\
\partial_{n-1,a} \mathbf{w} &= \partial_{n-1,a} \mathbf{v} + \frac{1}{\epsilon} \cos \frac{x_{n-1}}{\epsilon^4}\ \cos \frac{x_a}{\epsilon^2} \mathbf{E}_a \\
\partial_{n-1,n-1} \mathbf{w} &= \partial_{n-1,n-1} \mathbf{v} + \frac{1}{\epsilon^3} \left(\cos \frac{x_{n-1}}{\epsilon^4} \mathbf{E}_{n-1} - \sin \frac{x_{n-1}}{\epsilon^4} \sum_{a=1}^{n-2} \sin \frac{x_a}{\epsilon^2}\, \mathbf{E}_a \right),
\end{aligned}$$

where $\mathbf{E}_i$ are unit vectors in the x_i direction in $\mathbb{E}^n$. So if we take ϵ small and choose x' with $\|x'\| < \epsilon^5$, we may conclude that $\partial_j \mathbf{w}$ and $\partial_{jk} \mathbf{w}$ are linearly independent at every point of their domain of definition in $\mathbb{R}^{N-1}$. We also have

$$\partial_i \mathbf{w} \cdot \partial_j \mathbf{w} = \partial_i \mathbf{v} \cdot \partial_j \mathbf{v} + \partial_i \mathbf{V} \cdot \partial_j \mathbf{V}\,,$$

where

$$|\partial_i \mathbf{V} \cdot \partial_j \mathbf{V}| < C\epsilon^2\,, \tag{48}$$

for some positive constant C.

If we set now

$$\hat{g}_{ij} := \bar{g}_{ij} - \partial_i \mathbf{V} \cdot \partial_j \mathbf{V}\,,$$

then this metric is now positive-definite at the origin as a consequence of the estimate (48), so we may apply the Cartan-Janet Theorem as we did above to $\hat{g}_{ij}$ to conclude using (47) that the map $\mathbf{w}$ satisfies

$$\partial_i \mathbf{w} \cdot \partial_j \mathbf{w} = \bar{g}_{ij}\,.$$

Next we consider the embedding $\mathbf{u}_0 : \Sigma \to \mathbb{E}^{N-1} \times \mathbb{E}^{n-1} = \mathbb{E}^{N+n-2}$ given by

$$\mathbf{u}_0 := (\mathbf{w}, 0)\,. \tag{49}$$

The tangent space $T_{\mathbf{u}_0(x')}\bar{\Sigma}$ to $\bar{\Sigma} = \mathbf{u}_0(\Sigma) \subset \mathbb{E}^{N+n-2}$ at $\mathbf{u}_0(x')$ is the $(n-1)$-dimensional subspace given by the linear span of the vectors $\partial_j \mathbf{w}$, $1 \le j \le n-1$, that is

$$T_{\mathbf{u}_0(x')}\bar{\Sigma} = \langle \partial_j \mathbf{w}\,,\, 1 \le j \le n-1 \rangle\,. \tag{50}$$

We also have

$$\langle \partial_j \mathbf{w}, \partial_{jk}\mathbf{w}\,,\, 1 \leq j,k \leq n-1 \rangle = \langle \partial_j \mathbf{w}, \mathbf{N}_r\,,\, 1 \leq j \leq n-1,\, 1 \leq r \leq n(n-1)/2 \rangle\,,$$

where $\{\mathbf{N}_r,\, 1 \leq r \leq n(n-1)/2\}$ is a linearly independent set of unit normal vectors in $\mathbb{E}^{N-1} \subset \mathbb{E}^{N+n-2}$.

Denoting by $\{\mathbf{e}_j : 1 \leq j \leq n-1\}$ an orthonormal basis of the $\mathbb{E}^{n-1}$ factor in $\mathbb{E}^{N+n-2} = \mathbb{E}^{N-1} \times \mathbb{E}^{n-1}$, the normal space $N_{\mathbf{u}_0(x')}\bar{\Sigma}$ to $\bar{\Sigma} = \mathbf{u}_0(\Sigma) \subset \mathbb{E}^{N+n-2}$ at $\mathbf{u}_0(x')$ is, in view of the linear independence of set of vectors $\{\partial_j \mathbf{w}, \partial_{ab}\mathbf{w} : 1 \leq j \leq n-1,\, 1 \leq a,b \leq n-2\}$, given by the $(N-1)$-dimensional subspace

$$N_{\mathbf{u}_0(x')}\bar{\Sigma} = \langle \mathbf{N}_r, \mathbf{e}_j\,,\, 1 \leq r \leq n(n-1)/2, 1 \leq j \leq n-1 \rangle\,. \tag{51}$$

Consequently, there exists a unique vector field $\mathbf{N}$ along $\bar{\Sigma}$ of the form

$$\mathbf{N} = \sum_{r=1}^{n(n-1)/2} \alpha_r \mathbf{N}_r \,,$$

such that

$$\mathbf{N} \cdot \partial_{jk}\mathbf{w} = h_{ij} \,.$$

Note that the hypothesis (14) on g can be rewritten as $h_{ij} = O(\|x'\|^2)$, so we immediately infer that

$$\|\mathbf{N}\| = O(\|x'\|^2)\,. \tag{52}$$

We now set

$$\mathbf{u}_1 := \mathbf{N} + \sum_{j=1}^{N} x_j \, G \, \mathbf{e}_j \,, \tag{53}$$

where

$$G := \left(F - \frac{\|\mathbf{N}\|^2}{\|x'\|^2}\right)^{1/2} \tag{54}$$

is a now real-valued C^ω function near $x' = 0$ by the bound (52).

Next, a direct calculation using (50), (51), (53) and (54) shows that the initial data $\mathbf{u}_0, \mathbf{u}_1$ defined by (49), (53) satisfy the constraints (42), (43), (44) and (45). Finally, we have

$$\begin{aligned}
\Delta(x') &= \det(\partial_j \mathbf{u}_0(x'), \mathbf{u}_1(x'), \partial_{jk}\mathbf{u}_0(x'), \mathbf{e}_a)_{1\leq j,k\leq n-1,\ 2\leq a\leq n-1} \\
&= \det\left(\partial_j \mathbf{u}_0(x'), \mathbf{N}(x') + \sum_{l=1}^{N} x_l\, G(x')\, \mathbf{e}_l, \partial_{jk}\mathbf{u}_0(x'), \mathbf{e}_a\right)_{1\leq j,k\leq n-1,\ 2\leq a\leq n-1} \\
&= \det\left(\partial_j \mathbf{u}_0(x'), x_1\, G(x')\, \mathbf{e}_1, \partial_{jk}\mathbf{u}_0(x'), \mathbf{e}_a\right)_{1\leq j,k\leq n-1,\ 2\leq a\leq n-1} \\
&= x_1\, G(x')\, \det\left(\partial_j \mathbf{u}_0(x'), \mathbf{e}_1, \partial_{jk}\mathbf{u}_0(x'), \mathbf{e}_a\right)_{1\leq j,k\leq n-1,\ 2\leq a\leq n-1} \\
&=: x_1\, \Delta_0(x')\,,
\end{aligned}$$

where indeed $\Delta_0(0) \neq 0$ by the linear independence of the above vectors. This ends the proof of the proposition.

Proof of the main theorem

Since the system is underdetermined in that there are fewer equations ($N = n(n+1)/2$) than unknowns ($N + n - 2 = (n^2 + 3n - 4)/2$), let us augment the system by imposing that

$$\mathbf{e}_a \cdot \partial_{nn}\mathbf{u} = 0\,, \qquad 2 \leq a \leq n-1 \tag{55}$$

where the orthonormal vectors $\{\mathbf{e}_a\}_{a=2}^{n-1}$ are defined as before.

To construct a solution to the augmented system of PDEs (9)-(10)-(11) and (55), we employ Leray's Cauchy-Kovalevskaya theorem in the form given by Choquet-Bruhat for non-linear systems.

Let us consider the Cauchy surface $S := \{x \in \mathbb{R}^n : x_n = 0\}$, corresponding to the function $s(x) := x_n$.

One checks that $\mathcal{A}(x,p)$ is an $(N+n-2)\times(N+n-2)$ matrix of the form

$$\mathcal{A}(x,p) = p_n^3\,(\partial_j \mathbf{u}(x), \partial_n \mathbf{u}(x), \partial_{jk}\mathbf{u}(x), \mathbf{e}_a)_{1\leq j,k\leq n-1,\ 2\leq a\leq n-1} + \sum_\alpha p^\alpha \mathcal{M}_\alpha(x)\,,$$

where the sum ranges over the set of multi-indices with $|\alpha| = 3$ such that the monomial p^α is different from p_n^3 and $\mathcal{M}_\alpha(x)$ are matrices whose concrete expressions will not be needed.

With the Cauchy data

$$\mathbf{u}|_{x_n=0} = \mathbf{u}_0 \,, \qquad \partial_n \mathbf{u}|_{x_n=0} = \mathbf{u}_1 \,, \tag{56}$$

and using the fact that the gradient of $s(x) = x_n$ points in the n-th direction, we immediately obtain from the previous formula that, on S, the function $\mathcal{A}_*$ defined above is precisely

$$\mathcal{A}_*(x') = \Delta(x') \,,$$

where the function Δ was introduced in (46).

We have $\partial_1 \Delta(0) \neq 0$, so there is a direction tangent to the hyperplane S such that the corresponding directional derivative of Δ at 0 does not vanish. The origin is then a non-exceptional characteristic point of the system.

Choquet-Bruhat's nonlinear extension of Leray's theorem, then shows that, in a small deleted neighborhood of 0, the system (9)-(10)-(11)-(55) admits a unique ramified solution with the initial data (56), and that the singularities of $\mathbf{u}$ in a neighborhood of 0 are algebroid.

This implies that there is a finite Riemannian cover U' of $U \backslash \{0\}$ as in the statement of the theorem such that $\mathbf{u}$ defines an embedding $U' \to \mathbb{E}^{N+n-2}$.

This completes the proof.

5. Perspectives

Perspectives

- It is likely that Leray's Theorem can be used to handle the LIE problem for metrics with singularities which are more severe than the admissible singularities considered here. It would be interesting to explore this possibility.
- Cartan's proof of the Cartan-Janet Theorem is based on the Cartan-Kähler Theorem for the existence of integral manifolds of exterior differential systems in involution. The Cartan-Kähler Theorem has many other geometric applications (orthogonal coordinates, special submanifolds, G_2 structures, *etc*...). It would be an interesting but challenging problem to formulate a ramified version of the Cartan-Kähler Theorem that could be applied to these settings.

Happy birthday, Peter !