Special Symmetry Reductions for Tzitzeica Surfaces PDE

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Abstract

One of the most prestigious Romanian mathematicians, Gheorghe Tzitzeica(1873-1939), introduced two centro-affine invariants that carry his name today. A *Tzitzeica curve* is a spatial curve for which the ratio of its torsion and the square of the distance from the origin to the osculating plane at an arbitrary point of the curve is constant. A *Tzitzeica surface* is a surface for which the ratio of its Gaussian curvature and the fourth power of the distance from the origin to the tangent plane at any arbitrary point of the surface is constant. It may be shown that the asymptotic curves on a Tzitzeica surface with negative Gaussian curvature are Tzitzeica curves. The differential equations related to the Tzitzeica curves and Tzitzeica surfaces, respectively, are both nonlinear and challenging to solve. In this work, it is shown how one can use the Tzitzeica curves as symmetry reductions for the nonlinear Tzitzeica surfaces PDE. The method may be further explored in using special curves on a surface as symmetry reductions for the related surface's partial differential equation.

Introduction

Consider a regular surface S: $z = f(x,y)$, where f is a smooth real function on a domain $D \subset \mathbf{R}^2$, whose Gaussian curvature is given by

$$K = \frac{z_{xx}z_{yy} - z_{xy}^2}{\left(1+z_x^2+z_y^2\right)^2}. \tag{1}$$

The distance d from the origin to the tangent plane at an arbitrary point of S is

$$d = \frac{|xz_x + yz_y - z|}{\sqrt{1+z_x^2+z_y^2}}. \tag{2}$$

The surface S is a *Tzitzeica surface* ([6], 1907) if the following relation

$$\frac{K}{d^4} = a, \tag{3}$$

holds everywhere on D, where $a \neq 0$ is a real number. The relation (3) may be written as the nonlinear second-order PDE

$$z_{xx}z_{yy} - z_{xy}^2 = a\left(xz_x + yz_y - z\right)^4 \tag{4}$$

for which the condition $xz_x + yz_y - z \neq 0$ is satisfied. The equation (4) is called *the Tzitzeica surfaces PDE* and a is named *the surface's constant*.

Let $\boldsymbol{r}(t) = \left(x(t), y(t), z(t)\right)$, $t \in I$, define a smooth regular space curve C, where $I \subset \mathbf{R}$ is an interval. Assume that the curve's curvature and torsion given by, respectively,

$$k(t) = \frac{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3} \quad \text{and} \quad \tau(t) = \frac{\left(\boldsymbol{r}'(t), \boldsymbol{r}''(t), \boldsymbol{r}'''(t)\right)}{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|^2} \tag{5}$$

do not vanish on I. The distance $\tilde{d}$ from the origin to the osculating plane of the curve C at an arbitrary point $\boldsymbol{r}(t)$ is

$$\tilde{d} = \frac{1}{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}\begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}. \tag{6}$$

The curve C is a *Tzitzeica curve* ([7], 1911) if the condition

$$\frac{\tau}{\tilde{d}^2} = b \tag{7}$$

holds everywhere on I, where $b \neq 0$ is a real constant. The condition (7) may be written as a Wronskian-involving third-order nonlinear autonomous undetermined ODE

$$W(x', y', z')(t) = b\,[W(x,y,z)(t)]^2, \tag{8}$$

where Wronskians of the curve's defining functions $W(x,y,z)$ and the Wronskian of their first derivatives $W(x', y', z')$ are assumed nonzero on I. The equation (8) is called *the Tzitzeica curves ODE*, and b is named the *curve's constant.*

Symmetry Reductions

Tzitzeica has shown that (3) and (7) are centro-affine invariants [6]-[7]. The aim of this work is to revisit Tzitzeica' s theory from the point of view of symmetry analysis theory [5], and to show how the Tzitzeica surfaces may be obtained from *weak* or *strong* solutions of the Tzitzeica curves ODE (8).

Theorem 1 [8]. *The Lie algebra L of the symmetry group G associated with the Tzitzeica surfaces PDE (4) is generated by the following vector fields*

$$X_1 = x\partial_x - z\partial_z,\ \ X_2 = y\partial_y - z\partial_z,\ \ X_3 = y\partial_x,\ \ X_4 = z\partial_x, \tag{9}$$
$$X_5 = x\partial_y,\ \ X_6 = z\partial_y,\ \ X_7 = x\partial_z,\ \ X_8 = y\partial_z.$$

Since the Tzitzeica curves ODE (8) may be regarded as an equation in the unknown function $z(t)$ with $x(t)$ and $y(t)$ are arbitrary functions, then by including the arbitrary functions in the space of the dependent variable, one can find its related extended classical symmetries [4] that may be further used to find group-invariant solutions.

Theorem 2. *The Lie algebra $\tilde{L}$ of the extended symmetry group $\tilde{G}$ associated with the Tzitzeica curve ODE (8) is generated by the following vector fields*

$$X_1 = x\partial_x - z\partial_z,\ \ X_2 = y\partial_y - z\partial_z,\ \ X_3 = y\partial_x,\ \ X_4 = z\partial_x, \tag{10}$$
$$X_5 = x\partial_y,\ \ X_6 = z\partial_y,\ \ X_7 = x\partial_z,\ \ X_8 = y\partial_z\ , X_9 = F(t,x,y,z)\partial_t,$$

where F is an arbitrary smooth function in four variables.

Corollary. *The Tzitzeica surfaces PDE (4) and the Tzitzeica curves ODE (8) are invariant under the Lie group of transformations G whose infinitesimal generators are given by (9).*

The vector fields $X_1, \ldots, X_8$ correspond to the centro-affine unimodular group of transformations and generate group-invariant solutions for (4) and (8), respectively, on different jet spaces related to these DEs. The resulting pair of Tzitzeica curves and surfaces (C, S) will be invariant under the same infinitesimal generator. However, the side condition (11) in the Theorem 3 provides *weak* and *strong* solutions of (8) (here a weak solution is not G-invariant).

Theorem 3 [3] *Any three linearly independent solutions $x(t), y(t),$ and $z(t)$ of the third-order homogeneous ODE with variable coefficients*

$$u''' + \beta(t)u'' + \gamma(t)u' + \delta(t)u = 0, \tag{11}$$

with $\delta(t) = C\exp\left(-\int\beta(t)dt\right)$, $C \neq 0$, and *$\beta(t)$ and $\gamma(t)$ are arbitrary smooth functions, define a Tzitzeica curve for which the curve's constant is*

$$b = -\frac{\delta(t)}{W(x,y,z)(t)}, \tag{12}$$

where $W(x,y,z)(t)$ is their associated Wronskian.

If the coefficients $\beta(t), \gamma(t),$ and $\delta(t)$ are constant functions has been discussed in [1]. In particular, for $\delta \neq 0$ a real constant [2], the side condition (11) turns into the ODE $u''' + \gamma(t)u' + \delta u = 0$. In this case, new Tzitzeica curves may be found. For instance, the curve $x(t) = t^{-\frac{3}{2}}$, $y(t) = \left(1 - 2t^{-\frac{3}{2}}\right)\exp\left(t^{-\frac{3}{2}}\right)$, $z(t) = \left(1 + 2t^{-\frac{3}{2}}\right)\exp\left(-t^{-\frac{3}{2}}\right)$, with $t > 1$, is a weak solution of (8) lying on the Tzitzieca surface S: $yz = 1 - 4x^2$.

Examples

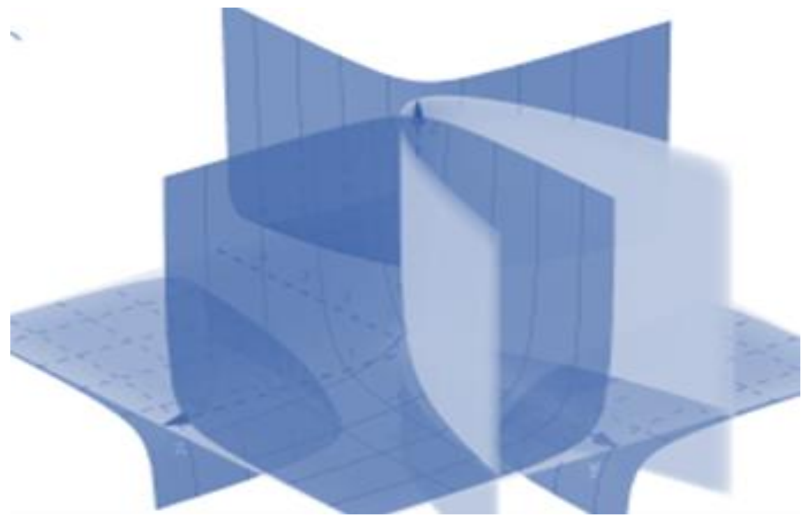

Fig.1. The Tzitzeica surface $z = \frac{1}{xy}$ is a group-invariant solution of the PDE (4) that corresponds to $X_1 + 2X_2$ and is obtained from the condition $I_1 = h(I_0)$ satisfied by the invariants $I_1 = xyz$ and $I_0 = \frac{y}{x^2}$. The Tzitzeica curve $x(t) = e^t$, $y(t) = e^{2t}$, $z(t) = e^{-3t}$ is an integral curve of the vector field $X_1 + 2X_2$ and is graphed as the intersection of the surfaces of equations given by $I_1 = 1$ and $I_0 = 1$. Here the constants in the DEs (4) and (8), respectively, are $a = \frac{1}{27}$ and $b = -\frac{3}{10}$.

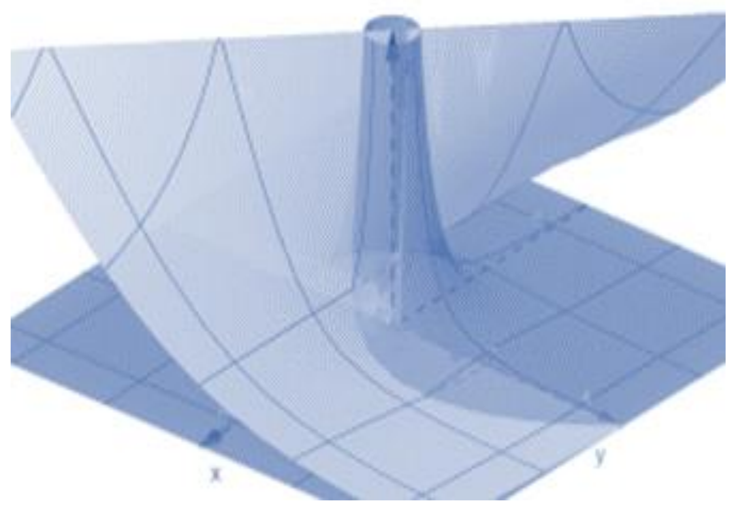

Fig.2. Tzitzeica surface $z = \frac{1}{x^2+y^2}$ is a group-invariant solution of the PDE (4) that corresponds to $X_1 + X_2 - X_3 + X_5$ and is obtained as $I_1 = 1$ satisfied by the invariant $I_1 = z(x^2+y^2)$. The Tzitzeica curve $x(t) = e^t\cos t$, $y(t) = e^t\sin t$, $z(t) = e^{-2t}$ is an integral curve of the vector field $X_1 + X_2 - X_3 + X_5$ and is graphed as the intersection of the surfaces $I_1 = 1$, $I_0 = 1$, for $I_0 = z\exp\left(2\arctan\frac{y}{x}\right)$. Here the constants in the DEs (4) and (8), respectively, are $a = -\frac{4}{27}$ and $b = -\frac{2}{5}$.

Remark. The side condition (11) may be further explored in similar problems when a PDE for a surface is coupled with an ODE for a specific curve. For a skew curve C, the relation (11) expresses that the third derivatives $\boldsymbol{r}'''$ of the curve's defining functions may be expressed linearly in terms of $\boldsymbol{r}, \boldsymbol{r}'$, and $\boldsymbol{r}''$. If the curve C is defined by an undetermined nonlinear Wronskian-involving ODE related to its torsion (for instance, ODE (8)), then (11) may be used to seek for weak or strong solutions, invariant or not under the symmetry group of the curve's ODE. Furthermore, these solutions may be used as symmetry reductions for the PDE related to the surface S on which the curve C lies on (e.g., PDE (4)).

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