Partial Difference Moving Frames

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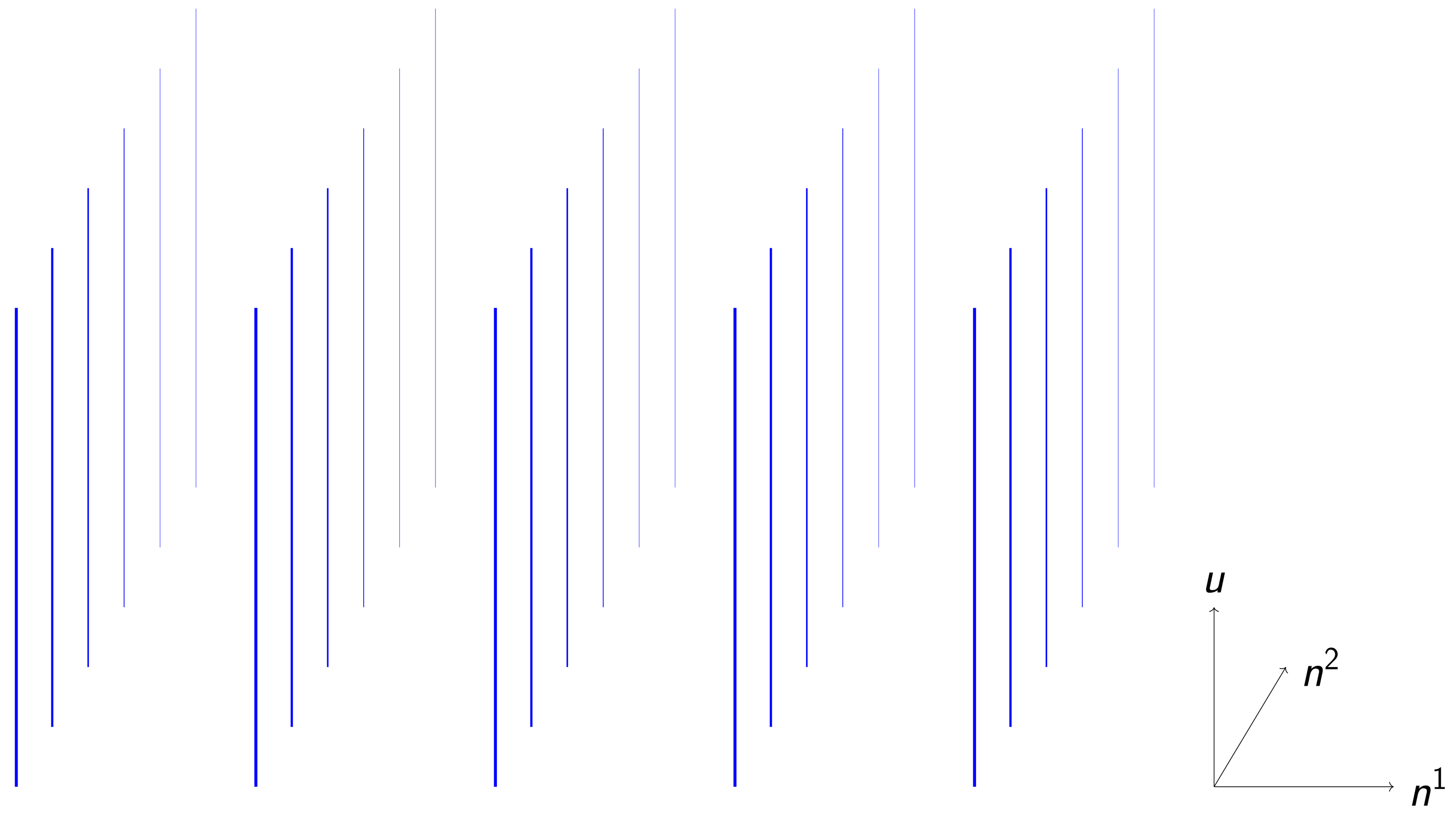
Total and prolongation spaces

For scalar PΔEs the total space is $\mathcal{T} = \mathbb{Z}^p \times \mathbb{R}$, where the independent variables are $\mathbf{n} := (n^1, n^2, \ldots, n^p) \in \mathbb{Z}^p$, and the dependent variable is $u \in \mathbb{R}$.



Let $u_{\mathbf{J}}$ denote the value of the dependent variable u on the fibre $\mathbf{n} + \mathbf{J}$. Then

- The translation operator $\mathrm{T}_{\mathbf{K}}$ acts on the total space in the following way

$$\mathrm{T}_{\mathbf{K}} : \mathbb{Z}^p \times \mathbb{R} \to \mathbb{Z}^p \times \mathbb{R}, \qquad \mathrm{T}_{\mathbf{K}} : (\mathbf{n}, (u_{\mathbf{J}})) \mapsto (\mathbf{n} + \mathbf{K}, (u_{\mathbf{J}})).$$

- The pullback operator $\mathrm{T}^*_{\mathbf{K}}$ takes the value of $u_{\mathbf{J}}$ on the fibre $\mathbf{n} + \mathbf{K}$ to $u_{\mathbf{J}+\mathbf{K}}$ on the fibre $\mathbf{n}$.
- The shift operator relates to the pullback in the following way $\mathrm{S}_{\mathbf{K}} f_{\mathbf{n}} := \mathrm{T}^*_{\mathbf{K}} f_{\mathbf{n}+\mathbf{K}}$.

The prolongation space over $\mathbf{n}$, denoted $P_{\mathbf{n}}(\mathbb{R})$, represents the values of u on all fibres on to a single fibre $\mathbf{n}$, (See [4]). The total space $\mathcal{T}$ is disconnected, but the prolongation space $P_{\mathbf{n}}(\mathbb{R})$ provides a connected representation of this space, over an arbitrary $\mathbf{n}$. This allows us to use difference moving frames (see OΔEs [3]). Let us omit equations that have singularities and treat $\mathbf{n}$ as fixed from now on.

Calculus of Variations

There are several ways to calculate the difference Euler–Lagrange equation for

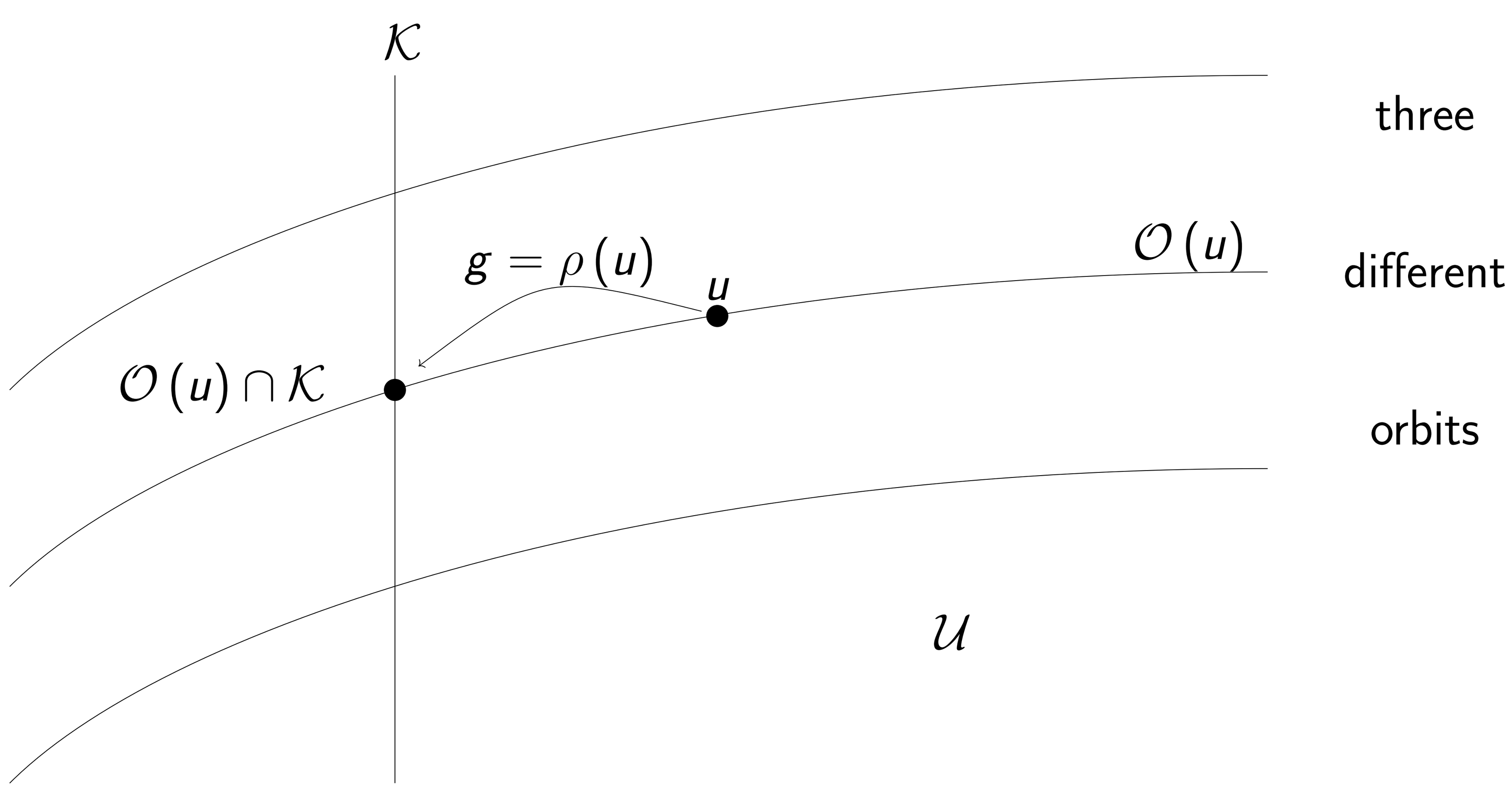
$$\mathcal{L}[u] = \sum \mathrm{L}(\mathbf{n}, [u]).$$

Kupershmidt [2] showed that the Euler–Lagrange equation is

$$\mathrm{E}_u(\mathrm{L}) := \mathrm{S}_{-\mathbf{J}} \left(\frac{\partial \mathrm{L}}{\partial u_{\mathbf{J}}} \right) = 0.$$

Moving Frames

On the difference prolongation space $P_{\mathbf{n}}(\mathbb{R})$, apply moving frames.



Given a smooth Lie group action $G \times M \to M$, a moving frame is an equivariant map $\rho : \mathcal{U} \subset M \to G$ with $\mathcal{U}$ the domain of the frame. To find a moving frame use the normalization equations $\psi(g \cdot u_{\mathbf{J}_r}) = c_r$ and solve for the parameters.

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Invariant Euler–Lagrange equation

To calculate an invariant formulation of the Euler–Lagrange equation (See [1] for PDEs) we write the Lagrangian in terms of generating invariants and their shifts

$$\mathcal{L}[u] = \sum L(\mathbf{n}, [\kappa^\beta]).$$

Then the vertical derivative is used as in the calculation for the original Euler–Lagrange equation. This introduces the Euler operators of L with respect to the generating invariants, that is,

$$\mathrm{E}_{\kappa^\beta}(L) = \mathrm{S}_{-\mathbf{J}} \frac{\partial L}{\partial \kappa^\beta_{\mathbf{J}}}.$$

In the calculation the invariant difference form defined by

$$\iota(\mathrm{d}_v u_{\mathbf{0}}) = \vartheta_{\mathbf{0}} \mathrm{d}_v u_{\mathbf{0}}, \quad \text{where} \quad \vartheta_{\mathbf{0}} = \left. \frac{\partial (g \cdot u_{\mathbf{0}})}{\partial u_{\mathbf{0}}} \right|_{g=\rho_0},$$

is shifted and its component is inverted to write all the forms in terms of $\iota(\mathrm{d}_v u_{\mathbf{0}})$. Finally, taking the adjoint of an operator $\mathcal{H}^\beta$ which occurs, the invariantization of the original Euler–Lagrange equations is

$$\iota(\mathrm{E}_u(\mathrm{L})) = \mathcal{H}^{\beta *} \mathrm{E}_{\kappa^\beta}(L),$$

where $\mathcal{H}^{\beta *}$ is the difference operator

$$\mathcal{H}^{\beta *} = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left(\iota \left(\frac{\partial \kappa^\beta}{\partial u_{\mathbf{J}}} \right) \iota\left(\vartheta_{\mathbf{J}}^{-1}\right) \right) \mathrm{S}_{-\mathbf{J}}.$$

Example

The Lagrangian

$$\mathrm{L} = \ln \left| \frac{u_{1,0} - u_{0,1}}{u_{1,1} - u_{0,0}} \right|,$$

has the Euler–Lagrange equation

$$\mathrm{E}_u(\mathrm{L}) = \frac{1}{u_{1,1} - u_{0,0}} - \frac{1}{u_{-1,1} - u_{0,0}} - \frac{1}{u_{1,-1} - u_{0,0}} + \frac{1}{u_{-1,-1} - u_{0,0}} = 0.$$

This Lagrangian is invariant under the two parameter Lie group action

$$g \cdot u = au + b, \quad \text{where} \quad a \in \mathbb{R}^+, \ b \in \mathbb{R}.$$

Using the normalization equations $g \cdot u_{0,0} = 0$ and $g \cdot u_{1,1} = 1$ gives the parameters on the frame

$$a = \frac{1}{u_{1,1} - u_{0,0}}, \quad b = \frac{-u_{0,0}}{u_{1,1} - u_{0,0}}.$$

Our invariants are, therefore,

$$\iota(u_{i,j}) = \frac{u_{i,j} - u_{0,0}}{u_{1,1} - u_{0,0}}, \quad \text{for} \quad i, j \in \mathbb{Z}.$$

Two generating invariants are $\kappa^1 := \iota(u_{1,0})$ and $\kappa^2 := \iota(u_{0,1})$ and along with all their shifts they enable us to write all possible invariants of this Lie group action. The invariant form of the Lagrangian is

$$L = \ln |\kappa^1 - \kappa^2|.$$

Using our formula above the invariant Euler–Lagrange equation is

$$1 + \frac{1 - \kappa^1_{-1,0}}{\kappa^2 \left(\kappa^1_{-1,0} - \kappa^2_{-1,0}\right)} - \frac{1 - \kappa^2_{0,-1}}{\kappa^1 \left(\kappa^1_{0,-1} - \kappa^2_{0,-1}\right)} + \frac{\left(\kappa^1_{-1,0} - 1\right)\left(\kappa^2_{-1,-1} - 1\right)}{\kappa^2 \kappa^1_{-1,0}} = 0.$$

References

[1] I A. Kogan and P J. Olver. "Invariant Euler–Lagrange equations and the invariant variational bicomplex". *Acta Appl. Math.* 76.2 (2003), pp. 137–193.

[2] B A. Kupershmidt. *Discrete Lax equations and differential-difference calculus*. Vol. 123. Société mathématique de France, 1985.

[3] E L. Mansfield et al. "Moving frames and Noether's finite difference conservation laws I". *Trans. Math. Appl.* 3.1 (2019), tnz004.

[4] L. Peng and P E. Hydon. "Transformations, symmetries and Noether theorems for differential-difference equations". *Proceedings of the Royal Society A* 478.2259 (2022), p. 20210944.

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