On the self-similar solutions to the curvature flow for curves

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Symmetry, invariants and their applications

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Outline of the talk

- **Curve shortening flow (CSF)**
- **Soliton solutions on the sphere.**
- **Soliton solutions on the hyperbolic space.**

Definition **A family of curves** $\hat{X}^t : I \longrightarrow (M^2, g)$**,** $t \in [0, T)$**, is said to be a solution to the Curve Shortening Flow (CSF) with a given initial condition** $X : I \longrightarrow M^2$**, if**

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{X}^t(\cdot) = \hat{k}^t(\cdot)\hat{N}^t(\cdot) \\ \hat{X}^0(\cdot) = X(\cdot), \end{cases}$$

$\hat{k}^t(\cdot)$ **is the geodesic curvature,** $\hat{N}^t(\cdot)$ **unit normal vector field**

The geodesics are trivial solutions.

- **Epstein-Gage (1987) showed that when $M^2 = \mathbb{R}^2$, the CSF is geometrically the same for**

$$\frac{\partial}{\partial t}\hat{X}^t(\cdot) = \hat{k}^t(\cdot)\hat{N}^t(\cdot) + \textbf{tangential components}$$

$$\Downarrow$$

one can define the CSF to satisfy $\left\langle \frac{\partial}{\partial t}\hat{X}^t(\cdot), \hat{N}^t(\cdot) \right\rangle = \hat{k}^t(\cdot)$.

- **The name curve shortening flow is justified: when the curves $\hat{X}^t$ are closed, it is a gradient type of flow for the arc length functional.**

- **When the flow evolves by isometries or homotheties then $X(s)$ is called a self-similar solution to the CSF.**

- **The curve is a soliton if the flow evolves just by isometries.**

- **On $\mathbb{R}^2$ the Grim Reaper: graph of $f(s) = \ln(\cos(s))$ evolves by translations. Giga (2006): unique such curve on $\mathbb{R}^2$.**
- **The yin-yang spiral evolves by isometries of $\mathbb{R}^2$.**
- **Abresch-Langer and Epstein-Weinstein investigated the closed curves, not necessarily simple, that evolve by homotheties.**
- **In the 80s, Gage, Gage-Hamilton and finally Grayson closed embedded curves evolve to circular curves and then they collapse into a point at a finite time.**
- **Halldorsson (2012) described all self-similar solutions on $\mathbb{R}^2$.**
- **Angenent (1991): under some general conditions, the CSF evolves in a sense into a self-similar flow, showing the importance of self-similar solutions.**

• **Halldorsson (2015) classified the self-similar solutions on the Minkowski plane.**

Soliton solutions to CSF on $\mathbb{S}^2$ H. dos Reis ____ (2019)

Theorem $X(s)$ **a non-geodesic curve p.a.l. on** $\mathbb{S}^2 \subset \mathbb{R}^3$ **is a soliton solution to the CSF**

$$\Updownarrow$$

$\exists\, \mathbf{v} \in \mathbb{R}^{\mathbf{3}} \setminus \{\mathbf{0}\}$ **such that** $\langle T(s), v\rangle = \kappa(s)$**, where** $T = X'$ **and** κ **is the geodesic curvature of** X **(w.l.o.g.** $v = ae$**,** $e = (0, 0, 1)$**,** $a > 0$**).**

$$\Updownarrow$$

The functions $\alpha = \langle X, e\rangle$**,** $\tau = \langle T, e\rangle$**,** $\nu = \langle N, e\rangle$ **satisfy**

$$\begin{cases} \alpha' = \tau, \\ \tau' = a\tau\nu - \alpha, \\ \nu' = -a\tau^2, \end{cases} \qquad \textbf{with} \quad \alpha^2(0) + \tau^2(0) + \nu^2(0) = 1.$$

Theorem

- **For any $v \in \mathbb{R}^3 \setminus \{0\}$, there is a 2-parameter family of non trivial soliton solutions to the CSF on the sphere, $X(s)$ $s \in \mathbb{R}$.**
- **The two ends are asymptotic to the geodesic Γ of $\mathbb{S}^2$ orthogonal to v.**
- **If $0 < \|v\| < 2$, then X intersects Γ at infinitely many points**
- **If $\|v\| \geq 2$, then X intersects Γ at most on a finite number of points. In this case, each end of X converges to Γ without self-intersections.**

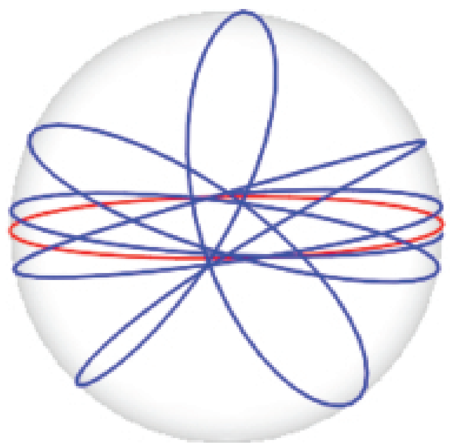


Figure 1: $|v| = 0.5$ infinite intersection points with Γ

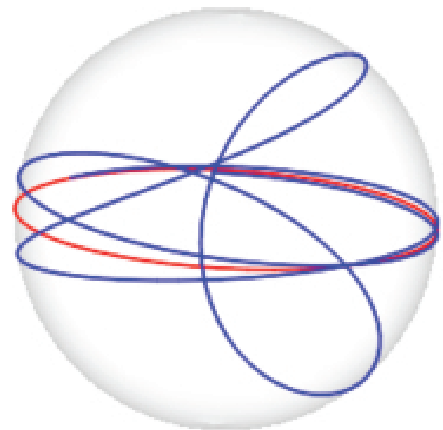


Figure 2: $|v| = 1$ infinite intersection points

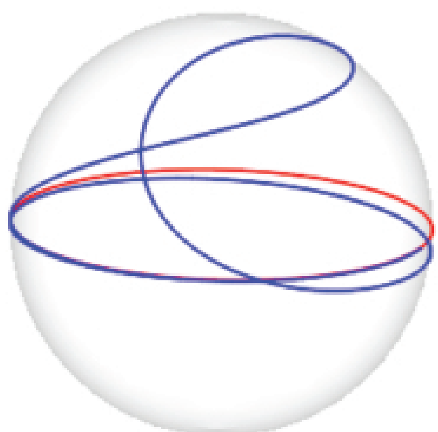


Figure 3: $|v| = 2$

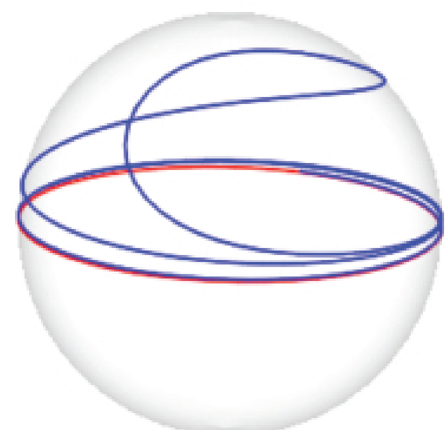


Figure 4: $|v| = 3$

Soliton solutions to the CSF on $\mathbb{H}^2$

Fábio N. da Silva, ______ (2022)

We consider $\mathbb{H}^2 \subset \mathbb{R}^3_1$, with the Minkowski metric on $\mathbb{R}^3_1$

$$\langle u, v \rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3.$$

- **The CSF on $\mathbb{H}^2$ is geometrically invariant if tangential components are added to the right hand side of $\frac{\partial}{\partial t}\hat{X}^t = \hat{k}^t \hat{N}^t$.**
- **It is a gradient type of flow for the arc length functional on $\mathbb{H}^2$.**

$X : I \to \mathbb{H}^2 \subset \mathbb{R}^3_1$ **a regular curve parametrized by arc length** s**.**

$T(s) = X'(s)$ **the tangent vector field,**

$N(s) = X(s) \times T(s)$ **the unit normal vector field**

$k(s) = \langle T'(s), N(s) \rangle$ **the geodesic curvature of** X**.**

A one parameter family of curves $\hat{X} : I \times J \to \mathbb{H}^2$ **is called a curve shortening flow (CSF) with initial condition** X**, if**

$$\begin{cases} \left\langle \dfrac{\partial}{\partial t} \hat{X}(s,t), \hat{N}(s,t) \right\rangle = \hat{k}(s,t), \\ \hat{X}(s,0) = X(s), \end{cases}$$

where $\hat{k}^t(\cdot) = \hat{k}(\cdot, t)$ **is the geodesic curvature**

$\hat{N}^t(\cdot) = \hat{N}(\cdot, t)$ **is the unit normal vector field of** $\hat{X}^t(\cdot) = \hat{X}(\cdot, t)$**.**

Solitons of the CSF on $\mathbb{H}^2$

Definition **Let $\hat{X} : I \times J \to \mathbb{H}^2 \subset \mathbb{R}^3_1$ be a solution to the CSF, with initial condition $X : I \to \mathbb{H}^2$. We say that X is a soliton solution to the CSF if**

$$\hat{X}^t(s) = M(t)X(s), \quad \textbf{for all} \quad t \in J,$$

where $M(t) : \mathbb{H}^2 \to \mathbb{H}^2$ is a 1-parameter family of isometries, such that $M(0) = Id$ is the identity map.

An isometry of $\mathbb{H}^2$ is an element of the Lie group $O_1(3)$ that preserves $\mathbb{H}^2$.

Theorem. **Let $X : I \to \mathbb{H}^2$ be a regular curve parametrized by arc length $s \in I$. Then X is a soliton solution to the CSF**

$$\Updownarrow$$

there is a vector $v \in \mathbb{R}^3_1 \setminus \{0\}$ such that

$$\langle T(s), v\rangle = k(s), \qquad \forall s \in I,$$

where $T(s)$ is the unit tangent vector field and $k(s)$ is the geodesic curvature of X.

Remark **w.l.o.g, up to isometries of $\mathbb{H}^2$,**

$$v \text{ is a multiple of } \begin{cases} w_1 = (1,0,0) & if \;\; v \text{ is timelike}, \\ w_2 = (1,1,0) & if \;\; v \text{ is lightlike}, \\ w_3 = (0,0,-1) & if \;\; v \text{ is spacelike}. \end{cases}$$

Depending on the type of the vector v, the curvature is given by

$$k_i(s) = \langle T(s), v_i \rangle \ \textbf{where}\ \mathbf{v_i = aw_i}, \quad a > 0 \quad i = 1, 2, 3.$$

The curve X evolves as $\hat{X}_i(s,t) = M_i(t)X(s)$, where

$$M_1(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(at) & \sin(at) \\ 0 & -\sin(at) & \cos(at) \end{pmatrix},$$

$$M_2(t) := \begin{pmatrix} 1 + \frac{(at)^2}{2} & -\frac{(at)^2}{2} & at \\ \frac{(at)^2}{2} & 1 - \frac{(at)^2}{2} & at \\ at & -at & 1 \end{pmatrix},$$

$$M_3(t) := \begin{pmatrix} \cosh(at) & \sinh(at) & 0 \\ \sinh(at) & \cosh(at) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

System of ODEs

Proposition: **A curve** $X : I \to \mathbb{H}^2$**, parametrized by arc length** s**, is a soliton solution to the** $\boxed{\text{CSF}}$ **i.e.** $\exists v \in \mathbb{R}^3_1 \setminus \{0\}$**, such that**

$$k_(s) = \langle T(s), v \rangle, \qquad \forall\, s \in I,$$

$$\Updownarrow$$

$v = ae_i$**, for** $a > 0$**,** $i \in \{1, 2, 3\}$ **and the functions** $\alpha_i(s) = \langle X(s), e_i \rangle$, $\tau_i(s) = \langle T(s), e_i \rangle$, $\eta_i(s) = \langle N(s), e_i \rangle$**, satisfy the system of ODEs**

$$\begin{cases} \alpha_i'(s) = \tau_i(s), \\ \tau_i'(s) = a\tau_i(s)\eta_i(s) + \alpha_i(s), \\ \eta_i'(s) = -a\tau_i^2(s), \end{cases}$$

with initial condition $(\alpha_i(0), \tau_i(0), \eta_i(0))$ **satisfying**

$$-\alpha_i^2(0) + \tau_i^2(0) + \eta_i^2(0) = \begin{cases} -1, & \textbf{if } i = 1, \\ 0, & \textbf{if } i = 2, \\ 1, & \textbf{if } i = 3. \end{cases}$$

Relating solutions of the system to soliton curves

Proposition. Let $(\alpha(s), \tau(s), \eta(s))$ **be a solution to the system**

$$\begin{cases} \alpha'(s) = \tau(s) \\ \tau'(s) = a\tau(s)\eta(s) + \alpha(s) \\ \eta'(s) = -a\tau^2(s), \end{cases} \qquad a > 0, \ \textbf{fixed} \ ,$$

and initial conditions at $s = 0$**, satisfying**

$$-\alpha^2(0) + \tau^2(0) + \eta^2(0) = -1 \qquad \textbf{(resp. } 0 \textbf{ and } 1\textbf{)},$$

$$\Downarrow$$

$\exists\, \mathbf{X} : \mathbf{I} \to \mathbb{H}^{\mathbf{2}}$ **p.a.l.** s**, such that** $k = a\tau$, T **and** N **satisfy**

$$\alpha(s) = \langle X(s), e\rangle, \quad \tau(s) = \langle T(s), e\rangle \ \textbf{ and } \ \eta(s) = \langle N(s), e\rangle,$$

where $e = (-1, 0, 0)$ **(resp.** $e = (-1, 1, 0)$ **and** $e = (0, 0, 1)$**).**

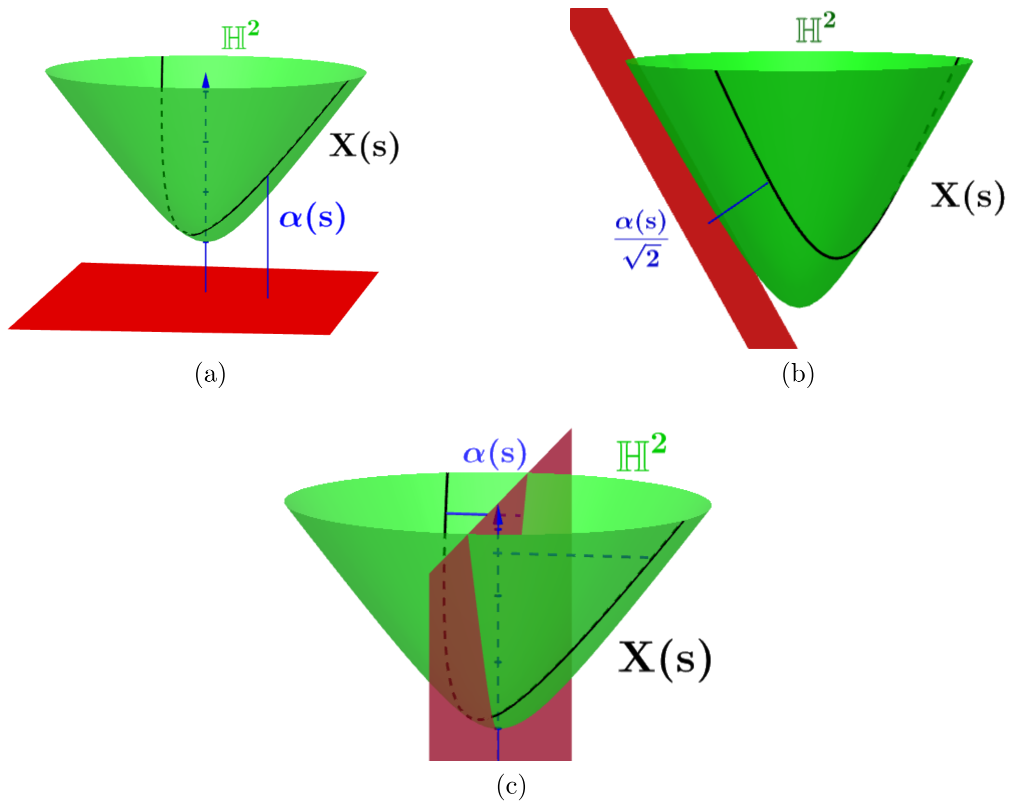


Figure 5: Geometric interpretation of the functions $\alpha(s)$, when $e = (-1, 0, 0)$, $e = (-1, 1, 0)$, $e = (0, 0, 1)$.

Investigating soliton curves to the CSF on $\mathbb{H}^2$ is equivalent to studying the solutions $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ of the system of ODEs, for each constant $a > 0$, defined on a maximal interval $I = (\omega_-, \omega_+)$, with initial condition $\psi(0) \in H \cup C \cup S \subset \mathbb{R}^3$, where

$$H := \{(\alpha, \tau, \eta) \in \mathbb{R}^3 : -\alpha^2 + \tau^2 + \eta^2 = -1, \alpha > 0\},$$
$$C := \{(\alpha, \tau, \eta) \in \mathbb{R}^3 \setminus \{0\} : -\alpha^2 + \tau^2 + \eta^2 = 0, \alpha > 0\},$$
$$S := \{(\alpha, \tau, \eta) \in \mathbb{R}^3 : -\alpha^2 + \tau^2 + \eta^2 = 1\}.$$

•Particular solutions: $\tau(\mathbf{s}) = \mathbf{b} \in \mathbb{R}$ constant $\Leftrightarrow$ $\psi(\mathbf{0}) \in \mathbf{S}$, $\mathbf{I} = \mathbb{R}$, and $\mathbf{b} \in \{-\mathbf{1}, \mathbf{0}, \mathbf{1}\}$, called trivial solutions. Moreover,

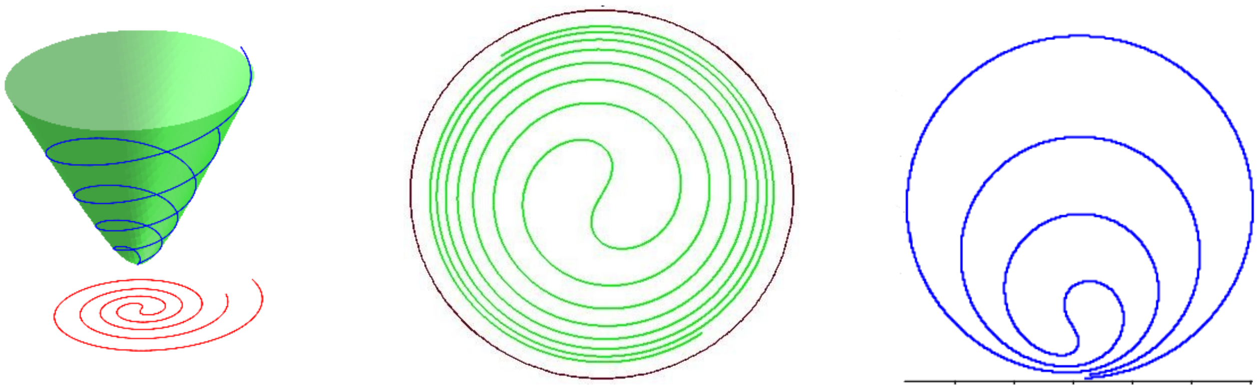
- **i) If $b = 0$, then $\psi(s) = (0, 0, \pm 1)$ are singular solutions. They correspond to geodesics on $\mathbb{H}^2$.**

- **ii) If $b^2 = 1$, then $a = 1$ and $\psi(s) = (\pm s + \alpha(0), \pm 1, -s \pm \alpha(0))$. They correspond to planar curves on $\mathbb{H}^2$.**

Theorem.

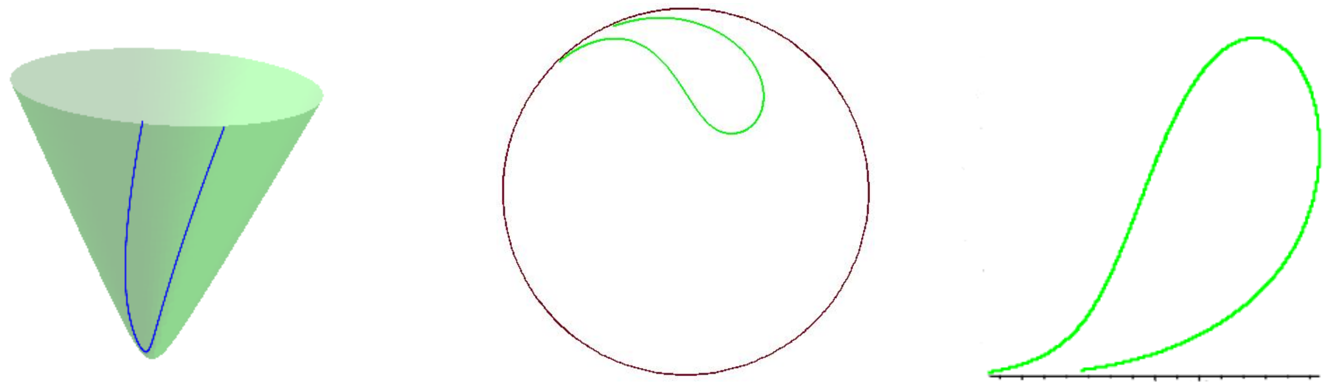
- **For any $v \in \mathbb{R}^3_1 \setminus \{0\}$, there is a 2-parameter family of non-trivial (non constant curvature) soliton solutions to the CSF on $\mathbb{H}^2$.**
- **There are three classes of soliton curves on $\mathbb{H}^2$, determined by the type of the vector v.**
- **A series of lemmas inply that each soliton**
  - **is defined on the whole real line;**
  - **at each end, the curvature tends to a constant $\in \{-1, 0, 1\}$.**
  - **is an embedded curve on $\mathbb{H}^2$;**

Visualizing some soliton solutions to the CSF on $\mathbb{H}^2$



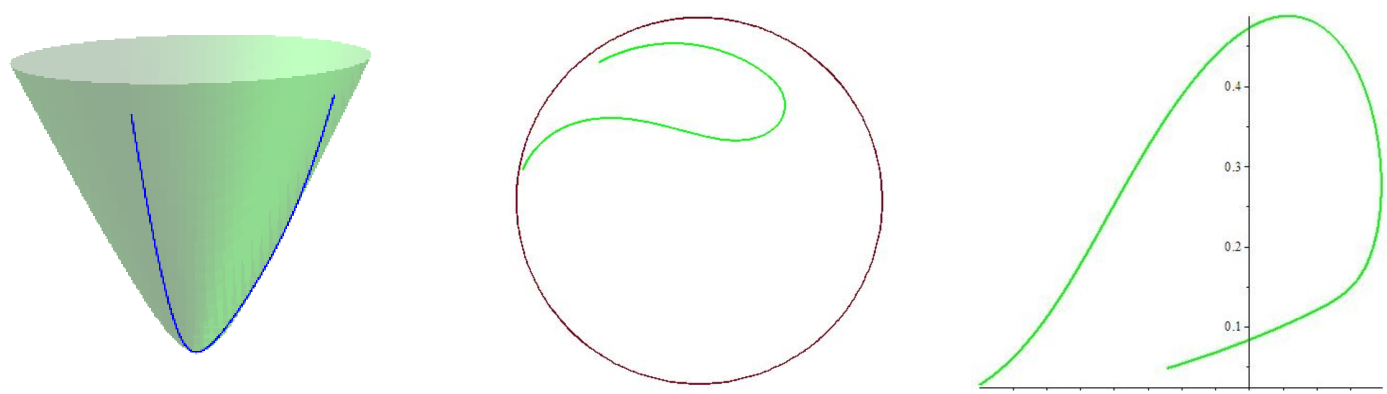
(a) (b) (c)

Figure 6: Soliton solution to the CSF on $\mathbb{H}^2$ with fixed vector $v = (-1, 0, 0)$ and $a = 1$.



(a) (b) (c)

Figure 7: Soliton solution to the CSF on $\mathbb{H}^2$ with fixed vector $(-1, 1, 0)$ and $a = 1$.



(a) (b) (c)

Figure 8: Soliton solution to the CSF on $\mathbb{H}^2$ with fixed vector $(0, 0, 1)$ and $a = 1$.

Remark:

- **Partial results on the $\mathbb{H}^2$ were also obtained, independently, by Woolgar-Xie.**
- **Fábio N. da Silva, ________ - "Self-similar solutions to the curvature flow and its inverse on the 2-dimensional light cone" 2022, just appeared on line.**

References:

1. dos Reis, H.F.S., Tenenblat, K., "Soliton solutions to the Curve Shortening Flow on the sphere", Proc. A.M.S. 147 (11) (2019) 4955-4967, DOI.org/10.1090/proc/14178.

2. Nunes da Silva, F., Tenenblat, K. "Soliton solutions to the curve shortening flow on the 2-dimensional hyperbolic space", Rev. Mat. Iberoam. 2022, DOI.org/10.4171/RMI/1343.

3. Nunes da Silva, F., Tenenblat, K. "Self-similar solutions to the curvature flow and its inverse on the 2-dimensional light cone", Annali di Matematica Pura ed Applicata 2022 DOI.org/10.1007/s10231-022-01240-8.

THANK YOU !

HAPPY BIRTHDAY PETER !!