On symmetries of weak solutions: a rigorous approach by diffeologies

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Outline of the presentation

Part 1 : On diffeologies

Part 2 : Weak and strong solutions of functional equations with parameters

Part 3 : Symmetry groups, infinitesimal symmetries

Part 4 : Invariants of the equations

Motivations for diffeologies

Define a "user-friendly" category for differential geometry including :

- infinite dimensional objects with no atlas
- objects with singularities of various kind : quotients, orbifolds, quasifolds etc.
- "embarrassing" quotients : $\mathbb{R}/\mathbb{Q}$, the irrational torus,

Diffeologies provide a proper framework for :

- calculus of variations
- defining smooth maps, operations, actions
- defining the de Rham complex, symplectic forms, (smooth) homotopy
- defining "bundles"

Necessary notions

Definition (Diffeology)

Let X be a set. A **parametrisation** of X is a map of sets $p\colon U \to X$ where U is an open subset of Euclidean space (no fixed dimension). A **diffeology** $\mathcal{P}$ on X is a set of parametrisations satisfying the following three conditions :

1. (Covering) $\forall x \in X$, $\forall n \in \mathbb{N}$, the constant function $p\colon \mathbb{R}^n \to \{x\} \subseteq X$ is in $\mathcal{P}$.
2. (Locality) Let $p\colon U \to X$ be a parametrisation such that for every $u \in U$ there exists an open neighbourhood $V \subseteq U$ of u satisfying $p|_V \in \mathcal{P}$. Then $p \in \mathcal{P}$.
3. (Smooth Compatibility) Let $(p\colon U \to X) \in \mathcal{P}$. Then for every n, every open subset $V \subseteq \mathbb{R}^n$, and every smooth map $F\colon V \to U$, we have $p \circ F \in \mathcal{P}$.

A set X equipped with a diffeology $\mathcal{P}$ is called a **diffeological space**, and the parametrisations $p \in \mathcal{P}$ are called **plots**.

Necessary notions (2)

Definition (Diffeologically Smooth Map)

Let $(X, \mathcal{P}_X)$ and $(Y, \mathcal{P}_Y)$ be two diffeological spaces, and let $F\colon X \to Y$ be a map. Then we say that F is **smooth** if $F \circ \mathcal{P}_X \subset \mathcal{P}_Y$.

With the above notations, we define

- the pull-back diffeology $F^*(\mathcal{P}_Y)$ as the diffeology, maximal for inclusion, for which F is smooth.
- the push-forward diffeology $F_*(\mathcal{P}_X)$ as the diffeology, minimal for inclusion, for which F is smooth. ($\Rightarrow$ any quotient $X/\sim$ of a diffeological space X carries a natural diffeology $\pi_*(X)$).
- Let $(X_i, \mathcal{P}_i)_I$ be a family of diffeological spaces, and let $\pi_k : \prod_I X_i \to X_k$, $\mathcal{P}_I = \bigcap_I \pi_i^*(\mathcal{P}_i)$ is the product diffeology.

Necessary diffeologies for this talk

- Diffeologies for (infinite dimensional) manifolds
  Let M be a smooth manifold and
  $$\mathcal{P}_\infty(M) = \coprod_O C^\infty(O, M)$$
- The Cauchy diffeology. Let Y be a complete topological vector space and let $X \subset Y$ as a diffeological space with smooth inclusion i in Y.
  $\mathcal{C}(X, Y)$ is the subset of $X^{\mathbb{N}}$ of sequences that converge in Y.
  The Cauchy diffeology is defined as
  $$\mathcal{P}_{\mathcal{C}} = \mathcal{P}(X^{\mathbb{N}})|_{\mathcal{C}(X,Y)} \cap \lim{}^* \mathcal{P}_\infty(Y)$$
- The group of diffeomorphisms of a diffeological space is a diffeological space for which composition and inversion is smooth.

Diffeological functional equations with parameters

Let Z be a LCTVS and let Y be a Fréchet space with dense subset X (with diffeology). Let Q be a diffeological space of parameters.

Definition

A **smooth functional equation** is defined by a smooth map $F : X \times Q \rightarrow Z$ and by the condition

$$F(u, q) = 0 \tag{1}$$

The set $Num_F(Y) \subset C^\infty(Q, \mathcal{C}(X, Y))$ of $Y-$**smooth numerical schemes** is such that such that, if $x(q) = (x_n)_{n\in\mathbb{N}} \in Num_F(Y)(q)$ for $q \in Q$, $\lim_{n\rightarrow+\infty} F(x_n, q) = 0$. The space of $Q-$parametrized solutions of (1) is

$$\mathcal{S}_Y(F) = \left\{ \lim_{n\rightarrow+\infty} x \in C^\infty(Q, Y) \,|\, x \in Num_F(Y) \right\}.$$

Strong solutions in X versus weak solutions in Y.

Let $S(F)$ be the set of strong solutions to (1) and let $S_X(F)$ the limits **in X** of sequences (u_n) such that $\exists q, \lim F(u_n, q) = 0$.
Since $X \subset Y$, then $S_F \subset S_X(F) \subset S_Y(F)$.

Lemma

(exercise) $S(F) = S_X(F)$ *as diffeological spaces.*

BUT There may happen that

$$\lim_* \mathcal{P}(\mathcal{C}(X, Y)) \neq \mathcal{P}_\infty(Y)$$

Counter-example : $X = \mathbb{R}^*$ and $Y = \mathbb{R}$

Open question : what diffeology is adequate for $S_Y(F)$?

What should be a symmetry ?

Starting point : a symmetry should transform a solution of (1) to another solution of (1)

Various possible approaches for strong solutions which all carry diffeologies :

- $C^\infty(S(F), S(F))$
- the smooth maps in $C^\infty(X, X)$ which restrict to smooth maps in $C^\infty(S(F), S(F))$
- the smooth maps (or diffeomorphisms) in $C^\infty(X, X)$ which restrict to diffeomorphisms $S(F)$
- $Diff(S(F))$

which have their pending parts for weak solutions in Y, but especially :

- $C^\infty(S_Y(F), S_Y(F))$ (with which diffeology on $S_Y(F)$?)
- $Diff(S_Y(F))$
- automorphisms of the diffeological fiber pseudo-bundle $Num_F(Y) \to S_Y(F)$ (with the push-forward diffeology on $S_Y(F)$ via the limit map).

On infinitesimal symmetries

Starting point : an infinitesimal symmetry should be tangent at the identity, considered as an element of the space of symmetries of (1).

- There exists actually 3 types of tangent spaces to a diffeological space : diff, kinetic or exterior,
- If the space of symmetries is not a group, the tangent space may not be a vector space,
- Even if the space of symmetries is a diffeological group, the infinitesimal symmetries may not act on X or Y,
- infinitesimal symmetries may not "integrate" to symmetries (no exponential map).

Documents :

- On the differential geometry of numerical schemes and weak solutions of functional equations. *Nonlinearity* **33**, No. 12, 6835-6867 (2020).
- On diffeological infinite dimensional bundles and pseudo-bundles : examples of interest, results and applications (in preparation)

Perspectives : apply these propositions to examples of interest, in which one of these propositions is **feasible** and **useful** in order to :

- study weak solutions
- describe efficiently invariants of the functional equations that can be defined through geometric invariants of the symmetries.

Thanks for your attention and for your interest